Microeconomics

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October, 2015

Course Outline (1)

Learning Objectives:

- This course covers key concepts of microeconomic theory. The main goal of this course is to provide students with both, a basic understanding and analytical traceability of these concepts.

- The main concepts are discussed in detail during the lectures. In addition students have to work through the textbooks and have to solve problems to improve their understanding and to acquire skills to apply these tools to related problems.

Course Outline (2)

Literature:

- Andreu Mas-Colell, A., Whinston, M.D., Green, J.R., *Microeconomic Theory*, Oxford University Press, 1995. (MWG in the following)
- Gravelle, H. and Rees, R., *Microeconomics*, 3^{rd} edition, Prentice Hall, 2004. (GR in the following)

Supplementary Literature:

- Gilboa, I., *Theory of Decision under Uncertainty*, Cambridge University Press, 2009.
- Gollier C., *The Economics of Risk and Time*, Mit Press, 2004.
- Jehle G.A. and P. J. Reny, *Advanced Microeconomic Theory*, Addison-Wesley Series in Economics, Longman, Amsterdam, 2000.
- Ritzberger, K., *Foundations of Non-Cooperative Game Theory*, Oxford University Press, 2002.

Course Outline (3)

- The slides you find in the university information system contain (most of) the topics covered in MWG Chapters 1-6, 10 and 15 and 16.

- For the slides I'm grateful to comments by my colleges Egbert Dierker and Martin Meier as well as to all former students in Bruno and Vienna.

- For the course in Brno I'm grateful for comments from and discussions with Rostislav Staněk, Josef Menšik, by Ondřej Krčál, and many others.

Course Outline (4)

1. Decision theory and the theory of the consumer:
   - Rationality, preference primitives and axioms, preference representations and utility (MWG 1-3, GR 2A-B).
   - Utility maximization, Walrasian demand and comparative statics (MWG, 2, 3D, GR 2C-D).
   - Indirect utility, expenditure function, Hicksian demand (MWG 3E,G, GR 3A).
   - Slutsky equation, substitution and wealth effect (MWG 3 G, GR 3B).

Course Outline (5)

2. Production and cost:

   – Production functions, returns to scale (GR 5).
   – Production set (MWG 5 B).
   – Cost minimization, conditional factor demands, cost function (MWG 5 C, GR 6.A,B,E).
   – Profit maximization, input demands, profit function, objectives of the firm (MWG 5 C,G, GR 7.A,C,D).

Course Outline (6)

3. General Equilibrium:
   - Introduction, Walrasian equilibrium (MWG 15, GR 12.A-D).
   - The Edgeworth box (MWG 15B, GR 12.E).
   - Welfare theorems (MWG16 A-D, GR 13).

4. Decisions under uncertainty:
   - Expected utility theorem, risk aversion (MWG 6A-C,GR 17 A-D).

Course Outline (7)

- Winter Term 2015 (see university information system)
  - First block: October 22-23; 2015.
  - Second block: November 19-20, 2015.
  - Third block: December 17-18, 2015.
- Contact hours (per semester): 12 units a 90 minutes.
- Practice session will be organized by Rostislav Staněk.

Course Outline (8)

Some more comments on homework and grading:

- Final test (80%).
- Homework and practice session (20%).
- Final test and retakes: tba

Consumer Theory (1)
Rationality and Preferences

- Consumption set X

- Rationality

- Preference relations and utility

- Choice correspondences and the weak axiom of revealed preference

- Relationship between the axiomatic approach and the revealed preference approach

MWG, Chapter 1.

Consumer Theory (1)
Rationality (1)

- We consider agents/individuals and goods that are available for purchase in the market.

- **Definition**: The set X of all possible mutually exclusive alternatives (complete consumption plans) is called **consumption set** or **choice set**.

- "Simplest form of a consumption set": We assume that each good, $x_l \in X$, $l = 1, \ldots, L$ can be consumed in infinitely divisible units, i.e. $x_l \in \mathbb{R}_+$. With L goods we get the **commodity vector** x in the commodity space $\mathbb{R}_+^L$.

Consumer Theory (1)
Rationality (2)

- Approach I: describe behavior by means of preference relations; preference relation is the primitive characteristic of the individual.

- Approach II: the choice behavior is the primitive behavior of an individual.

Consumer Theory 1
Rationality (3)

- Consider the binary relation "at least good as", abbreviated by the symbol $\succeq$.

- For $x, y \in X$, $x \succeq y$ implies that from a particular consumer's point of view x is preferred to y or that he/she is indifferent between consuming x and y.

- From $\succeq$ we derive the **strict preference relation** $\succ$: $x \succ y$ if $x \succeq y$ but not $y \succeq x$ and the **indifference relation** $\sim$ where $x \succeq y$ and $y \succeq x$.

Consumer Theory 1
Rationality (4)

- Often we require that pair-wise comparisons of consumption bundles are possible for all elements of X.

- **Completeness:** For all $x, y \in X$ either $x \succeq y$, $y \succeq x$ or both.

- **Transitivity:** For the elements $x, y, z \in X$: If $x \succeq y$ and $y \succeq z$, then $x \succeq z$

- **Definition**[D 1.B.1]: The preference relation $\succeq$ is called **rational** if it is complete and transitive.

- Remark: Reflexive $x \succeq x$ follows from completeness [D 1.B.1].

Consumer Theory 1
Rationality (5)

- Based on these remarks it follows that:

  Proposition [P 1.B.1]: If $\succeq$ is rational then,

  - $\succ$ is transitive and irreflexive.
  - $\sim$ is transitive, reflexive and symmetric.
  - If $x \succ y \succeq z$ then $x \succ z$.

Consumer Theory 1
Utility (1)

- **Definition** : A function $X \to \mathbb{R}$ is a **utility function** representing $\succeq$ if for all $x, y \in X$ $x \succeq y \;\Leftrightarrow\; u(x) \geq u(y)$. [D 1.B.2]

- Does the assumption of a rational consumer imply that the preferences can be represented by means of a utility function and vice versa?

- **Theorem**: If there is a utility function representing $\succeq$, then $\succeq$ must be complete and transitive. [P 1.B.2]

- The other direction requires more assumptions on the preferences - this comes later & in Micro II!

Consumer Theory 1
Choice Rules (1)

- Now we follow the behavioral approach (vs. axiomatic approach), where choice behavior is represented by means of a **choice structure**.

- **Definition - Choice structure**: A choice structure $(\mathcal{B}, C(.))$ consists: (i) of a family of nonempty subsets of X; with $B \subset X$. The elements of $\mathcal{B}$ are the *budget sets* B. and (ii) a choice rule $C(.)$ that assigns a nonempty set of chosen elements $C(B) \in B$ for every $B \in \mathcal{B}$.

- $\mathcal{B}$ is an exhaustive listing of all choice experiments that a restricted situation can pose on the decision maker. It need not include all subsets of X.

Consumer Theory 1
Choice Rules (2)

- The choice rule assigns a set $C(B)$ to every element B, i.e. it is a correspondence.

- Example [1.C.1,(i)]: $X = \{x, y, z\}$. $\mathcal{B} = \{B_1, B_2\}$ with $B_1 = \{x, y\}$ and $B_2 = \{x, y, z\}$. $C(B_1) = \{x\}$ and $C(B_2) = \{x\}$.

- Example [1.C.1,(ii)]: $X = \{x, y, z\}$. $\mathcal{B} = \{B_1, B_2\}$ with $B_1 = \{x, y\}$ and $B_2 = \{x, y, z\}$. $C(B_1) = \{x\}$ and $C(B_2) = \{x, y\}$.

- Put structure on choice rule - weak axiom of revealed preference.

Consumer Theory 1
Choice Rules (3)

- **Definition - Weak Axiom of Revealed Preference**: A choice structure $(\mathcal{B}, C(.))$ satisfies the weak axiom of revealed preference if for some $B \in \mathcal{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$. [D 1.C.1]

- I.e. if we have a budget set where both x and y are available and x is chosen, then there cannot be a budget set containing both bundles, for which y is chosen and x is not.

- Example: $C(\{x, y\}) = x$, then $C(\{x, y, z\}) = y$ is not possible.

Consumer Theory 1
Choice Rules (4)

- **Definition - Revealed Preference Relation $\succeq^*$ based on the Weak Axiom**: Given a choice structure $(\mathcal{B}, C(.))$, $\succeq^*$ is defined by:
  $x \succeq^* y \Leftrightarrow$ there is some $B \in \mathcal{B}$ such that $x, y \in B$ and $x \in C(B)$. [D 1.C.2]

- In other words: If x is revealed at least as good as y, then y cannot be revealed preferred to x.

- Example [1.C.1,(i)] satisfies the weak axiom, Example [1.C.1,(ii)] does not.

Consumer Theory 1
Choice Rules (5)

- If a decision maker has a rational preference ordering $\succeq$, does the choice structure satisfy the weak axiom of reveal preferences?

- Given a choice structure satisfying the weak axiom, does this result in a rational preference relation?

- **Definition**: Choice structure implied by a rational preference relation $\succeq$: $C^*(B, \succeq) = \{x \in B | x \succeq y$ for every $y \in B\}$. Assume that $C^*(B, \succeq)$ is nonempty.

- **Proposition**: If $\succeq$ is a rational preference relation, then the choice structure $C^*(B, \succeq)$ satisfies the weak axiom. [P 1.D.1]

Proof:

- Suppose that x, y in some $B \in \mathcal{B}$ and $x \in C^*(B, \succeq)$, then (i) $x \succeq y$.

- We have to show that if x, y are in some B' and $y \in C^*(B', \succeq)$, then also $x \in C^*(B', \succeq)$.

- Suppose that x, y in $B' \in \mathcal{B}$ and $y \in C^*(B', \succeq)$, then (ii) $y \succeq z$ for all $z \in B'$.

- The rational preference relation $\succeq$ is transitive. Hence (i), (ii) and transitivity imply that $x \succeq y \succeq z$. Therefore $x \in C^*(B, \succeq)$ by the definition of this set.

Choice Rules (7)

- Other direction. We want to know whether the choice rule $C(.)$ satisfying the weak axiom is equal to $C^*(., \succeq)$ for some rational preference relation $\succeq$.

- **Definition:** Given a choice rule $C(.)$. $\succeq$ **rationalizes the choice rule** if $C(B) = C^*(B, \succeq)$ for all $B \in \mathcal{B}$. [D 1.D.1]

- **Proposition**: If a choice structure $(\mathcal{B}, C(.))$ satisfies the weak axiom of revealed preference and $\mathcal{B}$ includes all subsets of X of up to three elements, then there is a rational preference relation $\succeq$ that rationalizes $C(.)$ relative to $\mathcal{B}$. That is, $C(B) = C^*(B, \succeq)$ for all $B \in \mathcal{B}$. The rational preference relation is the unique preference relation that does so. [P 1.D.2]

Consumer Theory 1
Choice Rules (8)

- Example [1.D.1]: Counterexample that choice structure cannot be rationalized if $\mathcal{B}$ does not contain all subsets up to three elements. $X = \{x, y, z\}$. $\mathcal{B} = \{B_1, B_2, B_3\}$ with $B_1 = \{x, y\}$, $B_2 = \{y, z\}$ and $B_3 = \{x, z\}$. $C(B_1) = \{x\}$, $C(B_2) = \{y\}$ and $C(B_3) = \{z\}$. Here the weak axiom is satisfied. To rationalize this choice structure it would be necessary that $x \succ y$, $y \succ z$ and $z \succ x$, which does not satisfy the transitivity property.

Consumer Theory 1
Choice Rules (9)

- Remark: The more subsets of X we consider, the stronger the restrictions implied by the weak axiom.

- Remark: If the choice structure is defined for all subsets of X, then the approaches based on the preference relation and on the weak axiom are equivalent. To consider all budget sets is a strong requirement. Another way to get equivalence, is to replace the weak axiom by the **strong axiom of revealed preference**.

Consumer Theory 2
Walrasian Demand (1)

- Competitive budget sets.

- Walrasian/Marshallian Demand

- Walras' law, Cournot and Engel aggregation.

MWG: Chapter 2

Consumer Theory 2
Consumption Set (1)

- We have already defined the consumption set: The set of all alternatives (complete consumption plans). We assumed $X = \mathbb{R}^L_+$.

- Each x represents a different consumption plan.

- Physical restrictions: divisibility, time constraints, survival needs, etc. might lead to a strict subset of $\mathbb{R}^L_+$ as consumption set X.

Consumer Theory 2
Budget Set (1)

Definition - Budget Set: B

- Due to constraints (e.g. income) we cannot afford all elements in X, problem of scarcity.

- The budget set B is defined by the elements of X, which are achievable given the economic realities.

- $B \subset X$.

Consumer Theory 2
Budget Set (2)

- By the consumption set and the budget set we can describe a consumer's alternatives of choice.

- These sets do not tell us what x is going to be chosen by the consumer.

- To describe the choice of the consumer we need a theory to model or describe the preferences of a consumer (or the choice structure).

Consumer Theory 2
Competitive Budgets (1)

- Assumption: All L goods are traded in the market (principle of completeness), the prices are given by the price vector p, $p_l > 0$ for all $l = 1, \ldots, L$. Notation: $p \gg 0$. Assumption - the prices are constant and not affected by the consumer.
- Given a wealth level $w \geq 0$, the set of affordable bundles is described by

$$p \cdot x = p_1 x_1 + \cdots + p_L x_L \leq w.$$

- **Definition - Walrasian Budget Set**: The set $B_{p,w} = \{x \in \mathbb{R}_+^L | p \cdot x \leq w\}$ is called **Walrasian** or **competitive budget set**. [D 2.D.1]
- **Definition - Consumer's problem**: Given p and w choose the optimal bundle x from $B_{p,w}$.

Consumer Theory 2
Competitive Budgets (2)

- **Definition - Relative Price**: The ratios of prices p_j/p_i are called **Relative Prices**.

- Here the price of good j is expressed in terms of good i. In other words: The price of good x_i is expressed in the units of good x_j.

- On the market we receive for one unit of x_j, $p_j/p_i \cdot 1$ units of x_i.

- Example: $p_j = 4$, $p_i = 2$. Then $p_j/p_i = 2$ and we get two units of x_i for one unit of x_j.

Consumer Theory 2
Competitive Budgets (3)

- The budget set B describes the goods a consumer is able to buy given wealth level w.

- **Definition - Numeraire Good**: If all prices p_j are expressed in the prices of good n, then this good is called numeraire. p_j/p_n, $j = 1, \ldots, L$. The relative price of the numeraire is 1.

- There are $L - 1$ relative prices.

Consumer Theory 2
Competitive Budgets (4)

- The set $\{x \in \mathbb{R}^L_+ | p \cdot x = w\}$ is called **budget hyperplane**, for $L = 2$ it is called **budget line**.

- Given x and x' in the budget hyperplane, $p \cdot x = p \cdot x' = w$ holds. This results in $p(x - x') = 0$, i.e. p and $(x - x')$ are orthogonal - see Figure 2.D.3 page 22.

- The budget hyperplane is a convex set. In addition it is closed and bounded $\Rightarrow$ compact. $0 \in B_{p,w}$ (given the assumption that $p \gg 0$).

Consumer Theory 2
Demand Functions (1)

- **Definition - Walrasian demand correspondence**: The correspondence assigning to a pair (p, w) a set of consumption bundles is called **Walrasian demand correspondence** $x(p, w)$; i.e. $(p, w) \twoheadrightarrow x(w, p)$. If $x(p, w)$ is single valued for all p, w, $x(., .)$ is called *demand function*.

- **Definition - Homogeneity of degree zero**: $x(., .)$ is homogeneous of degree zero if $x(\alpha p, \alpha w) = x(p, w)$ for all p, w and $\alpha > 0$. [D 2.E.1]

- **Definition - Walras law, budget balancedness:** $x(., .)$ satisfies Walras law if for every $p \gg 0$ and $w > 0$, we get $p \cdot x = w$ for all $x \in x(p, w)$. That is, the consumer spends all income w with her/his optimal consumption decision. [D 2.E.2]

Consumer Theory 2
Demand Functions (2)

- Assume that $x(.,.)$ is a function:
- With p fixed at $\bar{p}$, the function $x(\bar{p},.)$ is called **Engel** function.
- If the demand function is differentiable we can derive the gradient vector: $D_w x(p,w) = (\partial x_1(p,w)/\partial w, \ldots, x_L(p,w)/\partial w)$. If $\partial x_l(p,w)/\partial w \geq 0$, x_l is called **normal** or superior, otherwise it is **inferior**.
- See Figure 2.E.1, page 25
- Notation: $D_w x(p,w)$ results in a $1 \times L$ row matrix, $D_w x(p,w) = (\nabla_w x(p,w))^\top$.

Consumer Theory 2
Demand Functions (3)

- With w fixed, we can derive the $L \times L$ matrix of partial derivatives with respect to the prices: $D_p x(p, w)$.
- $\partial x_l(p, w)/\partial p_k = [D_p x(p, w)]_{l,k}$ is called the **price effect**.
- A **Giffen good** is a good where the own price effect is positive, i.e. $\partial x_l(p, w)/\partial p_l > 0$
- See Figure 2.E.2-2.E.4, page 26.

Consumer Theory 2
Demand Functions (4)

- **Proposition**: If a Walrasian demand function $x(.,.)$ is homogeneous of degree zero and differentiable, then for all p and w:
$$\sum_{k=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k} p_k + \frac{\partial x_l(p,w)}{\partial w} w = 0$$
for $l = 1, \ldots, L$;, or in matrix notation $D_p x(p,w)p + D_w x(p,w) = 0$. [P 2.E.1]

- Proof: By the Euler theorem (if $g(.)$ is homogeneous of degree r, then $\sum \partial g(x)/\partial x \cdot x = rg(x)$, [MWG, Theorem M.B.2, p. 929]), the result follows directly when using the stacked vector $x = (p^\top, w)^\top$. Apply this to $x_1(p,w), \ldots, x_L(p,w)$.

Consumer Theory 2
Demand Functions (5)

- **Definition - Price Elasticity of Demand**: $\eta_{ij} = \frac{\partial x_i(p,w)}{\partial p_j} \frac{p_j}{x_i(p,w)}$.

- **Definition - Income Elasticity**: $\eta_{iw} = \frac{\partial x_i(p,w)}{\partial w} \frac{w}{x_i(p,w)}$.

- **Definition - Income Share**:

$$s_i = \frac{p_i x_i(p, w)}{w} ,$$

where $s_i \geq 0$ and $\sum_{i=1}^{n} s_i = 1$.

Consumer Theory 3
The Axiomatic Approach (1)

- Axioms on preferences

- Preference relations, behavioral assumptions and utility (axioms, utility functions)

- The consumer's problem

- Walrasian/Marshallian Demand

  MWG, Chapter 3.A-3.D

Consumer Theory 3
The Axiomatic Approach (2)

- **Axiom 1 - Completeness:** For all $x, y \in X$ either $x \succeq y$, $y \succeq x$ or both.

- **Axiom 2 - Transitivity:** For the elements $x, y, z \in X$: If $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

- We have already defined a rational preference relation by completeness and transitivity [D 3.B.1].

- If the number of elements is finite it is easy to see that one can describe a preference relation by means of a function.

Consumer Theory 3
The Axiomatic Approach (3)

Sets arising from the preference relations:

- $\succeq (x) := \{y | y \in X, y \succeq x\}$ - at least as good (sub)set
- $\preceq (x) := \{y | y \in X, y \preceq x\}$ - the no better set
- $\succ (x) := \{y | y \in X, y \succ x\}$ - at preferred to set
- $\prec (x) := \{y | y \in X, y \prec x\}$ - worse than set
- $\sim (x) := \{y | y \in X, y \sim x\}$ - indifference set

Consumer Theory 3
The Axiomatic Approach (4)

- **Axiom 3.A - Local Nonsatiation**: For all $x \in X$ and for all $\varepsilon > 0$ there exists some $y \in X$ such that $||x - y|| \leq \varepsilon$ and $y \succ x$. [D 3.B.3]

- This assumptions implies that for every small distance ε there must exist at least one y, which is preferred to x.

- Indifference “zones“ are excluded by this assumption. See Figure 3.B.1 on page 43.

- **Axiom 3.B - Monotonicity**: For all $x, y \in \mathbb{R}^L_+$: If $x \geq y$ then $x \succeq y$ while if $x \gg y$ then $x \succ y$ (**weakly monotone**). It is **strongly/strict monotone** if $x \geq y$ and $x \neq y$ imply $x \succ y$. [D 3.B.2]

- Here $\geq$ means that at least one element of x is larger than the elements of y, while $x \gg y$ implies that all elements of x are larger than the elements of y.

- Remark: Local nonsatiation vs. monotonicity: The latter implies that more is always better, while Axiom 3.A only implies that in a set described by $||x - y|| \leq \varepsilon$ there has to exist a preferred alternative.

Consumer Theory 3
The Axiomatic Approach (6)

- Discuss the differences of Axioms 3.A and 3.B (what are their impacts on indifference sets?), e.g. by means Figures 3.B.1 and 3.B.2, page 43.

Consumer Theory 3
The Axiomatic Approach (7)

- Last assumption on taste - “mixtures are preferred to extreme realizations“

- See Figure 3.B.3, page 44.

- **Axiom 4.A - Convexity**: For every $x \in X$, if $y \succeq x$ and $z \succeq x$ then $\nu y + (1-\nu)z \succeq x$ for $\nu \in [0,1]$. [D 3.B.4]

- **Axiom 4.B - Strict Convexity**: For every $x \in X$, $y \succeq x$, $z \succeq x$ and $y \neq z$ then $\nu y + (1-\nu)z \succ x$ for $\nu \in (0,1)$. [D 3.B.5]

- Given these assumptions, indifference curves become (strict) convex.

Consumer Theory 3
The Axiomatic Approach (8)

- **Definition - Homothetic Preferences**: A monotone preference $\succeq$ on X is homothetic if all indifference sets are related by proportional expansions along rays. I.e. $x \sim y$ then $\alpha x \sim \alpha y$. [D 3.B.6]

- **Definition - Quasilinear Preferences**: A monotone preference $\succeq$ on $X = (-\infty, \infty) \times \mathbb{R}^{L-1}$ is quasilinear with respect to commodity 1 if : (i) all indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$ then $x + \alpha e_1 \sim y + \alpha e_1$ and $e_1 = (1, 0, \dots)$. (ii) Good one is desirable: $x + \alpha e_1 \succ x$ for all $\alpha > 0$. [D 3.B.7]

Consumer Theory 3
The Axiomatic Approach (9)

- With the next axiom we regularize our preference order by making it continuous:

- **Axiom 5 - Continuity:** A preference order $\succeq$ is continuous if it is preserved under limits. For any sequence $(x^{(n)}, y^{(n)})$ with $x^{(n)} \succeq y^{(n)}$ for all n, and limits x, y ($x = \lim_{n\to\infty} x^{(n)}$ and $y = \lim_{n\to\infty} y^{(n)}$) we get $x \succeq y$. [D 3.C.1]

- Equivalently: For all $x \in X$ the set "at least as good as" ($\succeq (x)$) and the set "no better than" ($\preceq (x)$) are closed in X.

Consumer Theory 3
The Axiomatic Approach (10)

- Topological property of the preference relation (important assumption in the existence proof of a utility function).

- By this axiom the set $\prec (x)$ and $\succ (x)$ are open sets (the complement of a closed set is open ...). $\succ (x)$ is the complement of $X \backslash \prec (x)$.

- The intersection of $\preceq (x) \cap \succeq (x)$ is closed (intersection of closed sets). Hence, indifference sets are closed.

- Consider a sequence of bundles $y^{(n)}$ fulfilling $y^{(n)} \succeq x$, for all n. For $y^{(n)}$ converging to y, Axiom 5 imposes that $y \succeq x$.

Consumer Theory 3
The Axiomatic Approach (11)

Lexicographic order/dictionary order:

- Given two partially order sets X_1 and X_2, an order is called lexicographical on $X_1 \times X_2$ if $(x_1, x_2) \prec (x_1', x_2')$ if and only if $x_1 < x_1'$ (or $x_1 = x_1',\ x_2 < x_2'$). That is, "good 1 is infinitely more desired than good 2".

- Example in $\mathbb{R}^2_+$ (Example of **Debreu**): $x = (0, 1)$ and $y^{(n)} = (1/n, 0)$, $y = (0, 0)$. For all n, $y^{(n)} \succ x$, while for $n \to \infty$: $y^{(n)} \to y = (0, 0) \prec (0, 1) = x$.

- The lexicographic ordering is a rational (strict) preference relation (we have to show completeness and transitivity).

Consumer Theory 3
The Axiomatic Approach (12)

- Axioms 1 and 2 guarantee that an agent is able to make consistent comparisons among all alternatives.

- Axiom 5 imposes the restriction that preferences do not exhibit “discontinuous behavior“ ; mathematically important

- Axioms 3 and 4 make assumptions on a consumer’s taste (satiation, mixtures).

Consumer Theory 3
Utility Function (1)

- **Definition: Utility Function**: A real-valued function $u : \mathbb{R}^L_+ \to \mathbb{R}$ is called utility function representing the preference relation $\succeq$ if for all $x, y \in \mathbb{R}^L_+$ $u(x) \geq u(y)$ if and only if $x \succeq y$.

- I.e. a utility function is a mathematical device to describe the preferences of a consumer.

- Pair-wise comparisons are replaced by comparing real valued functions evaluated for different consumption bundles.

- Function is of no economic substance (for its own).

Consumer Theory 3
Utility Function (2)

- First of all we want to know if such a function exists.

- **Theorem: Existence of a Utility Function**: If a binary relation $\succeq$ is complete, transitive and continuous, then there exists a continuous real valued function function $u(x)$ representing the preference ordering $\succeq$.

- Proof: by assuming monotonicity see page 47; Debreu's (1959) proof is more advanced.[P 3.C.1]

Consumer Theory 3
Utility Function (3)

- Consider $y = u(x)$ and the transformations $v = g(u(x))$; $v = \log y$, $v = y^2$, $v = a + by$, $v = -a - by$ (see MWG, page 49). Do these transformations fulfill the properties of a utility function?

- **Theorem: Invariance to Positive Monotonic Transformations**: Consider the preference relation $\succeq$ and the utility function $u(x)$ representing this relation. Then also $v(x)$ represents $\succeq$ if and only if $v(x) = g(u(x))$ is strictly increasing on the set of values taken by $u(x)$.

Proof:

- $\Rightarrow$ Assume that $x \succeq y$ with $u(x) \geq u(y)$: A strictly monotone transformation $g(.)$ then results in $g(u(x) \geq g(u(y))$. I.e. $v(x)$ is a utility function describing the preference ordering of a consumer.

Proof:

- $\Leftarrow$ Now assume that $g(u(x))$ is a utility representation, but g is not strictly positive monotonic on the range of $u(.)$: Then $g(u(x))$ is not $>$ to $g(u(y))$ for some pair x, y where $u(x) > u(y)$. Hence, for the pair x, y we have $x \succ y$ since $u(x) > u(y)$, but $g(u(x)) \leq g(u(y))$.

  This contradicts the assumption that $v(.) = g(u(.))$ is a utility representation of $\succeq$.

Consumer Theory 3
Utility Function (6)

- By Axioms 1,2,5 the existence of a utility function is guaranteed. By the further Axioms the utility function exhibits the following properties.

- **Theorem: Preferences and Properties of the Utility Function**:
  - $u(x)$ is strictly increasing if and only if $\succeq$ is strictly monotonic.
  - $u(x)$ is quasiconcave if and only if $\succeq$ is convex: $u(x^{\nu}) \geq \min\{u(x), u(y)\}$, where $x^{\nu} = \nu x + (1-\nu)y$.
  - $u(x)$ is strictly quasiconcave if and only if $\succeq$ is strictly convex. That is, $u(x^{\nu}) > \min\{u(x), u(y)\}$ for $x^{\nu} = \nu x + (1-\nu)y$, $x \neq y$ and $\nu \in (0,1)$.

Consumer Theory 3
Utility Function (7)

- **Definition: Indifference Curve**: Bundles where utility is constant (in $\mathbb{R}^2$).
- Marginal rate of substitution and utility: Assume that $u(x)$ is differentiable, then

$$\begin{aligned} du(x_1, x_2) &= \frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} dx_2 = 0 \\ \frac{dx_2}{dx_1} &= -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} \\ MRS_{12} &= \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}. \end{aligned}$$

- The marginal rate of substitution describes the trade-off between goods 1 and 2 that marginally keep the consumer indifferent at a given consumption bundle (x_1, x_2). That is, the "amount of good 2" the consumer has to obtain for giving up "one unit of good 1" while staying at the same utility level.

Consumer Theory 3
Utility Function (8)

- If $u(x)$ is differentiable and the preferences are strictly monotonic, then marginal utility is strictly positive.

- With strictly convex preferences the marginal rate of substitution is a strictly decreasing function (i.e. in $\mathbb{R}^2$ the slope of the indifference curve becomes flatter).

- For a quasiconcave utility function (i.e. $u(x^\nu) \geq \min\{u(x^1), u(x^2)\}$, with $x^\nu = \nu x^1 + (1-\nu)x^2$) and its Hessian $H(u(x))D^2(u(x))$ we get: $yH(u(x))y^\top \leq 0$ for all vectors y, where $grad(u(x)) \cdot y = 0$. That is, when moving from x to y that is tangent to the indifference surface at x utility does not increase (decreases if the equality is strict).

Consumer Theory 3
Consumer's Problem (1)

- The consumer is looking for a bundle x^* such that $x^* \in B$ and $x^* \succeq x$ for all x in the feasible set B.

- Assume that the preferences are complete, transitive, continuous, strictly monotonic and strictly convex. Then $\succeq$ can be represented by a continuous, strictly increasing and strictly quasiconcave utility function. Moreover we can assume that we can take first and second partial derivatives of $u(x)$. These are usual assumptions, we can also solve the **utility maximization problem** (UMP) with less stringent assumptions.

Consumer Theory 3
Consumer's Problem (1)

- We assume prices $p_i > 0$, $p = (p_1, \ldots, p_L)$ is the vector of prices. We assume that the prices are fixed from the consumer's point of view. (Notation: $p \gg 0$ means that all coordinates of p are strictly larger than zero.)

- The consumer is endowed with wealth w.

Consumer Theory 3
Consumer's Problem (2)

- Budget set induced by w: $B_{p,w} = \{x | x \in \mathbb{R}^L_+ \wedge p \cdot x \leq w\}$.

- With the constant w and the consumption of the other goods constant, we get:

$$\begin{aligned} dw &= p_1 dx_1 + p_2 dx_2 = 0 \\ \frac{dx_2}{dx_1} &= -\frac{p_1}{p_2} \quad \text{with other prices constant} \quad . \end{aligned}$$

- Budget line with two goods; slope $-p_1/p_2$. See Figure 2.D.1, page 21, MWG.

- **Definition - Utility Maximization Problem [UMP]**: Find the optimal solution for:

$$\max_x u(x) \quad s.t.\ x_i \geq 0\ , \quad p \cdot x \leq w.$$

The solution $x(p, w)$ is called **Walrasian demand** .

- Remark: Some textbooks call the UMP also **Consumer's Problem**.

- Remark: Some textbooks call $x(p, w)$ **Marshallian demand**.

Consumer Theory 3
Consumer's Problem (4)

- **Proposition - Existence**: If $p \gg 0$, $w > 0$ and $u(x)$ is continuous, then the utility maximization problem has a solution. [P 3.D.1]

- Proof: By the assumptions $B_{p,w}$ is compact. $u(x)$ is a continuous function. By the Weierstraß theorem (Theorem M.F.2(ii), p. 945, MWG; maximum value theorem in calculus), there exists an $x \in B_{p,w}$ maximizing $u(x)$.

Consumer Theory 3
Consumer's Problem (5)

- By altering the price vector p and income w, the consumer's maximization provides us with the correspondence $x(p, w)$, which is called **Walrasian demand correspondence**. If preferences are strictly convex we get **Walrasian demand functions** $x(p, w)$.

- What happens to the function if w or p_j changes? See MWG, Figure 3.D.1 - 3.D.4

Consumer Theory 3
Consumer's Problem (6)

- In a general setting demand need not be a smooth function.

- **Theorem - Differentiable Walrasian Demand Function**: Let $x^* \gg 0$ solve the consumers maximization problem at price $p_0 \gg 0$ and $w_0 > 0$. If $u(x)$ is twice continuously differentiable, $\partial u(x)/\partial x_i > 0$ for some $i = 1, \ldots, n$ and the bordered Hessian of $u(x)$,

$$\begin{pmatrix} D^2 u(x) & \nabla u(x) \\ \nabla u(x)^\top & 0 \end{pmatrix},$$

has a non-zero determinant at x^*, then $x(p, w)$ is differentiable at p_0, w_0.

- More details are provided in MWG, p. 94-95.

Consumer Theory 3
Consumer's Problem (7)

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  has a non-zero determinant at x^*, then $x(p, w)$ is differentiable at p_0, w_0.

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Correspondences (1)

- Generalized concept of a function.

- **Definition - Correspondence**: Given a set $A \in \mathbb{R}^n$, a correspondence $f : A \twoheadrightarrow \mathbb{R}^k$ is a rule that assigns a set $f(x) \subseteq Y \subset \mathbb{R}^k$ to every $x \in A$.

- If $f(x)$ contains exactly one element for every $x \in A$, then (up to abuse of notation) f is a function.

- $A \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^k$ are the domain and the codomain.

- Literture: e.g. MWG, chapter M.H, page 949.

Correspondences (2)

- The set $\{(x,y)|x \in A\ ,\ y \in \mathbb{R}^k\ ,\ y \in f(x)\}$ is called **graph** of the correspondence.

- **Definition - Closed Graph**: A correspondence has a closed graph if for any pair of sequences $x^{(m)} \to x \in A$, with $x^{(m)} \in A$ and $y^{(m)} \to y$, with $y^{(m)} \in f(x^{(m)})$, we have $y \in f(x)$.

Correspondences (3)

- Regarding continuity there are two concepts with correspondences.

- **Definition - Upper Hemicontinuous**: A correspondence is UHC if the graph is closed and the images of compact sets are bounded. That is, for every compact set $B \subseteq A$, the set $f(B) = \{y \in \mathbb{R}^k : y \in f(x)$ for some $x \in B\}$ is bounded.

- **Definition - Lower Hemicontinuous**: Given $A \subseteq \mathbb{R}^n$ and a compact set $Y \subseteq \mathbb{R}^k$, the correspondence is LHC if for every sequence $x^{(m)} \to x$, $x^{(m)}, x \in A$ for all m, and every $y \in f(x)$, we can find a sequence $y^{(m)} \to y$ and an integer M such that $y^{(m)} \in f(x^{(m)})$ for $m > M$.

Consumer Theory 3
Consumer's Problem (8)

- **Theorem - Properties of $x(p, w)$**: Consider a continuous utility function $u(x)$ representing a rational locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$. Then $x(p, w)$ has the following properties: [P 3.D.2]
  - Homogeneity of degree zero in (p, w).
  - Walras' law: $p \cdot x = w$ for all $x \in x(p, w)$.
  - Convexity/uniqueness: If $\succeq$ is convex, so that $u(x)$ is quasiconcave, then $x(p, w)$ is a convex set. If $\succeq$ is strictly convex, where $u(x)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.

Consumer Theory 3
Consumer's Problem (9)

Proof:

- Property 1 - Homogeneity in p, w: We have to show that $x(\mu p, \mu w) = \mu^0 x(p, w)$. Plug in μp and μw in the optimization problem $\Rightarrow B_{p,w} = B_{\mu p, \mu w}$. The result follows immediately.

Consumer Theory 3
Consumer's Problem (10)

Proof:

- Property 2- Walras' law: If $x \in x(p, w)$ and $p \cdot x < w$, then there exists a y in the neighborhood of x, with $y \succ x$ and $p \cdot y < w$ by local nonsatiation. Therefore x cannot be an optimal bundle. This argument holds for all interior points of $B_{p,w}$.

Consumer Theory 3
Consumer's Problem (10)

Proof:

- Property 3 - $x(p, w)$ is a convex set: If preferences are convex then $u(x^\nu) \geq \min\{u(x), u(y)\}$, where $x^\nu = \nu x + (1-\nu)y$; replace $\geq$ by $>$ if $\succeq$ is strictly convex. I.e. $u(x)$ is quasiconcave. We have to show that if $x, y \in x(p, w)$, then $x^\nu \in x(p, w)$. From the above property x, y and x^ν have to be elements of the budget hyperplane $\{x | x \in X \text{ and } p \cdot x = w\}$.

  Since x and y solve the UMP we get $u(x) = u(y)$, therefore $u(x^\nu) \leq u(x) = u(y)$. By quasiconcavity of $u(x)$ we get $u(x^\nu) \geq u(x) = u(y)$, such that $u(x^\nu) = u(x) = u(y)$ holds for arbitrary $x, y \in x(p, w)$. I.e. the set $x(p, w)$ has to be convex.

Consumer Theory 3
Consumer's Problem (11)

Proof:

- Property 3 - $x(p, w)$ is single valued if preferences are strictly convex: Assume, like above, the x and y solve the UMP; $x \neq y$. Then $u(x) = u(y) \geq u(z)$ for all $z \in B_{p,w}$. By the above result x, y are elements of the budget hyperplane.

- Since preferences are strictly convex, $u(x)$ is strictly quasiconcave $\Rightarrow u(x^\nu) > \min\{u(x), u(y)\}$. $x^\nu = \nu x' + (1-\nu)y'$ and x', y' are some arbitrary elements of the budget hyperplane; (a contradiction to strict convexity).

- Now $u(x^\nu) > \min\{u(x), u(y)\}$, also for x, y. Therefore the pair x, y cannot solve the UMP. Therefore, $x(p, w)$ has to be single valued.

Consumer Theory 4
Duality

- Instead of looking at $u(x)$, we'll have an alternative look on utility via prices, income and the utility maximization problem $\Rightarrow$ indirect utility

- Expenditure function, the dual problem and Hicksian demand

- Income- and substitution effects, Slutsky equation

  MWG, Chapter 3.D-3.H

Consumer Theory 4
Indirect Utility (1)

- We have already considered the direct utility function $u(x)$ in the former parts.

- Start with the utility maximization problem

$$\max_{x} u(x) \quad s.t.\ p \cdot x \leq w$$

  $x^* \in x(p, w)$ solves this problem for $(p, w) \gg 0$.

- **Definition** - **Indirect Utility**: By the highest levels of utility attainable with p, w, we define a maximal value function. This function is called **indirect utility function** $v(p, w)$. It is the maximum value function corresponding to the consumer's optimization problem (utility maximization problem).

Consumer Theory 4
Indirect Utility (2)

- $v(p, w)$ is a function, by Berge's theorem of the maximum $x(p, w)$ is upper hemicontinuous and $v(p, w)$ is continuous (see MWG, page 963, [M.K.6], more details in Micro II).

- If $u(x)$ is strictly quasiconcave such that maximum x^* is unique, we derive the demand function $x^* = x(p, w)$.

- In this case the indirect utility function is the composition of the direct utility function and the demand function $x(p, w)$, i.e. $v(p, w) = u(x^*) = u(x(p, w))$.

Consumer Theory 4
Indirect Utility (3)

- **Theorem: Properties of the Indirect Utility Function** $v(p, w)$: [P 3.D.3] Suppose that $u(x)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ on the consumption set $X = \mathbb{R}^L_+$. Then the indirect utility function $v(p, w)$ is
  - Continuous in p and w.
  - Homogeneous of degree zero in p, w.
  - Strictly increasing in w.
  - Nonincreasing in p_l, $l = 1, \ldots, L$.
  - Quasiconvex in (p, w).

Proof:

- Property 1 - Continuity: follows from Berge's theorem of the maximum.

- Property 2 - Homogeneous in (p, w): We have to show that $v(\mu p, \mu w) = \mu^0 v(p, w) = v(p, w)$; $\mu > 0$. Plug in μp and μw in the optimization problem $\Rightarrow$ $v(\mu p, \mu w) = \{\max_x u(x) \;\; s.t.\; \mu p \cdot x \leq \mu w\} \Leftrightarrow$ $\{\max_x u(x) \;\; s.t.\; p \cdot x \leq w\} = v(p, w)$.

Consumer Theory 4
Indirect Utility (5)

Proof:

- Property 3 - increasing in w: Given the solutions of the UMP with p and w, w', where $w' > w$: $x(p, w)$ and $x(p, w')$.

- The corresponding budget sets are $B_{p,w}$ and $B_{p,w'}$, by assumption $B_{p,w} \subset B_{p,w'}$ (here we have a proper subset).

- Define $S_{p,w} = \{x \in X | p \cdot x = w\}$ (Walrasian budget hyperplane). Then $B_{p,w}$ is still contained in $B_{p,w'} \setminus S_{p,w'}$.

- Therefore also $S_{p,w} \in (B_{p,w'} \setminus S_{p,w'})$. From the above consideration we know that for any $y \in S_{p,w}$, we have $p \cdot y < w'$. By local nonsatiation there are better bundles in $B_{p,w'}$.

Consumer Theory 4
Indirect Utility (6)

Proof:

- Since $v(p, w)$ is a maximal value function, it has to increase if w increases.

- In other words: By local nonsatiation Walras law has to hold, i.e. $x(p, w)$ and $x(p, w')$ are subsets of the budget hyperplanes $\{x|x \in X$ and $p \cdot x = w\}$, $\{x|x \in X$ and $p \cdot x = w'\}$, respectively. We know where we find the optimal bundles. The hyperplane for w is a subset of $B_{p,w}$ and $B_{p,w'}$ (while the hyperplane for w' is not contained in $B_{p,w}$). Interior points cannot be an optimum under local nonsatiation.

- If $v(p, w)$ is differentiable this result can be obtained by means of the envelope theorem.

Proof:

- Property 4 - non-increasing in p_l: W.l.g. $p'_l > p_l$, then we get $B_{p,w}$ and $B_{p',w}$, where $B_{p',w} \subseteq B_{p,w}$. But $S_{p',w}$ is not fully contained in $B_{p,w} \setminus S_{p,w}$. (Observe the "common point" in $\mathbb{R}^2$.) The rest is similar to Property 3.

Proof:

- Property 5 - Quasiconvex: Consider two arbitrary pairs p^1, x^1 and p^2, x^2 and the convex combinations $p^\nu = \nu p^1 + (1-\nu)p^2$ and $w_\nu = \nu w_1 + (1-\nu)w_2$; $\nu \in [0,1]$.

- $v(p,w)$ would be quasiconvex if $v(p^\nu, w_\nu) \leq \max\{v(p^1, w_1), v(p^2, w_2)\}$.

- Define the consumption sets: $B_j = \{x | p^{(j)} \cdot x \leq w_j\}$ for $j = 1, 2, \nu$.

Consumer Theory 4
Indirect Utility (9)

Proof:

- First we show: If $x \in B_\nu$, then $x \in B_1$ or $x \in B_2$.

  This statement trivially holds for ν equal to 0 or 1.

  For $\nu \in (0,1)$ we get: Suppose that $x \in B_\nu$ but $x \in B_1$ or $x \in B_2$ is not true (then $x \notin B_1$ and $x \notin B_2$), i.e.

$$p^1 \cdot x > w_1 \ \wedge \ p^2 \cdot x > w_2$$

  Multiplying the first term with ν and the second with $1-\nu$ results in

$$\nu p^1 \cdot x > \nu w_1 \ \wedge \ (1-\nu) p^2 \cdot x > (1-\nu) w_2$$

Proof:

- Summing up both terms results in:

$$(\nu p^1 + (1-\nu)p^2) \cdot x = p^\nu \cdot x > \nu w_1 + (1-\nu)w_2 = w_\nu$$

  which contradicts our assumption that $x \in B_\nu$.

- From the fact that $x_\nu \in x(p^\nu, w_\nu)$ is either $\in B_1$ or $\in B_2$, it follows that $v(p^\nu, w_\nu) \leq \max\{v(p^1, w_1), v(p^2, w_2)\}$. The last expression corresponds to the definition of a quasiconvex function.

Consumer Theory 4
Expenditure Function (1)

- With indirect utility we looked at maximized utility levels given prices and income.

- Now we raise the question a little bit different: what expenditures e are necessary to attain an utility level u given prices p.

- Expenditures e can be described by the function $e = p \cdot x$.

Consumer Theory 4
Expenditure Function (2)

- **Definition - Expenditure Minimization Problem [EMP]**: $\min_x p \cdot x \ s.t. \ u(x) \geq u$, $x \in X = \mathbb{R}^L_+$, $p \gg 0$. (We only look at $u \geq u(0)$. $U = \{u | u \geq u(0) \wedge u \in \text{Range}(u(x))\}$)

- It is the dual problem of the utility maximization problem. The solution of the EMP $h(p, u)$ will be called **Hicksian demand correspondence**.

- **Definition - Expenditure Function**: The minimum value function $e(p, u)$ solving the expenditure minimization problem $\min_x p \cdot x \ s.t. \ \ u(x) \geq u$, $p \gg 0$, is called expenditure function.

- Existence: The Weierstraß theorem guarantees the existence of an x^* s.t. $p \cdot x^*$ are the minimal expenditures necessary to attain an utility level u.

Consumer Theory 4
Expenditure Function (3)

- **Theorem: Properties of the Expenditure Function** $e(p, u)$:
  [P 3.E.2]
  If $u(x)$ is continuous utility function representing a locally nonsatiated preference relation. Then the expenditure function $e(p, u)$ is
  - Continuous in p, u domain $R^n_{++} \times U$.
  - $\forall p \gg 0$ strictly increasing in u.
  - Non-decreasing in p_l for all $l = 1, \ldots, L$.
  - Concave in p.
  - Homogeneous of degree one in p.

Consumer Theory 4
Expenditure Function (5)

Proof:

- Property 1 - continuous: Apply the theorem of the maximum.

Proof:

- Property 2 - increasing in u: We have to show that is $u_2 > u_1$ then $e(p, u_2) > e(p, u_1)$.

- Suppose that $h_1 \in h(p, u_1)$ and $h_2 \in h(p, u_2)$ solve the EMP for u_2 and u_1, but $e(p, u_2) \leq e(p, u_1)$. We show that this result in a contradiction. I.e. $u_2 > u_1$ but $0 \leq p \cdot h_2 \leq p \cdot h_1$.

- Then by continuity of $u(x)$ and local nonsatiation we can find an $\alpha \in (0, 1)$ such that αh_2 is preferred to h_1 (remember $u_2 > u_1$ is assumed) with expenditures $\alpha p \cdot h_2 < p \cdot h_1$. This contradicts that h_1 solves the EMP for p, u_1.

Consumer Theory 4
Expenditure Function (7)

Proof:

- Property 2 - with calculus: From
  $\min_x p \cdot x \;\; s.t. \;\; u(x) \geq u \; , \; x \geq 0$ we derive the Lagrangian:

$$L(x, \lambda) = p \cdot x + \lambda(u - u(x)) \; .$$

- From this Kuhn-Tucker problem we get:

$$\begin{aligned} \frac{\partial L}{\partial x_i} &= p_i - \lambda \frac{\partial u(x)}{\partial x_i} \geq 0 \; , \; \frac{\partial L}{\partial x_i} x_i = 0 \\ \frac{\partial L}{\partial \lambda} &= u - u(x) \leq 0 \; , \; \frac{\partial L}{\partial \lambda} \lambda = 0 \end{aligned}$$

Consumer Theory 4
Expenditure Function (8)

Proof:

- $\lambda = 0$ would imply that utility could be increased without increasing the expenditures (in an optimum) $\Rightarrow u = u(x)$ and $\lambda > 0$.

- Good x_i is demanded if the price does not exceed $\lambda \frac{\partial u(x)}{\partial x_i}$ for all $x_i > 0$.

- The envelope theorem tells us that

$$\frac{\partial e(p,u)}{\partial u} = \frac{\partial L(x,u)}{\partial u} = \lambda > 0$$

- Since $u(x)$ is continuous and increasing the expenditure function has to be unbounded.

Proof:

- Property 3 - non-decreasing in p_l: similar to property 3.

Consumer Theory 4
Expenditure Function (10)

Proof:

- Property 4 - concave in p: Consider an arbitrary pair p^1 and p^2 and the convex combination $p_\nu = \nu p^1 + (1-\nu)p^2$. The expenditure function is concave if $e(p_\nu, u) \geq \nu e(p^1, u) + (1-\nu)e(p^2, u)$.
- For minimized expenditures it has to hold that $p^1x^1 \leq p^1x$ and $p^2x^2 \leq p^2x$ for all x fulfilling $u(x) \geq u$.
- x^*_ν minimizes expenditure at a convex combination of p^1 and p^2.
- Then $p^1x^1 \leq p^1x^*_\nu$ and $p^2x^2 \leq p^2x^*_\nu$ have to hold.

Consumer Theory 4
Expenditure Function (11)

Proof:

- Multiplying the first term with ν and the second with $1-\nu$ and taking the sum results in $\nu p^1 x^1 + (1-\nu)p^2 x^2 \leq p_\nu x^*_\nu$.

- Therefore the expenditure function is concave in p.

Proof:

- Property 5 - homogeneous of degree one in p: We have to show that $e(\mu p, u) = \mu^1 e(p, u)$; $\mu > 0$. Plug in μp in the optimization problem $\Rightarrow e(\mu p, u) = \{\min_x \mu p \cdot x \;\; s.t. \; u(x) \geq u\}$. Objective function is linear in μ, the constraint is not affected by μ. With calculus we immediately see the μ cancels out in the first order conditions $\Rightarrow x^h$ remains the same $\Rightarrow$ $\mu\{\min_x p \cdot x \;\; s.t. \; u(x) \geq u\} = \mu e(p, u)$.

Consumer Theory 4
Hicksian Demand (1)

- **Theorem: Hicksian demand**: [P 3.E.3] Let $u(x)$ be continuous utility function representing a locally nonsatiatated preference order; $p \gg 0$. Then the Hicksian demand correspondence has the following properties:
  - Homogeneous of degree zero.
  - No excess utility $u(x) = u$.
  - Convexity/uniqueness: If $\succeq$ is convex, then $h(p, u)$ is a convex set. If $\succeq$ is strictly convex, then $h(p, u)$ is single valued.

Consumer Theory 4
Hicksian Demand (2)

Proof:

- Homogeneity follows directly from the EMP.
  $\min\{p \cdot x \ s.t. \ u(x) \geq u\} \Leftrightarrow \alpha \min\{p \cdot x \ s.t. \ u(x) \geq u\} \Leftrightarrow$
  $\min\{\alpha p \cdot x \ s.t. \ u(x) \geq u\}$ for $\alpha > 0$.

- Suppose that there is an $x \in h(p, u)$ with $u(x) > u$. By the continuity of u we find an $\alpha \in (0, 1)$ such that $x' = \alpha x$ and $u(x') > u$. But with x' we get $p \cdot x' < p \cdot x$. A contradiction that x solves the EMP.

- For the last property see the theorem on Walrasian demand or apply the forthcoming theorem.

Consumer Theory 4
Expenditure vs. Indirect Utility (1)

- With (p, w) the indirect utility function provides us with the maximum of utility u. Suppose $w = e(p, u)$. By this definition $v(p, e(p, u)) \geq u$.

- Given p, u and an the expenditure function, we must derive $e(p, v(p, w)) \leq w$.

- Given an x^* solving the utility maximization problem, i.e. $x^* \in x(p, w)$. Does x^* solve the EMP if $u = v(p, w)$?

- Given an h^* solving the EMP, i.e. $h^* \in h(p, u)$. Does h^* solve the UMP if $w = e(p, u)$?

Consumer Theory 4
Expenditure vs. Indirect Utility (2)

- **Theorem: Equivalence between Indirect Utility and Expenditure Function**: [P 3.E.1] Let $u(x)$ be continuous utility function representing a locally nonsatiatated preference order; $p \gg 0$.
  - If x^* is optimal in the UMP with $w > 0$, then x^* is optimal in the EMP when $u = u(x^*)$. $e(p, u(x^*)) = w$.
  - If h^* is optimal in the EMP with $u > u(0)$, then h^* is optimal in the UMP when $w = e(p, u)$. $v(p, e(p, u)) = u$.

Consumer Theory 4
Expenditure vs. Indirect Utility (3)

Proof:

- We prove $e(p, v(p, w)) = w$ by means of a contradiction. p, w $\in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$. By the definition of the expenditure function we get $e(p, v(p, w)) \leq w$. In addition $h^* \in h(p, u)$.

  To show equality assume that $e(p, u) < w$, where $u = v(p, w)$ and x^* solves the UMP: $e(p, u)$ is continuous in u. Choose ε such that $e(p, u + \varepsilon) < w$ and $e(p, u + \varepsilon) =: w_\varepsilon$.

  The properties of the indirect utility function imply $v(p, w_\varepsilon) \geq u + \varepsilon$. Since $w_\varepsilon < w$ and $v(p, w)$ is strictly increasing in w (by local nonstatiation) we get: $v(p, w) > v(p, w_\varepsilon) \geq u + \varepsilon$ but $u = v(p, w)$, which is a contradiction. Therefore $e(p, v(p, w)) = w$ and x^* also solves the EMP, such that $x^* \in h(p, u)$ when $u = v(p, u)$.

Consumer Theory 4
Expenditure vs. Indirect Utility (4)

Proof:

- Next we prove $v(p, e(p, u)) = u$ in the same way. $p, u \in \mathbb{R}^n_{++} \times U$. By the definition of the indirect utility function we get $v(p, e(p, u)) \geq u$.

  Assume that $v(p, w) > u$, where $w = e(p, u)$ and h^* solves the EMP: $v(p, w)$ is continuous in w. Choose ε such that $v(p, w - \varepsilon) > u$ and $v(p, w - \varepsilon) =: u_\varepsilon$.

  The properties of the expenditure function imply $e(p, u_\varepsilon) \leq w - \varepsilon$. Since $u_\varepsilon > u$ and $e(p, u)$ is strictly increasing in u we get: $e(p, u) < e(p, u_\varepsilon) \leq w - \varepsilon$ but $w = e(p, u)$, which is a contradiction. Therefore $v(p, e(p, u)) = u$. In addition h^* also solves the UMP.

Consumer Theory 4
Hicksian Demand (3)

- **Theorem: Hicksian/ Compensated law of demand**: [P 3.E.4] Let $u(x)$ be continuous utility function representing a locally nonsatiatated preference order and $h(p,u)$ consists of a single element for all $p \gg 0$. Then the Hicksian demand function satisfies the compensated law of demand: For all p' and p'':

$$(p'' - p')[h(p'', u) - h(p', u)] \leq 0.$$

Proof:

- By the EMP: $p'' \cdot h(p'', u) - p'' \cdot h(p', u) \leq 0$ and $p' \cdot h(p', u) - p' \cdot h(p'', u) \leq 0$ have to hold.

- Adding up the inequalities yields the result.

Consumer Theory 4
Shephard's Lemma (1)

- Investigate the relationship between a Hicksian demand function and the expenditure function.

- **Theorem - Shephard's Lemma**: [P 3.G.1] Let $u(x)$ be continuous utility function representing a locally nonsatiatated preference order and $h(p, u)$ consists of a single element. Then for all p and u, the gradient vector of the expenditure function with respect to p gives Hicksian demand, i.e.

$$\nabla_p e(p, u) = h(p, u).$$

Consumer Theory 4
Shephard's Lemma (2)

Proof by means of calculus:

- Suppose that the envelope theorem can be applied (see e.g. MWG [M.L.1], page 965):

- Then the Lagrangian is given by: $L(x, \lambda) = p \cdot x + \lambda(u - u(x))$.

- $\lambda > 0$ follows from $u = u(x)$.

Consumer Theory 4
Shephard's Lemma (3)

Proof with calculus:

- The Kuhn-Tucker conditions are:

$$\begin{aligned}
\frac{\partial L}{\partial x_i} &= p_i - \lambda \frac{\partial u(x)}{\partial x_i} \geq 0 \\
\frac{\partial L}{\partial x_i} x_i &= 0 \\
\frac{\partial L}{\partial \lambda} &= u - u(x) \geq 0 \\
\frac{\partial L}{\partial \lambda} \lambda &= 0
\end{aligned}$$

Consumer Theory 4
Shephard's Lemma (4)

Proof with calculus:

- Good x_i is demanded if the price does not exceed $\lambda \frac{\partial u(x)}{\partial x_i}$ for all $x_i > 0$.

- The envelope theorem tells us that

$$\frac{\partial e(p,u)}{\partial p_l} = \frac{\partial L(x,u)}{\partial p_l} = h_l(p,u)$$

for $l = 1, \ldots, L$.

Consumer Theory 4
Shephard's Lemma (5)

Proof:

- The expenditure function is the support function μ_k of the non-empty and closed set $K = \{x|u(x) \geq u\}$. Since the solution is unique by assumption, $\nabla \mu_K(p) = \nabla_p e(p,u) = h(p,u)$ has to hold by the Duality theorem.

- Alternatively: Assume differentiability and apply the envelope theorem.

Consumer Theory 4
Expenditure F. and Hicksian Demand (1)

- Furthermore, investigate the relationship between a Hicksian demand function and the expenditure function.

- **Theorem:**: [P 3.E.5] Let $u(x)$ be continuous utility function representing a locally nonsatiatated and strictly convex preference relation on $X = \mathbb{R}^L_+$. Suppose that $h(p,u)$ is continuously differentiable, then
  - $D_p h(p,u) = D_p^2 e(p,u)$
  - $D_p h(p,u)$ is negative semidefinite
  - $D_p h(p,u)$ is symmetric.
  - $D_p h(p,u) p = 0$.

Proof:

- To show $D_p h(p,u)p = 0$, we can use the fact that $h(p,u)$ is homogeneous of degree zero in prices ($r = 0$).

- By the Euler theorem [MWG, Theorem M.B.2, p. 929] we get

$$\sum_{l=1}^{L} \frac{\partial h(p,u)}{\partial p_l} p_l = r h(p,u).$$

Consumer Theory 4
Walrasian vs. Hicksian Demand (1)

- Here we want to analyze what happens if income w changes: normal vs. inferior good.
- How is demand effected by prices changes: change in relative prices - substitution effect, change in real income - income effect
- Properties of the demand and the law of demand.
- How does a price change of good i affect demand of good j.
- Although utility is continuous and strictly increasing, there might be goods where demand declines while the price falls.

Consumer Theory 4
Walrasian vs. Hicksian Demand (2)

- **Definition - Substitution Effect, Income Effect**: We split up the total effect of a price change into
  - an effect accounting for the change in the relative prices p_i/p_j (with constant utility or real income) $\Rightarrow$ **substitution effect**. Here the consumer will substitute the relatively more expensive good by the cheaper one.
  - an effect induced by a change in real income (with constant relative prices) $\Rightarrow$ **income/wealth effect**.

Consumer Theory 4
Walrasian vs. Hicksian Demand (3)

- Hicksian decomposition - keeps utility level constant to identify the substitution effect.

- The residual between the total effect and the substitution effect is the income effect.

- See Figures in MWG, Chapter 2 and

Consumer Theory 4
Walrasian vs. Hicksian Demand (4)

- Here we observe that the Hicksian demand function exactly accounts for the substitution effect.

- The difference between the change in Walrasian (total effect) demand induced by a price change and the change in Hicksian demand (substitution effect) results in the income effect.

- Note that the income effect need not be positive.

Consumer Theory 4
Walrasian vs. Hicksian Demand (5)

- Formal description of these effects is given by the Slutsky equation.

- **Theorem - Slutsky Equation**: [P 3.G.3] Assume that the consumer's preference relation $\succeq$ is complete, transitive, continuous, locally nonsatiated and strictly convex defined on $X = \mathbb{R}^L_+$. Then for all (p, w) and $u = v(p, w)$ we have

$$\underbrace{\frac{\partial x_l(p,w)}{\partial p_j}}_{TE} = \underbrace{\frac{\partial h_l(p,u)}{\partial p_j}}_{SE} \underbrace{-x_j(p,w)\,\frac{\partial x_l(p,w)}{\partial w}}_{IE} \quad l, j = 1, \ldots, L.$$

Consumer Theory 4
Walrasian vs. Hicksian Demand (6)

- Equivalently:

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^\top$$

- Remark: In the following proof we shall assume that $h(p,u)$ and $x(p,w)$ are differentiable. (Differentiability of $h(p,u)$ follows from duality theory presented in Section 3.F. This is a topic of the Micro II course.)

Consumer Theory 4
Walrasian vs. Hicksian Demand (7)

Proof:

- First, we use the Duality result on demand: $h_l(p,u) = x_l(p, e(p,u))$ and take partial derivatives with respect to p_j:

$$\frac{\partial h_l(p,u)}{\partial p_j} = \frac{\partial x_l(p,e(p,u))}{\partial p_j} + \frac{\partial x_l(p,e(p,u))}{\partial w}\frac{\partial e(p,u)}{\partial p_j}.$$

- Second: By the relationship between the expenditure function and the indirect utility it follows that $u = v(p,w)$ and $e(p,u) = e(p, v(p,w)) = w$.

Proof:

- Third: Shephard's Lemma tells us that $\frac{\partial e(p,u)}{\partial p_j} = h_j(p,u)$, this gives
$$\frac{\partial h_l(p,u)}{\partial p_j} = \frac{\partial x_l(p,w)}{\partial p_j} + \frac{\partial x_l(p,w)}{\partial w} h_j(p,u)$$

Consumer Theory 4
Walrasian vs. Hicksian Demand (9)

Proof:

- Forth: Duality between Hicksian and Walrasian demand implies that $h(p, v(p, w)) = x(p, w)$ with $v(p, w) = u$. Thus $\frac{\partial e(p,u)}{\partial p_j} = x_j(p, w)$.

- Arranging terms yields:

$$\frac{\partial x_l(p, w)}{\partial p_j} = \frac{\partial h_l(p, u)}{\partial p_j} - x_j(p, w) \, \frac{\partial x_l(p, w)}{\partial w}.$$

- From the Sultsky equation we can construct the following matrix:
  Definition - Slutsky Matrix:

$$S(p,w) := \begin{pmatrix} \frac{\partial x_1(p,w)}{\partial p_1} + x_1(p,w)\frac{\partial x_1(p,w)}{\partial w} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} + x_L(p,w)\frac{\partial x_1(p,w)}{\partial w} \\ \cdots & \ddots & \cdots \\ \frac{\partial x_L(p,w)}{\partial p_1} + x_1(p,w)\frac{\partial x_L(p,w)}{\partial w} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} + x_L(p,w)\frac{\partial x_L(p,w)}{\partial w} \end{pmatrix}$$

Consumer Theory 4
Walrasian vs. Hicksian Demand (11)

- **Theorem** Suppose that $e(p,u)$ is twice continuously differentiable. Then the Slutsky Matrix $S(p,w)$ is negative semidefinite, symmetric and satisfies $S(p,w)p = 0$.

Proof:

- Negative semidefiniteness follows from the negative semidefiniteness of $D_p h(p, u)$ which followed from the concavity of the expenditure function.

- Symmetry follows from the existence of the expenditure function and Young's theorem.

- $S(p, w) \cdot p = 0$ follows from an Euler theorem (see [MWG, Theorem M.B.2, p. 929]) argument already used in [P 3.G.2]

Consumer Theory 4
Roy's Identity (1)

- Goal is to connect Walrasian demand with the indirect utility function.

- **Theorem - Roy's Identity**: [P 3.G.4] Let $u(x)$ be continuous utility function representing a locally nonsatiatated and strictly convex preference relation $\succeq$ defined on $X = \mathbb{R}^L_+$. Suppose that the indirect utility function $v(p, w)$ is differentiable for any $p, w \gg 0$, then

$$x(p, w) = -\frac{1}{\nabla_w v(p, w)} \nabla_p v(p, w)$$

i.e.

$$x_l(p, w) = -\frac{\partial v(p, w)/\partial p_l}{\partial v(p, w)/\partial w}.$$

Consumer Theory 4
Roy's Identity (1)

Proof:

- Roy's Identity: Assume that the envelope theorem can be applied to $v(p, w)$.
- Let (x^*, λ^*) maximize $\{\max_x u(x) \;\; s.t. \; p \cdot x \leq w\}$ then the partial derivatives of the Lagrangian $L(x, \lambda)$ with respect to p_l and w provide us with:

$$\frac{\partial v(p, w)}{\partial p_l} = \frac{\partial L(x^*, \lambda^*)}{\partial p_l} = -\lambda^* x_l^* \;, \quad l = 1, \ldots, L.$$
$$\frac{\partial v(p, w)}{\partial w} = \frac{\partial L(x^*, \lambda^*)}{\partial w} = \lambda^*.$$

Consumer Theory 2
Indirect Utility (11)

Proof:

- Plug in $-\lambda$ from the second equation results in

$$\frac{\partial v(p,w)}{\partial p_l} = -\frac{\partial v(p,w)}{\partial w} x_l^*$$

such that

$$-\frac{\partial v(p,w)/\partial p_l}{\partial v(p,w)/\partial w} = x_l(p,w).$$

- Note that $\partial v(p,w)/\partial w$ by our properties on the indirect utility function.

Theorem of the Maximum (1)

- Consider a constrained optimization problem:

$$\max f(x) \ s.t. \ g(x,q) = 0$$

where $q \in Q$ is a vector of parameters. $Q \in \mathbb{R}^S$ and $x \in \mathbb{R}^N$. $f(x)$ is assumed to be continuous. $C(q)$ is the constraint set implied by g.

Theorem of the Maximum (1)

- **Definition**: $x(q)$ is the set of solutions of the problem, such that $x(q) \subset C(q)$ and $v(q)$ is the maximum value function, i.e. $f(x)$ evaluated at an optimal $x \in x(q)$.

- **Theorem of the Maximum**: Suppose that the constraint correspondence is continuous and f is continuous. Then the maximizer correspondence $x : Q \to \mathbb{R}^N$ is upper hemicontinuous and the value function $v : Q \to \mathbb{R}$ is continuous. [T M.K.6], page 963.

Duality Theorem (1)

- Until now we have not shown that $c(w, y)$ or $e(p, u)$ is differentiable when $u(x)$ is strictly quasiconcave.
- This property follows from the **Duality Theorem**.
- MWG, Chapter 3.F, page 63.

Duality Theorem (2)

- A set is $K \in \mathbb{R}^n$ is convex if $\alpha x + (1-\alpha)y \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.

- A half space is a set of the form $\{x \in \mathbb{R}^n | p \cdot x \geq c\}$.

- $p \neq 0$ is called the **normal vector**: if x and x' fulfill $p \cdot x = p \cdot x' = c$, then $p \cdot (x - x') = 0$.

- The boundary set $\{x \in \mathbb{R}^n | p \cdot x = c\}$ is called **hyperplane**. The half-space and the hyperplane are convex.

Duality Theorem (3)

- Assume that K is convex and closed. Consider $\bar{x} \notin K$. Then there exists a half-space containing K and excluding $\bar{x}$. There is a p and a c such that $p \cdot \bar{x} < c \leq p \cdot x$ for all $x \in K$ (**separating hyperplane theorem**).

- Basic idea of duality theory: A closed convex set can be equivalently (dually) described by the intersection of half-spaces containing this set.

- MWG, figure 3.F.1 and 3.F.2 page 64.

Duality Theorem (4)

- If K is not convex the intersection of the half-spaces that contain K is the smallest, convex set containing K. (closed convex hull of K, abbreviated by $\bar{K}$).

- For any closed (but not necessarily convex) set K we can define the **support function** of K:

$$\mu_K(p) = \inf\{p \cdot x | x \in K\}$$

- When K is convex the support function provides us with the dual description of K.

- $\mu_K(p)$ is homogeneous of degree one and concave in p.

Duality Theorem (5)

- **Theorem - Duality Theorem**: Let K be a nonempty closed set and let $\mu_K(p)$ be its support function. Then there is a unique $\bar{x} \in K$ such that $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$ if and only if $\mu_K(p)$ is differentiable at $\bar{p}$. In this case $\nabla_p \mu_K(\bar{p}) = \bar{x}$.

- Proof see literature. E.g. see section 25 in R.T. Rockafellar, *Convex Analysis*, Princeton University Press, New York 1970.

Consumer Theory 4
Revealed Preference Theory (1)

- Weak Axiom of Revealed Preference.
- Strong Axiom of Revealed Preference.
- Revealed preferences and utility maximization.

MWG, Chapter 1.C and 2.F., 3.J.

Consumer Theory 4
Revealed Preference Theory (2)

- Samuelson's idea: Cannot we start with observed behavior instead of assumptions on preferences.

- Idea: if a consumer buys a bundle x^0 instead of an other affordable bundle x^1, then the first bundle is called **revealed preferred** to x^1 (see Consumer Theory 1).

- **Definition - Weak Axiom on Revealed Preference**: [D 2.F.1] A Walrasian demand function $x(p, w)$ satisfies the weak axiom of revealed preference if for any two wealth price situations (p,w) and (p',w') the following relationship holds: If $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$ then $p' \cdot x(p, w) > w'$.

Consumer Theory 4
Revealed Preference Theory (3)

- Interpret the weak axiom by means for Figure 2.F.1, page 30.
- We assume that $x(p, w)$ is a function, which is homogeneous of degree zero and Walras' law holds.

- From the former parts we already know:

  Theorem - Weak Revealed Preference and Utility maximization: If $x(p, w)$ solves the utility maximization problem with strictly increasing and strictly quasiconcave utility function, then the weak axiom of revealed preference has to hold.

- See also P 1.D.1

Proof:

- Consider a pair x^0 and x^1 where $x^0 = x(p^0, w)$ solves the utility maximization problem for p^0, x^1 for p^1.

- Assume $u(x^0) > u(x^1)$: $w = p^0 \cdot x^0 \geq p^0 \cdot x^1$. Then $p^1 \cdot x^0 > p^1 \cdot x^1 = w$. Otherwise a consumer would have chosen x^0 if it were affordable in the second maximization problem.

- I.e. $p^1 \cdot x^0 > p^1 \cdot x^1$ has to be fulfilled. Since x^0 and x^1 are arbitrary pairs, the weak axiom of revealed preference has to hold.

Consumer Theory 4
Slutsky Compensation (1)

- **Definition - Slutsky compensation**: Given a bundle $x^0 = x(p, w)$ and income is compensated such that the consumer can always buy the bundle x^0, i.e. $w' = p' \cdot x(p, w)$. Then demand is called Slutsky compensated demand $x^S(p, w(x^0))$.

- Discuss this concept by means of Figure 2.F.2, page 31.

Consumer Theory 4
Slutsky Compensation (2)

- **Proposition**: Suppose that the Walrasian demand function $x(p, w)$ is homogeneous of degree zero and satisfies Walras' law. Then $x(p, w)$ satisfies the weak axiom if and only if the following property holds:

  For any compensated price change form the initial situation (p, w) to a new pair (p', w'), where $w' = p' \cdot x(p, w)$, we have

  $$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$$

  with strict inequality whenever $x(p, w) \neq x(p', w')$. [P 2.F.1]

- Remark: $x(p', w') = x^S(p', w(x^0))$.

Consumer Theory 4
Slutsky Compensation (3)

Proof:

- (i) The weak axiom implies $(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$ with strict inequality for different demands: If $x(p', w') = x(p, w)$ then $[x(p', w') - x(p, w)] = 0$.

- Suppose $x(p', w') \neq x(p, w)$ and expand $(p' - p) \cdot [x(p', w') - x(p, w)]$ to $p' \cdot [x(p', w') - x(p, w)] - p \cdot [x(p', w') - x(p, w)]$. By Walras' law and the construction of compensated demand the first term is 0.

- By compensated demand we get $p' \cdot x(p, w) = w'$. I.e. $x^0 = x(p, w)$ can be bought with p', w'. By the weak axiom $x(p', w') \notin B_{p,w}$, such that $p \cdot x(p', w') > w$. Walras' law implies $p \cdot x(p, w) = w$. This yields $p \cdot [x(p', w') - x(p, w)] > 0$, such that ... holds.

Consumer Theory 4
Slutsky Compensation (4)

Proof:

- (ii) $(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$ implies the weak law if $x(p', w') \neq x(p, w)$:

- If we consider compensated demand, then the weak axiom has to hold (replace Walrasian demand by compensated demand in the Theorem - Weak Revealed Preference and Utility maximization.

- It is necessary that the weak axiom holds for all compensated demand changes: Assume $u(x^0) > u(x^1)$: $w = p^0 \cdot x^0 \geq p^0 \cdot x^1$. Suppose that $p^1 \cdot x^0 \leq p^1 \cdot x^1 = w$. Then x^0 cannot be an optimum by local non-satiation.

- By these arguments the weak law holds if $p' \cdot x(p', w') > w$ whenever $p \cdot x(p, w) = w$ and $x(p', w') \neq x(p, w)$.

Consumer Theory 4
Slutsky Compensation (5)

Proof:

- By this argument we can test for the weak axiom by looking at compensated price changes. (We show that $\neg H \Rightarrow \neg C$.) If the weak law does not hold, there is a compensated price change such that $p'x(p, w') \leq w' \Rightarrow p'x(p, w) \leq w'$, $<$ for different x $(p \cdot x(p, w) = w)$. By Walras law we get

$$p \cdot [x(p', w') - x(p, w)] \leq 0$$

and

$$p' \cdot [x(p', w') - x(p, w)] \geq 0.$$

- This results in $(p' - p) \cdot [x(p', w') - x(p, w)] \geq 0$; $>$ for $x(p', w') \neq x(p, w)$. This contradicts that $(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$ holds.

- **Definition - Strong Axiom of Revealed Preference**: [3.J.1] The market demand satisfies the strong axiom of revealed preference if for any list

$$(p^1, w^1), \ldots, (p^N, w^N)$$

  with $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$ for all $n \leq N-1$, we have $p^N x(p^1, w^1) > w^N$ whenever $p^n \cdot x(p^{n+1}, w^{n+1}) \leq w^n$ for all $n \leq N-1$.

- I.e. if $x(p^1, w^1)$ is directly or indirectly revealed preferred to $x(p^N, w^N)$, then $x(p^N, w^N)$ cannot be directly or indirectly be revealed preferred to $x(p^1, w^1)$. Or for different bundles $x^1, x^2, \ldots, x^k$: If x^q is revealed preferred to x^2 and x^2 is preferred to x^3, then x^1 is revealed preferred to x^3.

Consumer Theory 4
Revealed Preference Theory (7)

- **Theorem - Revealed Preference and Demand (II)**: If the Walrasian demand function $x(p, w)$ satisfies the strong axiom of revealed preference then there is a rational preference relation $\succeq$ that rationalizes $x(p, w)$. I.e. for all (p, w), $x(p, w) \succ y$ for every $y \neq x(p, w)$ with $y \in B_{p,w}$. [P 3.J.1].

- Proof - see page 92.

Consumer Theory 5
Welfare Analysis (1)

- Measurement of Welfare

- Concept of the Equivalent Variation, the Compensating Variation and the Consumer Surplus.

- Pareto improvement and Pareto efficient

  Literature: MWG, Chapter 3.I, page 80-90.

Consumer Theory 5
Welfare Analysis (2)

- From a social point of view - can we judge that some market outcomes are better or worse?
- Positive question: How will a proposed policy affect the welfare of an individual?
- Normative question: How should we weight different effects on different individuals?

Consumer Theory 5
Welfare Analysis (3)

- **Definition - Pareto Improvement**: When we can make someone better off and no one worse off, then a Pareto improvement can be made.

- **Definition - Pareto Efficient**: A situation where there is no way to make somebody better off without making someone else worth off is called **Pareto efficient**. I.e. there is no way for Pareto improvements.

- Strong criterion.

Consumer Theory 5
Consumer Welfare Analysis (1)

- Preference based consumer theory investigates demand from a descriptive perspective.

- Welfare Analysis can be used to perform a normative analysis.

- E.g. how do changes of prices or income affect the well being of a consumer.

Consumer Theory 5
Consumer Welfare Analysis (2)

- Given a preference relation $\succeq$ and Walrasian demand $x(p, w)$, a price change from p^0 to p^1 increases the well-being of a consumer if indirect utility increases. I.e. $v(p^1, w) > v(p^0, w)$.

- Here we are interested in so called **money metric indirect utility functions**. E.g. expressing indirect utility in terms of monetary units.

Consumer Theory 5
Consumer Welfare Analysis (3)

- Suppose $u_1 > u_0$, $u_1 = v(p_1, w)$ arises from p^1, w and $u_0 = v(p_0, w)$ from p^0, w.

- With p fixed at $\bar{p}$, the property of the expenditure function that $e(p, u)$ is increasing in u yields:
  $e(\bar{p}, u_1)) = e(\bar{p}, v(\bar{p}, \tilde{w}^1)) = \tilde{w}^1 > e(\bar{p}, v(\bar{p}, \tilde{w}^0)) = e(\bar{p}, u_0) = \tilde{w}^0$
  - i.e. it is an indirect utility function which measures the degree of well-being in money terms.

- See Figure 3.I.1, page 81.

Consumer Theory 5
Consumer Welfare Analysis (4)

- Based on these considerations we set $\bar{p} = p^0$ or p^1 **and** $w = e(p^0, u^0) = e(p^1, u^1)$; we define:
  - **Definition - Equivalent Variation**: "old prices"

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - e(p^1, u^1) = e(p^0, u^1) - w$$

  - **Definition - Compensating Variation**: "new prices"

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = e(p^0, u^0) - e(p^1, u^0) = w - e(p^1, u^0)$$

Consumer Theory 5
Consumer Welfare Analysis (5)

- EV measures the money amount that a consumer is indifferent between accepting this amount and the status after the price change (i.e. to attain a utility level u^1).

- CV measures the money amount a consumer is willing to pay to induce the price change from p^0 to p^1 (i.e. to obtain utility level u^0 at the new price p^1). This money amount can be negative as well.

- Discuss Figure 3.1.2, page 82; if p_1 falls then the consumer is prepared to pay the amount CV, i.e. $CV > 0$.

Consumer Theory 5
Consumer Welfare Analysis (6)

- Both measures are associated with Hicksian demand.

- Suppose the only p_1 changes, then $p_1^0 \neq p_1^1$ and $p_l^0 = p_l^1$ for $l \geq 2$. With $w = e(p^0, u^0) = e(p^1, u^1)$ and $h_1(p, u) = \partial e(p, u)/\partial p_1$ we get

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1((p_1, p_-), u^1) dp_1$$

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1((p_1, p_-), u^0) dp_1$$

Consumer Theory 5
Consumer Welfare Analysis (7)

- Discuss these integrals - MWG, Figure 3.1.3, page 83. Here the following case is considered. p^0 and p^1 are L dimensional price vectors. Only the first component p_1 is changed. The other prices $p_- := (p_2, \ldots, p_L)$ are kept constant. w is constant as well.
- EV, CV increase if utility increases and vice versa.
- If x_1 is a normal good, then the slope of the Walrasian demand function $x_1(p, w)$ is smaller than the slopes of $h_1(p, .)$ (in absolute terms).
- We get $EV(p^0, p^1, w) > CV(p^0, p^1, w)$ if the good is normal (in absolute value), the converse is true for inferior goods.
- $EV(p^0, p^1, w) = CV(p^0, p^1, w)$ with zero income effect for good 2. This is the case with quasilinear preferences for good two (see [D 3.B.7]).

- $EV(p^0, p^1, w) = CV(p^0, p^1, w)$ with zero income effect for good 1.

  In this case $EV(p^0, p^1, w) = CV(p^0, p^1, w)$ is also equal to the change in **Marshallian Consumer Surplus**.

- **Definition - Marshallian Consumer Surplus**:
  $MCS_l(p, w) = \int_p^{\infty} x_l((p_l, p_-), w) dp_l$

- **Definition - Area Variation**:
  $AV(p^0, p^1, w) = \int_{p_l^1}^{p_l^0} x(p_l, p_-, w) dp_l$.

Consumer Theory 5
Area Variation Measure (1)

- **Definition - Area Variation**:
  $AV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} x(p_1, p_-, w) dp_1$.

- It measures the change in Marshallian consumer surplus.

- If the income effect is zero this measure corresponds to EV and CV. (see **Marshallian Consumer Surplus**)

- The argument that AV provides are good approximation of EV or CV can but need not hold. See MWG, Figure 3.1.8, page 90.

  Jehle/Reny, 1st edition, Theorem 6.3.2, page 278: Willing's upper and lower bounds on the difference between CS and CV.

Consumer Theory 5
Partial Information (1)

- Consider a bundle x^0, price vectors p^0, p^1 and wealth w. Often a complete Walrasian demand function cannot be observed, however:

- **Theorem - Welfare and Partial Information I:** Consider a consumer with complete, transitive, continuous, and locally non-satiated preferences. If $(p^1 - p^0) \cdot x^0 < 0$, then the consumer is strictly better of with (p^1, w) compared to (p^0, w). [P 3.I.1]

Proof:

- With non-satiation the consumer chooses a set on the boundary of the budget set, such that $p^0 \cdot x = w$. Then $p^1 \cdot x < w$.

- $\Rightarrow$ x is affordable within the budget set under p^1. By the assumption of local non-satiation, there exists a closed set with distance $\leq \varepsilon$ including a better bundle which remains within the budget set. Then the consumer is strictly better off with p^1.

- What happens if $(p^1 - p^0) \cdot x^0 > 0$? This implies $(\alpha p^1 + (1-\alpha)p^0 - p^0) \cdot x^0 > 0$ for $\alpha > 0$.

- **Theorem - Welfare and Partial Information II:** Consider a consumer with a twice differentiable expenditure function. If $(p^1 - p^0) \cdot x^0 > 0$, then there exists an $\bar{\alpha} \in (0,1)$ such that for all $0 < \alpha \leq \bar{\alpha}$, we have $e((1-\alpha)p^0 + \alpha p^1), u^0) > w$ the consumer is strictly better off under p^0, w than under $(1-\alpha)p^0 + \alpha p^1, w$. [P 3.I.2]

Consumer Theory 5
Partial Information (4)

Proof:

- We want to show that CV is negative, if we move from p^0 to p^1. Let $p^\alpha = (1-\alpha)p^0 + \alpha p^1$. We want to show that $CV = e(p^0, u^0) - e(p^\alpha, u^0) < 0$ for some $\bar{\alpha} \geq \alpha > 0$. In other words $e(p^\alpha, u^0) - e(p^0, u^0) > 0$.

- Taylor expand $e(p, u)$ at p^0, u^0:

$$e(p^\alpha, u^0) = e(p^0, u^0) + (p^\alpha - p^0)^\top \nabla_p e(p^0, u^0) + R(p^0, p^\alpha)$$

  where $R(p^0, p^\alpha)/||p^\alpha - p^0|| \to 0$ if $p^\alpha \to p^0$. $e(.,.)$ has to be at least C^1. (fulfilled since second derivatives are assumed to exist).

Proof:

- By the properties of this approximation, there has to exist an $\bar{\alpha}$, where the Lagrange residual can be neglected. Then $sgn(e(p^\alpha, u^0) - e(p^0, u^0)) = sgn\left((p^\alpha - p^0)^\top \nabla_p e(p^0, u^0)\right)$ for all $\alpha \in [0, \bar{\alpha}]$.

- This results in $e(p^\alpha, u^0) - e(p^0, u^0) > 0$ by the assumption that $(p^\alpha - p^0)^\top \nabla_p e(p^0, u^0) > 0$ and the fact that $\nabla_p e(p^0, u^0) = h(p^0, u^0) = x(p^0, e(p^0, u^0))$.

Consumer Theory 5
Partial Information (6)

- Remark: Note that with a differentiable expenditure function the second order term is non-positive, since the expenditure function is concave.

- Remark: We can show the former theorem also in this way (differentiability assumptions have to hold in addition). There the non-positive second order term does not cause a problem, since there we wanted to show that $e(p^1, u^0) - e(p^0, u^0) < 0$ if $(p^1 - p^0) \cdot x^0 < 0$.

Consumer Theory 6
Aggregate Demand

- Aggregate Demand
- Aggregate Welfare
- Aggregate Demand and the Weak Axiom
- Existence of a Representative Consumer

Literature: MWG, Chapter 4

Consumer Theory 6
Motivation (1)

- We already know that individual demand can be expressed as a function of prices and the individual wealth level.

- Can aggregate demand be expressed as a function of prices and the aggregate wealth level?

Consumer Theory 6
Motivation (2)

- Individual demand derived from a rational preference relation satisfies the weak axiom of revealed preference.

- Does aggregate demand satisfy the weak axiom?

Consumer Theory 6
Motivation (3)

- Consider the welfare measures (CV,EV,AV).

- When does aggregate demand have welfare significance. What is the meaning of welfare measures calculated from aggregate demand.

Consumer Theory 6
Aggregate Demand and Wealth (1)

- Consider individual Walrasian demands $x_i(p, w_i)$, $i = 1, \ldots, I$.

- **Definition - Aggregate Demand**:
  $x(p, (w_1, \ldots, w_I)) = \sum_{i=1}^{I} x_i(p, w_i)$.

- **Definition - Aggregate Wealth**: $w = \sum_{i=1}^{I} w_i$.

- When is it possible to write aggregate demand in the simpler form $x(p, \sum_i w_i)$?

- Consider $(w_1, \ldots, w_I)$ and $(w'_1, \ldots, w'_I)$ with $\sum_{i=1}^{I} w_i = \sum_{i=1}^{I} w'_i$. Are the demands $x(p, \sum_i w_i)$ and $x(p, \sum_i w'_i)$ equal for arbitrary pairs of wealth levels with equal aggregate wealth?

Consumer Theory 6
Aggregate Demand and Wealth (2)

- Start with $(w_1, \ldots, w_I)$ and the differential changes $(dw_1, \ldots, dw_I)$ such that $\sum_i dw_i = 0$.

- With a differentiable aggregate demand function, the requirement that aggregate demand does not change requires

$$\sum_{i=1}^{I} \frac{\partial x_{li}(p, w_i)}{\partial w_i} dw_i = 0.$$

Consumer Theory 6
Aggregate Demand and Wealth (3)

- The assumption that $(w_1, \ldots, w_I)$ and $(dw_1, \ldots, dw_I)$ are arbitrary (with $\sum dw_i = 0$) implies: $\sum_{i=1}^{I} \frac{\partial x_{li}(p,w_i)}{\partial w_i} dw_i = 0$ if and only if
$$\frac{\partial x_{li}(p, w_i)}{\partial w_i} = \frac{\partial x_{lj}(p, w_j)}{\partial w_j}$$
for every l, $l = 1, \ldots, L$ and $i, j \in I$, and every $(w_1, \ldots, w_I)$.

- This condition implies that the wealth effect is the same for each consumer and each wealth level.

Consumer Theory 6
Aggregate Demand and Wealth (4)

- This last issue can be satisfied if the wealth expansion paths are parallel straight lines. See MWG, Figure 4.B.1, page 107.

- This is the case with homothetic & identical preferences or quasilinear preferences with respect to the same good (see MWG, D 3.B.6,7 at page 45).

- More general result - next theorem.

Consumer Theory 6
Aggregate Demand and Wealth (5)

- **Proposition** [P 4.B.1] A necessary and sufficient condition for the set of consumers to exhibit parallel straight wealth expansion paths at the price vector p is that preferences admit indirect utility functions of the Gorman form, with the coefficients on w_i the same for all consumers i. That is

$$v_i(p, w_i) = a_i(p) + b(p) w_i \ .$$

- Proof: see exercise and reference in MWG, page 107, 123.

Consumer Theory 6
Aggregate Demand and Wealth (6)

- Assumption on preferences is quite restrictive in MWG [P 4.B.1]. The requirements were quite general - we have considered arbitrary wealth distributions.

- Simpler approach works via wealth distribution rules.

- **Definition - Wealth Distribution Rule**: A family of functions $(w_1(p,w), \ldots, w_I(p,w))$ assigning to each individual i a wealth level $w_i(p,w)$, fulfilling $\sum_{i=1}^{I} w_i = w$, is called wealth distribution rule.

- When we plug in $w_i(p,w)$ into $x_i(p,w_i)$ we get Walrasian demands $x_i(p,w_i(p,w))$ which are functions of p and w. In this case aggregate demand must be a function of p and w.

Consumer Theory 6
Aggregate Demand & Weak Axiom (1)

- Consider $x(p, w_1, \ldots, w_I) = \sum_{i=1}^{I} x_i(p, w_i)$.

- Individual demand: Continuity, homogeneity of degree zero and Walras law. These properties directly carry over to $x(p, w_1, \ldots, w_I)$.

- Individual demand derived from a rational preference relation satisfies the weak axiom (see Chapter 3). MWG "arguably the most central positive property of the individual Walrasian demand"

- Does aggregate demand satisfy the weak axiom?

Consumer Theory 6
Aggregate Demand & Weak Axiom (2)

- Consider $x(p, w_1, \ldots, w_I) = \sum_{i=1}^{I} x_i(p, w_i)$ and a wealth distribution rule, such that aggregate demand can be written as $x(p, w) = \sum_{i=1}^{I} x_i(p, w_i(p, w))$ where $w = \sum_{i=1}^{I} w_i$.

- Assume that $w_i(p, w) = \alpha_i w$ in the following. I.e. the distribution rule is independent from prices.

- **Definition - Weak Axiom and Aggregate Demand**:[D 4.C.1] The aggregate demand function $x(p, w)$ satisfies the weak axiom if $p \cdot x(p', w') \leq w$ and $x(p, w) \neq x(p', w')$ imply $p' \cdot x(p, w) > w'$ for any (p, w) and (p', w').

Consumer Theory 6
Aggregate Demand & Weak Axiom (3)

- Counterexample MWG, [4.C.1], page 110: Individual demands fulfill weak axiom but aggregate does not.

- Consider 2 consumers and 2 goods with $w_1 = w_2 = w/2$. For p and p', we get the demands $x_1(p, w/2)$, etc. fulfilling the weak axiom. Here $x_1(p, w)$ is weakly preferred to $x_1(p', w)$ and $x_2(p', w)$ is weakly preferred to $x_2(p, w)$, see Figure 4.C.1.

- Aggregate demands are $x_1(p, w/2) + x_2(p, w/2)$ and $x_1(p', w/2) + x_2(p', w/2)$. A convex combination of the individual demands is within the budget hyperplains $B_{p,w}$ and $B_{p',w}$. Assume a mixture-weight of $1/2$.

Consumer Theory 6
Aggregate Demand & Weak Axiom (4)

- This implies that $p \cdot x_1(p, w/2) + p \cdot x_2(p, w/2) = w$ and $p' \cdot x_1(p', w/2) + p' \cdot x_2(p', w/2) = w$.

- But $1/2p \cdot x(p', w) < w/2$ and $1/2p' \cdot x(p, w) < w/2$. Multiply both sides with 2, and we observe that $x(p, w)$ does not satisfy the weak axiom of revealed preference. ($p \cdot x(p', w') \leq w$ requires $p' \cdot x(p, w) > w'$ for any (p, w) and (p', w').)

Consumer Theory 6
Aggregate Demand & Weak Axiom (5)

- For individual demands we know that the weak axiom holds if demand satisfies the compensated law of demand (see P 2.F.1): If for any (p, w_i) and price change p' the wealth $w_i' = p' \cdot x(p, w_i)$ and
$$(p' - p)\left[x_i(p', w_i') - x_i(p, w_i)\right] \leq 0$$
with strict inequality for $x(p, w_i) \neq x(p', w_i')$.

- In the notation of this section this results in: $\alpha_i w' = p' x(p, \alpha_i w)$ then
$$(p' - p)\left[x_i(p', \alpha_i w') - x_i(p, \alpha_i w_i)\right] \leq 0$$
with strict inequality for $x_i(p, \alpha_i w) \neq x_i(p', \alpha_i w')$.

Consumer Theory 6
Aggregate Demand & Weak Axiom (6)

- Problem: price-wealth change compensated in the aggregate need not be compensated individually. I.e. if $w' = p' \cdot x(p, w)$ this does not imply that $\alpha_i w' = p' x_i(p, \alpha_i w)$. Problems arises due to differences in the wealth effects. That is, although w' is sufficient to buy the aggregate bundle $x(p, w)$ with the new price vector p', this property need not hold on an individual level.

- To get a condition that the weak axiom holds in the aggregate we consider non-compensated demands.

Consumer Theory 6
Aggregate Demand & Weak Axiom (7)

- **Definition - Uncompensated Law of Demand (ULD)**[D 4.C.2] The individual demand function $x_i(p, w_i)$ satisfies the uncompensated law of demand if

$$(p' - p)\left[x_i(p', w_i) - x_i(p, w_i)\right] \leq 0$$

for any p, p' and w_i with strict inequality for $x(p, w_i) \neq x(p', w_i)$. The analogous definition applies to the aggregate demand function.

- Similar to the weak axiom we get the following results: If $x_i(p, w_i)$ satisfies the ULD property then $D_p x_i(p, w)$ is negative semidefinite and vice versa. The same argument holds for aggregate demand.

- The ULD property aggregates.

- **Proposition**[P 4.C.1] If every consumers' Walrasian demand function $x_i(p, w_i)$ satisfies the uncompensated law of demand, so does aggregate demand $x(p, w) = \sum_{i=1}^{I} x_i(p, \alpha_i w)$. As a consequence the aggregate demand function satisfies the weak axiom.

Consumer Theory 6
Aggregate Demand & Weak Axiom (9)

Proof:

- Step 1: Show that aggregate demand satisfies ULD if individual demands do.
- We start with (p, w), (p', w') and $x_i(p, \alpha_i w) \neq x_i(p', \alpha_i w)$.
- By ULD we get $(p' - p)\left[x_i(p', \alpha_i w) - x_i(p, \alpha_i w)\right] < 0$. Summing up over i, results in $(p' - p)\left[x(p', w) - x(p, w)\right] < 0$. This holds for all p, p' and w.

Consumer Theory 6
Aggregate Demand & Weak Axiom (10)

Proof:

- Step 2: Show that the WA holds, i.e. if $p \cdot x(p', w') \leq w$ then $p' \cdot x(p, w) > w'$.

- To show that ULD for aggregate demand implies the weak axiom we take (p, w), (p', w'), $x(p, w) \neq x(p', w')$ and $p \cdot x(p', w') \leq w$. Define $p'' = (w/w')p'$. Since demand is homogeneous of degree zero in p, w we get $x(p'', w) = x(p', w')$.

- From $(p'' - p)\left[x(p'', w) - x(p, w)\right] < 0$, $p \cdot x(p'', w) \leq w$ and Walras' law if follows that $p'' \cdot x(p, w) > w$. By the definition of p'' this implies $p' \cdot x(p, w) > w'$ such the weak axiom is satisfied.

Consumer Theory 6
Aggregate Demand & Weak Axiom (11)

- How restrictive is Proposition 4.C.1?

- Textbook provides sufficient conditions where ULD is satisfied.

- **Proposition**[P 4.C.2] If $\succeq_i$ is homothetic, then $x_i(p, w_i)$ satisfies the uncompensated law of demand property.

- In the homothetic case the income effects are well behaved. This is the only case that does so. (Here we can write demands as $x_i(p, w_i) = f(p)w_i$. By the Euler theorem $\frac{\partial x_i(p,w_i)}{\partial w_i} w_i = x_{il}(p, w_i)$, etc.)

Consumer Theory 6
Aggregate Demand & Weak Axiom (12)

- ULD requires that the substitution effect is strong compared to the income effect.

- **Proposition**[P 4.C.3] Suppose that $\succeq_i$ is defined on the consumption set $X = \mathbb{R}^L_+$ and is representable by a twice continuously differentiable concave function $u_i(.)$. If

$$-\frac{x_i^\top \cdot D^2 u_i(x_i) \cdot x_i}{x_i^\top \nabla_x u_i(x_i)} < 4$$

for all x_i, then $x_i(p, w_i)$ satisfies the uncompensated law of demand property.

Consumer Theory 6
Representative Consumer (1)

- When does the computation of an aggregate welfare measure make sense? $\Rightarrow$ normative representative consumer.

- When can we treat the aggregate demand function as if it were generated by a representative consumer whose preferences can be used to measure social welfare?

- When can we construct a fictional individual whose utility maximization problem given the society's budget set results in aggregate demand? $\Rightarrow$ positive representative consumer.

- We shall observe that a normative representative consumer need not exist even if a positive representative consumer exists.

Consumer Theory 6
Representative Consumer (2)

- When does the computation of an aggregate welfare measure make sense? $\Rightarrow$ normative representative consumer.

- We start with individual demands $x_i(p, w_i)$ and a wealth distribution rule $(w_1(p, w), \ldots, w_I(p, w))$ with $w_i(p, w)$ homogeneous of degree one in w and continuous.

- **Definition - Positive Representative Consumer**[D 4.D.1] A positive representative consumer exists if there is a rational preference relation $\succeq$ on $\mathbb{R}^L_+$ such that the aggregate demand function $x(p, w)$ is the Walrasian demand function generated by this preference relation ($max\ u(x)$ s.t. $p \cdot x(p, w) = w$). That is $x(p, w) \succ x$ whenever $x \neq x(p, w)$ and $p \cdot x \leq w$.

Consumer Theory 6
Representative Consumer (3)

- Existence: The aggregate demand function satisfies Walras' law, homogeneity of degree zero and symmetric and negative definite Slutsky matrix. By the integrability theorem there exists a utility function $u(x)$ generating $x(p, w)$, where $u(x)$ goes back to a rational, continuous and monotone preference relation.

- $\Rightarrow$ a positive representative consumer exists under fairly mild conditions.

Consumer Theory 6
Representative Consumer (4)

- For welfare comparisons this question becomes more difficult.
- **Definition - Bergson-Samuelson Social Welfare Function**
  [D 4.D.2] A Bergson-Samuelson social welfare function $W : \mathbb{R}^I \to \mathbb{R}$ assigns an utility value to each possible vector of utility levels $(u_1, \ldots, u_I)$ for the I consumers in the economy.
- Expresses the society's judgment how individual utilities have to be compared to produce an ordering of possible social outcomes.
- Assume that W is strictly increasing, strictly concave and differentiable. $u_i(.)$ is strictly concave.

Consumer Theory 6
Representative Consumer (5)

- **Definition - Welfare Maximization Problem (WMP)**: Suppose that there is a benevolent social planner choosing a wealth distribution $(w_1(p,w),\ldots,w_I(p,w))$ for each (p,w), such that W is maximized. I.e.

$$max_{w_1,\ldots,w_I} \; W(v_1(p,w_1),\ldots,v_I(p,w_I)) \;\; s.t. \; \sum_{i=1}^{I} w_i \leq w$$

- Under the above assumptions a solution of this problem exists. The maximized value is denoted by $v(p,w)$.

- The next proposition shows that this indirect utility function provides us with a positive representative consumer.

Consumer Theory 6
Representative Consumer (6)

- **Proposition**[P 4.D.1] Suppose that for each p, w the wealth distribution rule $(w_1(p,w), \dots, w_I(p,w))$ solves the maximization problem WLP. The optimum value function $v(p,w)$ is an indirect utility function of a positive representative consumer for the aggregate demand function $x(p,w) = \sum_i x_i(p, w_i(p,w))$

Proof:

- By Berge's theorem of the maximum a continuous value function exists.

- Since the wealth distribution rule, $u_i(.)$ and $W(u_1, \ldots, u_I)$ are differentiable, and $v(p, w)$ is a composition of these functions, $v(p, w)$ has to be differentiable.

- Idea of the proof: We use Roy's identity to derive $x_R(p, w)$ from the indirect utility function $v(p, w)$. Then we establish that $x_R(p, u)$ is equal to aggregate demand $x(p, w)$.

Proof:

- Consider the optimization problem:

$$max_{w_1,\ldots,w_I} \;\; W(v_1(p,w_1),\ldots,v_I(p,w_I)) \;\; s.t. \; \sum_{i=1}^{I} w_i \leq w$$

with Lagrangeian
$L(w_1,\ldots,w_I) = W(v_1(p,w_1),\ldots,v_I(p,w_I)) + \lambda(w - \sum_i w_i)$

Consumer Theory 6
Representative Consumer (9)

Proof:

- From the first order conditions we get:

$$\lambda = \frac{\partial W}{\partial v_1}\frac{\partial v_1}{\partial w_1} = \cdots = \frac{\partial W}{\partial v_I}\frac{\partial v_I}{\partial w_I}.$$

- Since $v(p, w)$ is a value function, the envelope theorem applied to the social indirect utility function $v(p, w)$ results in

$$\lambda = \frac{\partial v(p, w)}{\partial w}.$$

Consumer Theory 6
Representative Consumer (10)

Proof:

- We already know that Roy's identity takes us back from indirect utility to demand:
$$x_R(p,w) = -\frac{\nabla_p v(p,w)}{\nabla_w v(p,w)}$$
- In addition the envelope theorem applied to the social indirect utility function $v(p,w)$ results in:

$$\frac{\partial v(p,w)}{\partial p_l} = \sum_{i=1}^{I} \left(\frac{\partial W(.)}{\partial v_i} \frac{\partial v_i(p,w)}{\partial p_l} + \lambda \frac{\partial w_i}{\partial p_l} \right) = \sum_{i=1}^{I} \frac{\partial W(.)}{\partial v_i} \frac{\partial v_i(p,w)}{\partial p_l}.$$

for $l = 1, \ldots, L$.

Consumer Theory 6
Representative Consumer (11)

Proof:

- The second term is zero because $\sum_i w_i(p, w) = w$ and Walras' law holds.

- In matrix notation this expression results in:

$$\nabla_p v(p, w) = \sum_{i=1}^{I} \frac{\partial W(.)}{\partial v_i} \nabla_p v_i(p, w).$$

- The above FOC implies that

$$\nabla_p v(p, w) = \sum_{i=1}^{I} \frac{\partial W(.)}{\partial v_i} \nabla_p v_i(p, w) = \sum_{i=1}^{I} \frac{\lambda}{(\partial v_i / \partial w_i)} \nabla_p v_i(p, w).$$

Consumer Theory 6
Representative Consumer (12)

Proof:

- The first order condition, x_R, the fact that $\frac{\partial v(p,w)}{\partial w} = \lambda$, the wealth distribution rule $w_i(p, w)$ and Roy's identity (for individual demands $x_i(p, w_i) = -\frac{1}{\partial v_i / \partial w_i} \nabla_p v_i(p, w_i)$) yields:

$$\begin{aligned} x_R(p, w) &= -\frac{1}{\lambda} \sum_i \frac{\lambda}{\partial v_i / \partial w_i} \nabla_p v_i(p, w_i(p, w)) \\ &= \sum_i x_i(p, w_i(p, w)) = x(p, w) \ . \end{aligned}$$

- **Definition - Normative Representative Consumer**[D 4.D.3]
  The positive representative consumer $\succeq$ for the aggregate demand $x(p, w)$ is a normative representative consumer relative to the social welfare function $W(.)$ if for every (p, w) the distribution of wealth $(w_1(p, w), \ldots, w_I(p, w))$ solves the welfare maximization problem (WMP) and therefore the maximum value function of this problem is an indirect utility function for $\succeq$.

Consumer Theory 6
Representative Consumer (14)

- If a normative representative consumer exists then welfare comparisons based on aggregate demand make sense.

- Example where positive and normative representative consumer exist - [E 4.D.1], page 118, where individual preferences are homothetic and $W(u_1, \ldots, u_2) = \sum \alpha_i \log u_i$, $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

- If aggregate demand can be represented by a representative consumer this does not imply the existence of a normative representative consumer. This depends on $W(.)$ and whether this function is maximized by the corresponding wealth distribution rule. Note that we have assumed this in proposition P 4.D.1. For a counterexample see Example page 120 (in small print).

Production 1
Motivation

- Production
- Production possibility sets and the production function
- Marginal product, marginal rate of substitution and returns to scale.

MasColell, Chapter 5

Production 1
Firms (1)

- In this section we treat the firm as a black box. We abstract from ownership, management, organization, etc.

- **Assumption**: A firm maximizes its profit.

- How can we justify this assumption?

Production 1
Production Possibility Set (1)

- **Definition - Production**: The process of transforming inputs to outputs is called production.

- The state of technology restricts what is possible in combining inputs to produce output (**technological feasibility**).

- **Definition - Production Possibility Set**: A set $Y \in \mathbb{R}^L$ describing possible production plans is called production possibility set, $Y = \{y \in \mathbb{R}^L |\ y$ is a feasible production plan$\}$. $y_i < 0$ are called **inputs**, $y_i > 0$ **outputs**.

Production 1
Production Possibility Set (2)

- Often the production possibility set is described by a function $F(.)$ called **transformation function**. This function has the property $Y = \{y \in \mathbb{R}^L | F(y) \leq 0\}$ and $F(y) = 0$ if and only if we are on the boundary of the set Y. $\{y \in \mathbb{R}^L | F(y) = 0\}$ is called **transformation frontier**.

- **Definition - Marginal Rate of Transformation**: If $F(.)$ is differentiable and $F(\bar{y}) = 0$, then for commodities k and l the ration
$$MRT_{lk}(\bar{y}) = \frac{\partial F(\bar{y})/\partial y_l}{\partial F(\bar{y})/\partial y_k}$$
is called marginal rate of transformation of good l for good k.

Production 1
Production Possibility Set (3)

- If l and k are outputs we observe how output of l increases if k is decreases.

- With inputs .... In this case the marginal rate of transformation is called **marginal rate of technical substitution**.

- With a single output q, production is often described by means of a production function $q = f(z_1, \ldots, z_m)$, where the inputs $z_i \geq 0$, $i = 1, \ldots, m$. In this case $Y = \{(-z_1, \ldots, -z_m, q)^\top | q - f(z_1, \ldots, z_m) \leq 0$ and $z_1, \ldots, z_m \geq 0\}$.

Production 1
Production Possibility Set (4)

- Assumption and Properties of production possibility sets

P1 Y is non-empty.
P2 Y is closed. I.e. Y includes its boundary, if $y_n \in Y$ converges to y then $y \in Y$.
P3 No free lunch. If $y_l \geq 0$ for $l = 1, \ldots, L$, then $y = \mathbf{0}$. It is not possible to produce something from nothing. Therefore $Y \cap \mathbf{R}^L_+ = \mathbf{0} \in Y$ (note that $\mathbf{0} \in Y$ has to be assumed here). See Figure 5.B.2, page 131.

Production 1
Production Possibility Set (5)

P4 Possibility of inaction: $\mathbf{0} \in Y$. This assumption hold at least ex-ante, before the setup of the firm. If we have entered into some irrevocable contracts, then a sunk cost might arise.

P5 Free Disposal: New inputs can be acquired without any reduction of output. If $y \in Y$ and $y' \leq y$ then $y' \in Y$. For any $y \in Y$ and $x \in \mathbb{R}^L_+$, we get $y - x \in Y$. See Figure 5.B.4, page 132.

P6 Irreversibility: If $y \in Y$ and $y \neq 0$, then $-y \notin Y$. It is impossible to reverse a possible production vector. We do not come from output to input.

Production 1
Production Possibility Set (6)

P7 Nonincreasing returns to scale: If $y \in Y$, then $\alpha y \in Y$ for all $\alpha \in [0,1]$. I.e. any feasible input-output vector y can be scaled down. See Figure 5.B.5.

P8 Nondecreasing returns to scale: If $y \in Y$, then $\alpha y \in Y$ for any scale $\alpha \geq 1$. I.e. any feasible input-output vector y can be scaled up. See Figure 5.B.6.

P9 Constant returns to scale: If $y \in Y$, then $\alpha y \in Y$ for any scale $\alpha \geq 0$. I.e. any feasible input-output vector y can be scaled up and down.

Production 1
Production Possibility Set (7)

P10 Additivity - free entry: If $y \in Y$ and $y' \in Y$, then $y + y' \in Y$. This implies that $ky \in Y$ for any positive integer k.

- Example: Output is an integer. If y and y' are possible, additivity means that $y + y'$ is still possible and the production of y has no impact on y' and vice versa. E.g. we have two independent plants.

- As regards free-entry: If the *aggregate production set* Y is additive, then unrestricted entry is possible. To see this, if $y \in Y$ is produced by firm A and $y' \in Y$ by firm B, then $y + y' \in Y$ if additivity holds. That is, the production plans of firm A do not interfere with the production plans of firm B (and vice versa). In other words, the aggregate production set has to satisfy additivity whenever unrestricted entry is possible.

Production 1
Production Possibility Set (8)

P11 Convexity: Y is a convex set. I.e. if $y \in Y$ and $y' \in Y$, then $\alpha y + (1 - \alpha) y' \in Y$.

- Convexity implies nonincreasing returns to scale.

- We do not increase productivity by using unbalanced input combinations. If y and y' produce the same output, then a convex combination of the correspond inputs must at least produce an output larger or equal to the output with y and y'.

Production 1
Production Possibility Set (9)

P12 Y is convex cone: Y is a convex cone if for any $y, y' \in Y$ and constants $\alpha, \beta \geq 0$, $\alpha y + \beta y' \in Y$. Conjunction between convexity and constant returns to scale property.

Production 1
Production Possibility Set (10)

- **Proposition**[P 5.B.1]: The production set Y is additive and satisfies the nonincreasing returns to scale property if and only if it is is convex cone.

Proof:

- If Y is a convex cone then Y is additive and satisfies the nonincreasing returns to scale by the definition of a convex cone.

- We have to show that with additivity and nonincreasing returns to scale we get $\alpha y + \beta y' \in Y$ for any y, y' and $\alpha, \beta > 0$ (note that with $\alpha \geq 0, \beta = 0$, $\alpha = 0, \beta \geq 0$ the relation $\alpha y + \beta y' \in Y$ follows from RTS and additivity): Let $\gamma = \max\{\alpha, \beta\} > 0$. By additivity $\gamma y \in Y$ and $\gamma y' \in Y$.

- α/γ and β/γ are ≤ 1. Due to nonincreasing returns to scale $\alpha y = (\alpha/\gamma)\gamma y$ and $\beta y' = (\beta/\gamma)\gamma y' \in Y$. By additivity $\alpha y + \beta y' \in Y$.

Production 1
Production Possibility Set (12)

- **Proposition**[P 5.B.2]: For any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant returns to scale convex production set $Y' \in \mathbb{R}^{L+1}$ such that $Y = \{y \in \mathbb{R}^L | (y, -1) \in Y'\}$. Y' is called **extended production set**.

- Proof: Let $Y' = \{y' \in \mathbb{R}^{L+1} | y' = \alpha(y, -1),\ y \in Y,\ \alpha \geq 0\}$. If $y' \in Y'$, then the first L components are in Y by construction. Since $\beta y^1 + (1-\beta) y^2 \in Y$ we get $\beta(y^1, -1) + (1-\beta)(y^2, -1) \in Y'$. $\alpha y' \in Y'$ by construction.

Production 1
Production Function (1)

- Often it is sufficient to work with an output $q \geq 0$ and inputs $z = (z_1, \ldots, z_m)$ where $z_i \geq 0$.
- **Definition - Production Function**: A function describing the the relationship between q and z is called production function f.
- **Remark**: The production functions assigns the maximum of output q that can be attained to an input vector z.
  $f(z) = \max\{q \geq 0 | z \in \mathbb{R}^m_+\}$; (*output efficient production*).

Production 1
Production Function (2)

- **Assumption PF on Production Function**: The production function $f : \mathbb{R}^m_+ \to \mathbb{R}_+$ is continuous, strictly increasing and strictly quasiconcave on $\mathbb{R}^m_+$; $f(\mathbf{0}) = 0$.

- **Assumption PF' - Production Function**: The production function $f : \mathbb{R}^m_+ \to \mathbb{R}_+$ is continuous, increasing and quasiconcave on $\mathbb{R}^m_+$; $f(\mathbf{0}) = 0$.

- How can we motivate these assumptions?

Production 1
Production Function (3)

- Considering production functions two approaches are common: (i) variation one factor, (ii) variation all factors in the same proportion.

- **Definition** - **Marginal Product**: If f is differentiable then $\frac{\partial f(z)}{\partial z_i} = MP_i(z)$ is called marginal product of the input factor z_i.

- By Assumption P5 all marginal products are strictly larger than zero, with P5' $MP_i(z) \geq 0$.

- **Definition** - **Average Product**: The fraction $f(z)/z_i = AP_i(z)$ is called average product of the input factor z_i.

Production 1
Production Function (4)

- **Definition - Isoquant**: The set $Q(q)$ where output is constant is called q-level isoquant. I.e. $Q(q) = \{z \geq 0 | f(z) = q\}$.

- In addition to $Q(q)$ we can define the the contour set $\bar{S}(q) = \{z \geq 0 | f(z) \geq q\}$. Since f is quasiconcave, this set is convex $\Rightarrow$ isoquants are convex curves.

- In addition, by means of the isoquant we can observe how input factors can be substituted to remain on the same level of output.

- **Definition - Marginal Rate of Technical Substitution**:

$$MRTS_{ij}(z) = \frac{MP_i}{MP_j}$$

- The slope of the isoquant is given by $-\frac{dz_j}{dz_i} = \frac{MP_i}{MP_j}$

- Discuss: $\frac{MP_i}{MP_j} > 0$ (≥ 0) and the concept of **technical efficiency**: To remain on the same level of output at least one input has to be increased if one input factor has been decreased.

Production 1
Production Function (6)

- In general the MRTS of two input depends on all other inputs (note that the MP_i depends on z).

- In applied work it is often assumed that inputs can be classified, such that the MRTS within a class is not affected by inputs outside this class.

Production 1
Production Function (7)

- **Definition - Separable Production Function**: Suppose that the inputs can be partitioned into $S > 1$ classes $N_1, \ldots, N_S$; $N = \{1, \ldots, n\}$ is an index set. The production function is called **weakly separable** if the MRTS between inputs within the same group is independent of the inputs used in the other groups:
$$\frac{\partial(MP_i/MP_j)}{\partial z_k} = 0$$
for all $i, j \in N_s$ and $k \notin N_s$. For $S > 2$ it is **strongly separable** if the MRTS between two inputs from different groups is independent of all inputs outside those groups:
$$\frac{\partial(MP_i/MP_j)}{\partial z_k} = 0$$
for all $i \in N_s$, $j \in N_t$ and $k \notin N_s \cup N_t$.

Production 1
Production Function (8)

- Since $MRTS_{ij}$ is sensitive to the dimension of the measurements of z_i and z_j an elasticity can be used.

- **Definition - Elasticity of Substitution**: For a differentiable production function the elasticity of substitution between inputs z_i and z_j is defined by

$$\sigma_{ij} := \frac{d(z_j/z_i)}{d(MP_i/MP_j)} \frac{(MP_i/MP_j)}{(z_j/z_i)} = \frac{d \log(z_j/z_i)}{d \log(MP_i/MP_j)} \ .$$

- With a quasiconcave production function $\sigma_{ij} \geq 0$

- **Theorem - Linear Homogeneous Production Functions are Concave**: Let f satisfy Assumption P5'. If f is homogenous of degree one, then $f(z)$ is concave in z.

Production 1
Production Function (10)

Proof:

- We have to show $f(z^\nu) \geq \nu f(z^1) + (1-\nu) f(z^2)$, where $z^\nu = \nu z^1 + (1-\nu) z^2$.

- Step 1: By assumption $f(\mu z) = \mu f(z) = \mu y$. Then $1 = f(z/y)$. I.e. $f(z^1/y^1) = f(z^2/y^2) = 1$. (Set $\mu = 1/y$.)

- Since $f(z)$ is quasiconcave: $f(z^\nu) \geq min\{f(z^1), f(z^2)\}$.

- Therefore $f\left(\nu(z^1/y^1) + (1-\nu)(z^2/y^2)\right) \geq 1$.

Production 1
Production Function (11)

Proof:

- Choose $\nu^* = y^1/(y^1+y^2)$. Then $f\left((z^1+z^2)/(y^1+y^2)\right) \geq 1$.

- By the homogeneity of f we derive:

$$f\left(z^1+z^2\right) \geq y^1+y^2 = f(z^1)+f(z^2)\ .$$

Production 1
Production Function (12)

Proof:

- Step 2: Now we show that $f(z^\nu) \geq \nu f(z^1) + (1-\nu)f(z^2)$ holds.
- By homogeneity $f(\nu z^1) = \nu f(z^1)$ and $f((1-\nu)z^2) = (1-\nu)f(z^2)$
- Insert into the above expressions:

$$f\left(\nu z^1 + (1-\nu)z^2\right) \geq f(\nu z^1) + f((1-\nu)z^2)$$

$$f(\nu z^1) + f((1-\nu)z^2) = \nu f(z^1) + (1-\nu)f(z^2)$$

Production 1
Production Function (13)

- Another way to look at the properties of production is to alter inputs proportionally. I.e. z_i/z_j remains constant.

- Discuss: This analysis is of interest especially for the long run behavior of a firm.

- **Definition - Returns to Scale**. A production function $f(z)$ exhibits
  - Constant returns to scale if $f(\mu z) = \mu f(z)$ for $\mu > 0$ and all z.
  - Increasing returns to scale if $f(\mu z) > \mu f(z)$ for $\mu > 1$ and all z.
  - Decreasing returns to scale if $f(\mu z) < \mu f(z)$ for $\mu > 1$ and all z.

Production 1
Production Function (15)

- With constant returns the scale the production function has to be homogeneous of degree one.
- Homogeneity larger than one is sufficient for increasing returns to scale but not necessary.
- Most production function/technologies often exhibit regions with constant, increasing and decreasing returns to scale.

- **Definition - Local Returns to Scale**. The elasticity of scale at z is defined by

$$LRTS(z) := \lim_{\mu \to 1} \frac{d \log(f(\mu z))}{d \log \mu} = \frac{\sum_{i=1}^{n} MP_i z_i}{f(z)} \ .$$

  A production function $f(z)$ exhibits

  – local constant returns to scale if $LRTS(z)$ is equal to one.
  – local increasing returns to scale if $LRTS(z)$ is larger than one.
  – local decreasing returns to scale if $LRTS(z)$ is smaller than one.

Production 2
Profits and Cost (1)

- Profit Maximization
- Cost minimization
- Price taking
- Cost, profit and supply function

MasColell, Chapter 5.C

Production 2
Profits (1)

- Assume that $p = (p_1, \ldots, p_L)$ are larger than zero and fixed (**price taking** assumption).
- We assume that firms maximize profits.
- Given an Input-Output vector y, the profit generated by a firm is $p \cdot y$.
- We assume that Y is non-empty, closed and free disposal holds.

Production 2
Profits (2)

- **Definition**: Given the production possibility set Y, we get the **profit maximization problem**

$$\max_y \; p \cdot y \quad s.t. \quad y \in Y.$$

- If Y can be described by a transformation function F, this problem reads as follows:

$$\max_y \; p \cdot y \quad s.t. \quad F(y) \leq 0.$$

- Define $\pi(p) = \sup_y p \cdot y \; s.t. \; y \in Y$.

Production 2
Profits (3)

- **Definition - Profit function** $\pi(p)$: The maximum value function associated with the profit maximization problem is called profit function. The firm's supply correspondence $y(p)$ is the set of profit maximizing vectors $\{y \in Y | p \cdot y = \pi(p)\}$.

- The value function $\pi(p)$ is defined on extended real numbers ($\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$). The set $S_p = \{p \cdot y | y \in Y\}$ is a subset of $\mathbb{R}$. $\{p \cdot y | y \in Y\}$ has an upper bound in $\bar{\mathbb{R}}$. For p where S_p is unbounded (from above) in $\mathbb{R}$ we set $\pi(p) = \infty$.

- If Y is compact a solution (and also the max) for the profit maximization problem exits. If this is not the case $\pi(p) = \infty$ is still possible. The profit function exists by Bergs theorem of the maximum if the constraint correspondence is continuous.

- We follow MWG and write $\max_y p \cdot y \ s.t. \ y \in Y$, although ....; Jehle/Reny call $\pi(p, w)$ well defined if $\pi(p, w) < \infty$.

Production 2
Profits (4)

- Suppose that $F(.)$ is differentiable, then we can formulate the profit maximization problem as a Kuhn-Tucker problem:

- The Lagrangian is given by: $L(y, \lambda) = p \cdot y - \lambda F(y)$

- Then the Kuhn-Tucker conditions are given by:

$$\begin{aligned}
\frac{\partial L}{\partial y_l} &= p_l - \lambda \frac{\partial F(y)}{\partial y_l} \leq 0 \;, \quad \frac{\partial L}{\partial y_l} y_l = 0 \\
\frac{\partial L}{\partial \lambda} &= -F(y) \geq 0 \\
\frac{\partial L}{\partial \lambda}\lambda &= 0
\end{aligned}$$

Production 2
Profits (5)

- For those inputs and output different from zero we get:

$$p = \lambda \nabla_y F(y)$$

  This implies that

$$\frac{p_l}{p_k} = \frac{\partial F / \partial y_l}{\partial F / \partial y_k} = MRT_{lk}.$$

- Since the left hand side is positive by assumption, the fraction of the right hand side and λ have to be positive.

Production 2
Profits (6)

- If $y_l, y_k > 0$, i.e. both goods are outputs, then y_l, y_k have to be chosen such that the fraction of marginal rates of transformation is equal to the ratio of prices.

- If $y_l, y_k < 0$, i.e. both goods are inputs, then y_l, y_k have to be chosen such that the fraction of marginal rates of transformation (= **marginal rate of technical substitution**) is equal to the ratio of prices.

- If $y_l > 0, y_k < 0$, i.e. y_l is an output and y_k is an input, then $p_l = \frac{\partial F/\partial y_l}{\partial F/\partial y_k} p_k$. Later on we shall observe that $\frac{\partial F/\partial y_l}{\partial F/\partial y_k} p_k$ is the marginal cost of good l. See Figure 5.C.1. page 136.

Production 2
Profits - Single Output Case (1)

- Suppose the there is only one output $q \geq 0$ and input $z \geq 0$. The relationship between q and z is described by a differentiable production function. The price of q is $p > 0$. Input factor prices are $w \gg 0$. We assume that the second order conditions are met.

- The profit maximization problem now reads as follows:

$$\pi(p, w) := \{\max_{z,q \geq 0} pf(z) - w \cdot z \ s.t. \ f(z) \geq q\}$$

- The input factor demand arising from this problem $z = z(w, q)$ is called **input factor demand**.

Production 2
Profits - Single Output Case (2)

- Is the profit function well defined?

- Suppose that $pf(z) - w \cdot z \geq 0$. What happens if $f(z)$ exhibits increasing returns to scale?

- Here $pf(\mu z) - w \cdot \mu z > p\mu f(z) - w \cdot \mu z$ for all $\mu > 1$.

- That is, the profit can always be increased when increasing μ.

- With constant returns to scale no problem arises when $\pi(w, p) = 0$. Then $pf(\mu z) - w \cdot \mu z = p\mu f(z) - w \cdot \mu z = 0$ for all μ.

Production 2
Profits - Single Output Case (3)

- From these remarks we get the (long run) problem:

$$\max\{pq - w \cdot z\} \quad s.t \quad f(z) \geq q$$

- The Lagrangian is now given by:

$$L(q, z, \lambda) = pq - w \cdot z + \lambda(f(z) - q)$$

- The marginal product will be abbreviated by $MP_i = \frac{\partial f(z)}{\partial z_i}$.

Production 2
Profits - Single Output Case (4)

- Then the Kuhn-Tucker conditions are given by:

$$
\begin{aligned}
\frac{\partial L}{\partial y} &= p + \lambda \leq 0 \ , \quad \frac{\partial L}{\partial q} q = 0 \\
\frac{\partial L}{\partial z_i} &= -w_i - \lambda M P_i \leq 0 \ , \quad \frac{\partial L}{\partial z_i} z_i = 0 \\
\frac{\partial L}{\partial \lambda} &= f(z) - q \geq 0 \ , \quad \frac{\partial L}{\partial \lambda} \lambda = 0
\end{aligned}
$$

Production 2
Profits - Single Output Case (5)

- This yields:

$$w_i = p\frac{\partial f(z)}{\partial z_i}\ ,\ \forall z_i > 0$$

- **Definition - Marginal Revenue Product**: $p\frac{\partial f(z)}{\partial z_i}$.

- For inputs i and j we derive:

$$\frac{\partial f(z)/\partial z_i}{\partial f(z)/\partial z_j} = \frac{w_i}{w_j}$$

Production 2
Profit Function (1)

- By means of $\pi(p)$ we can reconstruct $-Y$, if $-Y$ is a convex set.

- That is to say: $\pi(p)$ follows from $\{\max_y \; p \cdot y \; s.t. \; y \in Y\}$, which is equivalent to $\{min_y \; -p \cdot y \; s.t. \; y \in Y\}$ and $\{min_{-y} \; p \cdot (-y) \; s.t. \; (-y) \in -Y\}$.

- Remember the concept of a support function: By means of the support function $\mu_X(p)$ we get by means of $\{x | p \cdot x \geq \mu_X(p)\}$ a dual representation of the closed and convex set X.

- Here $-\pi(p) = \mu_{-Y}(p)$ where $\mu_{-Y}(p) = \min_y \{p \cdot (-y) | y \in Y\}$ such that $-\pi(p)$ is a support function of $-Y$.

Production 2
Profit Function (2)

- **Proposition**: [5.C.1] Suppose that $\pi(p)$ is the profit function of the production set Y and $y(p)$ is the associated supply correspondence. Assume that Y is closed and satisfies the the free disposal property. Then

  1. $\pi(p)$ is homogeneous of degree one.
  2. $\pi(p)$ is convex.
  3. If Y is convex, then $Y = \{y \in \mathbb{R}^L | p \cdot y \leq \pi(p)\ , \quad \forall p \gg 0\}$
  4. $y(p)$ is homogeneous of degree zero.
  5. If Y is convex, then $y(p)$ is convex for all p. If Y is strictly convex, then $y(p)$ is single valued.
  6. Hotelling's Lemma: If $y(\bar{p})$ consists of a single point, then $\pi(p)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$.
  7. If y is differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semidefinite matrix with $Dy(\bar{p})\bar{p} = 0$.

Production 2
Profit Function (3)

Proof:

- $\pi(p)$ is homogeneous of degree one and $y(p)$ is homogeneous of degree zero follow from the structure of the optimization problem. If $y \in y(p)$ solves $\{\max p \cdot y \ s.t. \ F(y) \leq 0\}$ then it also solves $\alpha\{\max p \cdot y \ s.t. \ F(y) \leq 0\}$ and $\{\max \alpha p \cdot y \ s.t. \ F(y) \leq 0\}$, such that $y \in y(\alpha p)$ for any $\alpha > 0$.

- This hold for every $y \in y(p) \Rightarrow y(p)$ is homogeneous of degree zero and $\pi(p)$ is homogeneous of degree one by the structure of the profit equation.

Production 2
Profit Function (4)

Proof:

- $\pi(p)$ is convex: Consider p^1 and p^2 and the convex combination p^ν. y^1, y^2 and y^ν are arbitrary elements of the optimal supply correspondences.

- We get $p^1y^1 \geq p^1y^\nu$ and $p^2y^2 \geq p^2y^\nu$

- Multiplying the first term with ν and the second with $1-\nu$, where $\nu \in [0,1]$ results in $\nu p^1y^1 + (1-\nu)p^2y^2 \geq \nu p^1y^\nu + (1-\nu)p^2y^\nu = p^\nu y^\nu$ which implies

$$\nu\pi(p^1) + (1-\nu)\pi(p^2) \geq \pi(p^\nu)$$

Production 2
Profit Function (5)

Proof:

- If Y is convex then $Y = \{y \in \mathbb{R}^L | p \cdot y \leq \pi(p)\ ,\ \text{for all } p \gg 0\}$: If Y is convex, closed and free disposal holds, then $\pi(p)$ provides a dual description of the production possibility set.

Production 2
Profit Function (6)

Proof:

- If Y is convex then $y(p)$ is a convex set, with strict convexity $y(p)$ is a function: If Y is convex then $y^\nu = \nu y^1 + (1-\nu)y^2 \in Y$.

- If y^1 and y^2 solve the PMP for p, then $\pi(p) = p \cdot y^1 = p \cdot y^2$. A rescaling of the production vectors has to result in $y^\nu = \nu y^1 + (1-\nu)y^2$ where $p \cdot y^\nu = \pi(p)$ has to hold.

  This follows from $p \cdot y^1 = p \cdot y^2 = \pi(p) = \nu\pi(p) + (1-\nu)\pi(p) = \nu p \cdot y^1 + (1-\nu)p \cdot y^2 = p\nu \cdot y^1 + p(1-\nu) \cdot y^2 = p(\nu \cdot y^1 + (1-\nu) \cdot y^2)$.

Production 2
Profit Function (7)

Proof:

- Suppose that y^α solves the PMP and Y is strictly convex (every point on the boundary is an extreme point, i.e. this point is not a convex combination of other points in Y). y^α is an element of $Y \cap H(p, \pi(p))$. $H(p, \pi(p))$ stands for an isoprofit hyperplane. Suppose that there is another solution y' solving the profit maximization problem (PMP). So y, y' are elements of this hyperplane. Since $y, y' \in Y$ this implies that Y cannot be strictly convex.

- Remark by Proposition P 5.F.1, page 150, $y(p)$ cannot be an interior point of y. Suppose that an interior point y'' solves the PMP, then $\pi(p) = p \cdot y''$. For any interior point, there is an y such that $y \geq y''$ and $y \neq y''$. Since $p \gg 0$ this implies $p \cdot y > p \cdot y''$ such that an interior point cannot be optimal.

Production 2
Profit Function (8)

Proof:

- Hotellings lemma: Follows directly from the duality theorem: $\nabla_p \pi(\bar{p}) = y(\bar{p})$; (see [P 3.F.1], page 66).

Production 2
Profit Function (9)

Proof:

- Property 7: If $y(p)$ and π are differentiable, then $D_p y(\bar{p}) = D_p^2 \pi(p)$. By Young's theorem this matrix is symmetric, since $\pi(p)$ is convex in p the matrix has to be positive semidefinite (see Theorem M.C.2, p.. 932).

- $D_p y(p) p = \mathbf{0}_{L \times 1}$ follows from the Euler theorem (see MWG, Theorem M.B.2, p. 929).

Production 2
Profit Function (10)

- By Hotellings lemma inputs and outputs react in the same direction as the price change: Output increases if output prices increase, while inputs decrease if its prices increase (**law of supply**), i.e.:
$$(p-p')[y(p)-y(p')] \geq 0$$
- This law holds for any price change (there is no budget constraint, therefore any form of compensation is not necessary. We have no wealth effect but only substitution effects).
- We can also show that the law of supply holds also for the non-differentiable case. (We know that $p^1y^1 \geq p^1y$ for any $y^1 \in y(p^1)$ and $p^2y^2 \geq p^2y$ for any $y^2 \in y(p^1)$, sum up ....)

Production 2
Cost Function (1)

- Profit maximization implies cost minimization!
- Production does not tell us anything about the minimal cost to get output.
- On the other hand side - if the firm is not a price taker in the output market, we cannot use the profit function, however the results on the cost function are still valid.
- With increasing returns to scale where the profit function can only take the values 0 or $+\infty$, the cost function is better behaved since the output is kept fixed there.

Production 2
Cost Function (2)

- Assume that the input factor prices $w \gg 0$ are constant. In addition we assume that the production function is at least continuous.

- **Definition - Cost**: Expenditures to acquire input factors z to produce output q; i.e. $w \cdot z$.

- **Definition - Cost Minimization Problem** (CMP): $\min_z w \cdot z$ s.t. $f(z) \geq q$. The minimal value function $C(w, q)$ is called **cost function**. The optimal input factor choices are called **conditional factor demand correspondence** $z(w, q)$.

Production 2
Cost Function (3)

- Existence: Construct the set $\{z|f(z) \geq q\}$. Under the usual assumptions on the production function the set is closed. By compactifying this set by means of $\{z|f(z) \geq q, z_i \leq w \cdot \bar{z}/w_i\}$ for some $\bar{z}$ with $f(\bar{z}) = q$ we can apply the Weierstraß theorem.

- By Berge's theorem of the maximum we get a continuous cost function $C(w, q)$, if the constraint correspondence is continuous.

Production 2
Cost Function (4)

- **Definition - Marginal Cost**: $LMC(q) = \frac{\partial c(w,q)}{\partial q}$ is called marginal cost.

- **Definition - Average Cost**: $LAC(q) = \frac{c(w,q)}{q}$ is called average cost.

Production 2
Cost Function (5)

- **Theorem: Properties of the Cost Function** $c(w, q)$: [P 5.C.2] Suppose that $c(w, q)$ is a cost function of a single output technology Y with production function $f(z)$ and $z(w, q)$ is the associated conditional factor demand correspondence. Assume that Y is closed and satisfies the free disposal property. Then

(i) $c(w, q)$ is homogeneous of degree one in w and nondecreasing in q.

(ii) Concave in w.

(iii) If the set $\{z \geq 0 | f(z) \geq q\}$ is convex for every q, then $Y = \{(-z, q) | w \cdot z \geq c(w, q)$ for all $w \gg 0\}$.

(iv) $z(w, q)$ is homogeneous of degree zero in w.

(v) If the set $\{z \geq 0 | f(z) \geq q\}$ is convex then $z(w, q)$ is a convex set, with strict convexity $z(w, q)$ is a function.

Production 2
Cost Function (6)

- **Theorem: Properties of the Cost Function** $c(w, q)$: [P 5.C.2] Suppose that $c(w, q)$ is a cost function of a single output technology Y with production function $f(z)$ and $z(w, q)$ is the associated conditional factor demand correspondence. Assume that Y is closed and satisfies the free disposal property. Then

(vi) Shepard's lemma: If $z(\bar{w}, q)$ consists of a single point, then $c(.)$ is differentiable with respect to w at $\bar{w}$ and $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$.

(vii) If $z(.)$ is differentiable at $\bar{w}$ then $D_w z(\bar{w}, q) = D^2 c(\bar{w}, q)$ is symmetric and negative semidefinite with $D_w z(\bar{w}, q)\bar{w} = 0$.

(viii) If $f(.)$ is homogeneous of degree one, then $c(.)$ and $z(.)$ are homogeneous of degree one in q.

(ix) If $f(.)$ is concave, then $c(.)$ is a convex function of q (marginal costs are nondecreasing in q).

Production 2
Cost Function (7)

- By means of the cost function we can restate the profit maximization problem (PMP):

$$\max_{q \geq 0} pq - C(w, q)$$

- The first order condition becomes:

$$p - \frac{\partial C(w, q)}{\partial q} \leq 0$$

with $(p - \frac{\partial C(w,q)}{\partial q}) = 0$ if $q > 0$.

Production 3
Aggregate Supply and Efficiency

- Aggregate Supply

- Joint profit maximization is a result of individual profit maximization

- Efficient Production

  Mas-Colell Chapters 5.D, 5.E

Production 3
Aggregate Supply (1)

- Consider J units (firms, plants) with production sets $Y_1, \dots, Y_J$ equipped with profit functions $\pi_j(p)$ and supply correspondences $y_j(p)$, $j = 1, \dots, J$.

- **Definition - Aggregate Supply Correspondence**: The sum of the $y_j(p)$ is called aggregate supply correspondence:

$$y(p) := \sum_{j}^{J} y_j(p) = \{y \in \mathbb{R}^L | y = \sum_{j}^{J} y_j \text{ for some } y_j \in y_j(p)\},\ j = 1, \dots, J\}$$

- **Definition - Aggregate Production Set**: The sum of the individual Y_j is called aggregate production set:

$$Y = \sum_{j}^{J} Y_j = \{y \in \mathbb{R}^L | y = \sum_{j}^{J} y_j \text{ for some } y_j \in Y_j,\ j = 1, \dots, J\}$$

Production 3
Aggregate Supply (2)

- **Proposition** The law of supply also holds for the aggregate supply function.

- Proof: Since $(p - p')[y_j(p) - y_j(p')] \geq 0$ for all $j = 1, \ldots, J$ it has also to hold for the sum.

- Definition: $\pi^*(p)$ and $y^*(p)$ are the profit function and the supply correspondence of the aggregate production set Y.

Production 3
Aggregate Supply (3)

- **Proposition**[5.E.1] For all $p \gg 0$ we have
  - $\pi^*(p) = \sum_j^J \pi_j(p)$
  - $y^*(p) = \sum_j^J y_j(p)$ $(= \{\sum_j^J y_j | y_j \in y_j(p)\})$

- Suppose that prices are fixed, this proposition implies that the aggregate profit obtained by production of each unit separately is the same as if we maximize the joint profit.

Production 3
Aggregate Supply (4)

Proof:

- $\pi^*(p) = \sum_j^J \pi_j(p)$: Since π^* is the maximum value function obtained from the aggregate maximization problem, we have $\pi^*(p) \geq p \cdot (\sum_j y_j) = \sum_j p \cdot y_j$ which implies $\pi^*(p) \geq \sum_j \pi_j(p)$.

- To show equality, note that there are y_j in Y_j such that $y = \sum_j y_j$. Then $p \cdot y = \sum_j p \cdot y_j \leq \sum_j \pi_j(p)$ for all $y \in Y$.

Production 3
Aggregate Supply (5)

Proof:

- $y^*(p) = \sum_j^J y_j(p)$: Here we have to show that $\sum_j y_j(p) \subset y^*(p)$ and $y^*(p) \subset \sum_j y_j(p)$. Consider $y_j \in y_j(p)$, then $p \cdot (\sum_j y_j) = \sum_j \pi_j(p) = \sum_j py_j = \sum_j \pi_j(p) = \pi^*(p)$ (the last step by the first part of [5.E.1]).

- From this argument is follows that $\sum y_j(p) \subset y^*(p)$.

- To get the second direction we start with $y \in y^*(p)$. Then $y = \sum_j y_j$ with $y_j \in Y_j$. Since $p \cdot y = p \cdot (\sum_j \cdot y_j) = \pi^*(p)$ and $\pi^*(p) = \sum_j^J \pi_j(p)$, it must be that $p \cdot y_j = \pi_j(p)$ (because $y_j' \in Y_j$ implies $p \cdot y_j' \leq \pi_j(p)$). Hence, we get $y^*(p) \subset \sum_j y_j(p)$.

Production 3
Aggregate Supply (6)

- The same aggregation procedure can also be applied to derive aggregate cost.

Production 3
Efficiency (1)

- We want to check whether or what production plans are wasteful.

- **Definition**:[D 5.F.1] A production vector is **efficient**, if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

- There is no way to increase output with given inputs or to decrease input with given output (sometimes called technical efficiency).

- Discuss MWG, Figure 5.F.1, p. 150.

Production 3
Efficiency (2)

- **Proposition**[P 5.F.1] If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient.

- Version of the first fundamental theorem of welfare economics. See MWG, Chapter 16 C, p. 549.

- It also tells us that a profit maximizing firm does not choose interior points in the production set.

Proof:

- We show this by means of a contradiction: Suppose that there is a $y' \in Y$ such that $y' \neq y$ and $y' \geq y$. Because $p \gg 0$ we get $p \cdot y' > p \cdot y$, contradicting the assumption that y solves the PMP.

- For interior points suppose that y'' is the interior. By the same argument we see that this is neither efficient nor optimal.

Production 3
Efficiency (4)

- This result implies that a firm chooses y in the convex part of Y (with a differentiable transfer function $F(.)$ this follows immediately from the first order conditions; otherwise we choose 0 or ∞).

- The result also holds for nonconvex production sets - see Figure 5.F.2, page 150.

- Generally it is not true that every efficient production vector is profit maximizing for some $p \geq 0$, this only works with convex Y.

Production 3
Efficiency (6)

- **Proposition**[P 5.F.2] Suppose that Y is convex. Then every efficient production $y \in Y$ is profit maximizing for some $p \geq 0$ and $p \neq 0$.

Production 3
Efficiency (7)

Proof:

- Suppose that y is efficient. Construct the set $P_y = \{y' \in \mathbb{R}^L | y' \gg y\}$. This set has to be convex. Since y is efficient the intersection of Y and P_y has to be empty.

- This implies that we can use the separating hyperplane theorem [T M.G.2], page 948: There is some $p \neq 0$ such that $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$ and $y'' \in Y$. This implies $p \cdot y' \geq p \cdot y$ for every $y' \gg y$. Therefore, we also must have $p \geq 0$. If some $p_l < 0$ then we could have $p \cdot y' < p \cdot y$ for some $y' \gg y$ with $y'_l - y_l$ sufficiently large. This procedure works for each arbitrary y. $p \neq 0$.

Production 3
Efficiency (8)

Proof:

- It remains to show that y maximizes the profit: Take an arbitrary $y'' \in Y$, y was fixed, p has been derived by the separating hyperplane theorem. Then $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$. $y' \in P_y$ can be chosen arbitrary close to y, such that $p \cdot y \geq p \cdot y''$ still has to hold. I.e. y maximizes the profit given p.

- Regarding this proof see also MWG, Figure 5.F.3, p. 151.

Production 4
Objectives of the Firm (1)

- Until now we have assumed that the firm maximizes its profit.
- The price vector p was assumed to be fixed.
- We shall see that although preference maximization makes sense when we consider consumers, this need not hold with profit maximization with firms.
- Only if p is fixed we can rationalize profit maximization.

Production 4
Objectives of the Firm (2)

- The objectives of a firm should be a result of the objectives of the owners controlling the firm. That is to say, firm owners are also consumers who look at their preferences. So profit maximization need not be clear even if the firm is owned by one individual.

- MWG argue ("optimistically") that the problem of profit maximization is resolved, when the prices are fixed. This arises with firms with no market power.

Production 4
Objectives of the Firm (3)

- Consider a production possibility set Y owned by consumers $i = 1, \ldots, I$. The consumers own the shares θ_i, with $\sum_{i=1}^{I} \theta_i = 1$. $y \in Y$ is a production decision. w_i is non-profit wealth.

- Consumer i maximizes utility $\max_{x_i \geq 0} \; u(x_i)$, s.t. $p \cdot x_i \leq w_i + \theta_i p \cdot y$.

- With fixed prices the budget set described by $p \cdot x_i \leq w_i + \theta_i p \cdot y$ increases if $p \cdot y$ increases.

- With higher $p \cdot y$ each consumer i is better off. Here maximizing profits $p \cdot y$ makes sense.

Production 4
Objectives of the Firm (4)

- Problems arise (e.g.) if
  - Prices depend on the action taken by the firm.
  - Profits are uncertain (risk attitude plays a role).
  - Firms are not controlled by its owners.

Production 4
Objectives of the Firm (5)

- Suppose that the output of a firm is uncertain. It is important to know whether output is sold before or after uncertainty is resolved.

- If the goods are sold on a spot market (i.e. after uncertainty is resolved), then also the owner's attitude towards risk will play a role in the output decision. Maybe less risky production plans are preferred (although the expected profit is lower).

- If there is a futures market the firm can sell the good before uncertainty is resolved and the consumers bear the risk. Profit maximization can still be optimal.

Production 4
Objectives of the Firm (6)

- Consider a two good economy with goods x_1 and x_2; $L = 2$, non-profit wealth $w_i = 0$. Suppose that the firm can influence the price of good 1, $p_1 = p_1(x_1)$. We normalize the price of good 2, such that $p_2 = 1$. z units of x_2 are used to produce x_1 with production function $x_1 = f(z)$. The cost is given by $p_2 z = z$.

- We consider the maximization problem $\max_{x_i \geq 0} \; u(x_{i1}, x_{i2})$, s.t. $p \cdot x_i \leq w_i + \theta_i p \cdot y$.

  Given the above notation $p = (p_1(x_1), 1)$, $y = (f(z), -z)$. $w_i = 0$ by assumption. The profit is $p \cdot y = p_1(x_1)x_1 - p_2 z = p_1(f(z))f(z) - z$.

Production 4
Objectives of the Firm (7)

- Assume that the preferences of the owners are such that they are only interested in good 2.

- The aggregate amount of x_2 the consumers can buy is $\frac{1}{p_2}(p_1(f(z))f(z) - p_2 z) = p_1(f(z))f(z) - z$ (since $p_2 = 1$).

- Hence, $\max_{x_i \geq 0} \; u(x_{i2})$, s.t. $p \cdot x_i \leq w_i + \theta_i p \cdot y$ results in $\max p_1(f(z))f(z) - z$.

Production 4
Objectives of the Firm (8)

- Assume that the preferences of the owners are such that they only look at good 1.
- The aggregate amount of x_1 the consumers can buy is $\frac{1}{p_1(f(z))}(p_1(f(z))f(z) - z) = f(z) - z/p_1(f(z))$.
- Then $\max_{x_i \geq 0}\ u(x_{i1})$, s.t. $p \cdot x_i \leq w_i + \theta_i p \cdot y$ results in $\max f(z) - z/p_1(f(z))$.
- We have two different optimization problems - solutions are different.

Production 4
Objectives of the Firm (9)

- Example: Let $p_1(f(z)) = \sqrt{z}$, then the first order conditions are different, i.e. $\frac{1}{2\sqrt{z}} f(z) + \sqrt{z} f'(z) - 1 = 0$ and $f'(z) - \frac{1}{2\sqrt{z}} = 0$.

- We have considered two extreme cases: all owners prefer (i) good 2, (ii) good 1. There is no unique output decision based on $max\ p \cdot y$.

- If the preferences become heterogeneous things do not become better.

Expected Utility
Uncertainty (1)

- Preferences and Lotteries.

- Von Neumann-Morgenstern Expected Utility Theorem.

- Attitudes towards risk.

- State Dependent Utility, Subjective Utility

  MWG, Chapter 6.

Expected Utility
Lotteries (1)

- A risky alternative results in one of a number of different **states of the world**, ω_i.
- The states are associated with **consequences** or **outcomes**, z_n. Each z_n involves no uncertainty.
- Outcomes can be money prices, wealth levels, consumption bundles, etc.
- Assume that the set of outcomes is finite. Then $Z = \{z_1, \ldots, z_N\}$.
- E.g. flip a coin: States $\{H, T\}$ and outcomes $Z = \{-1, 1\}$, with head H or tail T.

Expected Utility
Lotteries (2)

- **Definition - Simple Gamble/Simple Lottery**: [D 6.B.1] With the consequences $\{z_1, \ldots, z_N\} \subseteq Z$ and N finite. A simple gamble assigns a probability p_n to each outcome z_n. $p_n \geq 0$ and $\sum_{n=1}^{N} p_n = 1$.
- Notation: $L = (p_1, \ldots, p_N)$. $p_i \geq 0$ is the probability of consequence z_i, for $i = 1, \ldots, N$.
- Let us fix the set of outcomes Z: Different lotteries correspond to a different set of probabilities.
- **Definition - Set of Simple Gambles**: The set of simple gambles on Z is given by

$$\mathbf{L}_S = \{(p_1, \ldots, p_N) | p_n \geq 0\,,\ \sum_{n=1}^{N} p_n = 1\} = \{L | p_n \geq 0\,,\ \sum_{N=1}^{N} p_n = 1\}$$

Expected Utility
Lotteries (3)

- **Definition - Degenerated Lottery**:
  $\tilde{L}^n = (0, \ldots, 1, \ldots, 0) = e_n$.

- '$Z \subseteq \mathbf{L}_S$', since one can identify z_n with $\tilde{L}^n$.

Expected Utility
Lotteries (4)

- With N consequences, every simple lottery can be represented by a point in a $N-1$ dimensional simplex

$$\Delta^{(N-1)} = \{p \in \mathbb{R}^N_+ | \sum p_n = 1\} \ .$$

- At each corner n we have the degenerated case that $p_n = 1$.

- With interior points $p_n > 0$ for all i.

- See Ritzberger, p. 36,37, Figures 2.1 and 2.2 or MWG, Figure 6.B.1, page 169.

- Equivalent to Machina's triangle; with $N = 3$; $\{(p_1, p_3) \in [0,1]^2 | 0 \leq 1 - p_1 - p_3 \leq 1\}$.

Expected Utility
Lotteries (5)

- The consequences of a lottery need not be a $z \in Z$ but can also be a further lottery.

- **Definition - Compound Lottery**:[D 6.B.2] Given K simple lotteries L_k and probabilities $\alpha_k \geq 0$ and $\sum \alpha_k = 1$, the compound lottery $L_C = (L_1, \ldots, L_k, \ldots, L_K; \alpha_1, \ldots, \alpha_k, \ldots, \alpha_K)$. It is the risky alternative that yields the simple lottery L_k with probability α_k.

- The support of the compound lottery (the set of consequences with positive probability) is the union of the supports generating this lotteries.

Expected Utility
Lotteries (6)

- **Definition - Reduced Lottery**: For any compound lottery L_C we can construct a **reduced lottery/simple gamble** $L' \in \mathbf{L}_S$. With the probabilities p^k for each L^k we get $p' = \sum \alpha_k p^k$, such that probabilities for each $z_n \in Z$ are $p'_n = \sum_{k=1}^{K} \alpha_k p^k_n$.

- Examples: Example 2.5, Ritzberger p. 37

- A reduced lottery can be expressed by a convex combination of elements of compound lotteries (see Ritzberger, Figure 2.3, page 38). I.e. $\alpha p^{l1} + (1-\alpha)p^{l2} = p^{lreduced}$.

Expected Utility
von Neumann-Morgenstern Utility (1)

- Here we assume that any decision problem can be expressed by means of a lottery (simple gamble).

- Only the outcomes matter.

- Consumers are able to perform calculations like in probability theory, gambles with the same probability distribution on Z are equivalent.

Expected Utility
von Neumann-Morgenstern Utility (2)

- **Axiom vNM1 - Completeness**: For two gambles L_1 and L_2 in $\mathbf{L}_S$ either $L_1 \succeq L_2$, $L_2 \succeq L_1$ or both.

- Here we assume that a consumer is able to rank lotteries (risky alternatives). I.e. Axiom vNM1 is stronger than Axiom 1 under certainty.

- **Axiom vNM2 - Transitivity**: For three gambles L_1, L_2 and L_3: $L_1 \succeq L_2$ and $L_2 \succeq L_3$ implies $L_1 \succeq L_3$.

Expected Utility
von Neumann-Morgenstern Utility (3)

- **Axiom vNM3 - Continuity**: [D 6.B.3] The preference relation on the space of simple lotteries is continuous if for any L_1, L_2, L_3 the sets $\{\alpha \in [0,1] | \alpha L_1 + (1-\alpha)L_2 \succeq L_3\} \subset [0,1]$ and $\{\alpha \in [0,1] | L_3 \succeq \alpha L_1 + (1-\alpha)L_2\} \subset [0,1]$ are closed.

- Later we show: for any gambles $L \in \mathbf{L}_S$, there exists some probability α such that $L \sim \alpha \bar{L} + (1-\alpha)\underline{L}$, where $\bar{L}$ is the most preferred and $\underline{L}$ the least preferred lottery.

- This assumption rules out a lexicographical ordering of preferences (safety first preferences).

Expected Utility
von Neumann-Morgenstern Utility (4)

- Consider the outcomes $Z = \{1000, 10, death\}$, where $1000 \succ 10 \succ death$. L_1 gives 10 with certainty.

- If vNM3 holds then L_1 can be expressed by means of a linear combination of 1000 and $death$. If there is no $\alpha \in [0, 1]$ fulfilling this requirement vNM3 does not hold.

Expected Utility
von Neumann-Morgenstern Utility (5)

- **Monotonicity**: For all probabilities $\alpha, \beta \in [0, 1]$,

$$\alpha \bar{L} + (1 - \alpha)\underline{L} \succeq \beta \bar{L} + (1 - \beta)\underline{L}$$

  if and only if $\alpha \geq \beta$.

- Monotonicity is implied by the axioms vNM1-vNM4.

Expected Utility
von Neumann-Morgenstern Utility (6)

- **Axiom vNM4 - Independence, Substitution**: For all probabilities L_1, L_2 and L_3 in $\mathbf{L}_S$ and $\alpha \in (0,1)$:

$$L_1 \succeq L_2 \Leftrightarrow \alpha L_1 + (1-\alpha)L_3 \succeq \alpha L_2 + (1-\alpha)L_3 \ .$$

- This axiom implies that the preference orderings of the mixtures are independent of the third lottery.

- This axiom has no parallel in consumer theory under certainty.

Expected Utility
von Neumann-Morgenstern Utility (7)

- Example: consider a bundle x^1 consisting of 1 cake and 1 bottle of wine $x^1 = (1, 1)$, $x^2 = (3, 0)$; $x^3 = (3, 3)$. Assume that $x^1 \succ x^2$.

  Axiom vNM4 requires that $\alpha x^1 + (1 - \alpha)x^3 \succ \alpha x^2 + (1 - \alpha)x^3$; here $\alpha > 0$.

Expected Utility
von Neumann-Morgenstern Utility (8)

- **Lemma - vNM1-4 imply monotonicity**: Moreover, if $L_1 \succeq L_2$ then $\alpha L_1 + (1-\alpha)L_2 \succeq \beta L_1 + (1-\beta)L_2$ for arbitrary $\alpha, \beta \in [0,1]$ where $\alpha \geq \beta$. For every $L_1 \succeq L \succeq L_2$, there is unique $\gamma \in [0,1]$ such that $\gamma L_1 + (1-\gamma)L_2 \sim L$.

- See steps 2-3 of the vNM existence proof.

Expected Utility
von Neumann-Morgenstern Utility (9)

- **Definition - von Neumann Morgenstern Expected Utility Function**: [D 6.B.5] A real valued function $U : \mathbf{L}_S \to \mathbb{R}$ has expected utility form if there is an assignment of numbers $(u_1, \ldots, u_N)$ (with $u_n = u(z_n)$) such that for every lottery $L \in \mathbf{L}_S$ we have $U(L) = \sum_{z_n \in Z} p(z_n)u(z_n)$. A function of this structure is said to satisfy the **expected utility property** - it is called **von Neumann-Morgenstern** (expected) utility function.

- Note that this function is linear in the probabilities p_n.

- $u(z_n)$ is called **Bernoulli utility function**.

Expected Utility
von Neumann-Morgenstern Utility (10)

- **Proposition - Linearity of the von Neumann Morgenstern Expect Utility Function**: [P 6.B.1] A utility function has expected utility form if and only if it is linear. That is to say:

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

Expected Utility
von Neumann-Morgenstern Utility (11)

Proof:

- Suppose that $U(\sum_{k=1}^{K} \alpha_k L_k) = \sum_{k=1}^{K} \alpha_k U(L_k)$ holds. We have to show that U has expected utility form, i.e. if $U(\sum_k \alpha_k L_k) = \sum_k \alpha_k U(L_k)$ then $U(L) = \sum p_n u(z_n)$.

- If U is linear then we can express any lottery L by means of a compound lottery with probabilities $\alpha_n = p_n$ and degenerated lotteries $\tilde{L}^n$. I.e. $L = \sum p_n \tilde{L}^n$. By linearity we get $U(L) = U(\sum p_n \tilde{L}^n) = \sum p_n U(\tilde{L}^n)$.

- Define $u(z_n) = U(\tilde{L}^n)$. Then $U(L) = U(\sum p_n \tilde{L}^n) = \sum p_n U(\tilde{L}^n) = \sum p_n u(z_n)$. Therefore $U(.)$ has expected utility form.

Expected Utility
von Neumann-Morgenstern Utility (12)

Proof:

- Suppose that $U(L) = \sum_{n=1}^{N} p_n u(z_n)$ holds. We have to show that utility is linear, i.e. if $U(L) = \sum p_n u(z_n)$ then $U(\sum_k \alpha_k L_k) = \sum_k \alpha_k U(L_k)$

- Consider a compound lottery $(L_1, \ldots, L_K, \alpha_1, \ldots, \alpha_K)$. Its reduced lottery is $L' = \sum_k \alpha_k L_k$.

- Then $U(\sum_k \alpha_k L_k) = \sum_n \left(\sum_k \alpha_k p_n^k\right) u(z_n) = \sum_k \alpha_k \left(\sum_n p_n^k u(z_n)\right) = \sum_k \alpha_k U(L_k)$.

Expected Utility
von Neumann-Morgenstern Utility (13)

- **Proposition - Existence of a von Neumann Morgenstern Expect Utility Function**: [P 6.B.3] If the Axioms vNM 1-4 are satisfied for a preference ordering $\succeq$ on $\mathbf{L}_S$. Then $\succeq$ admits an expected utility representation. I.e. there exists a real valued function $u(.)$ on Z which assigns a real number to each outcome z_n, $n = 1, \ldots, N$, such that for any pair of lotteries $L_1 = (p_1, \ldots, p_N)$ and $L_2 = (p'_1, \ldots, p'_N)$ we get

$$L_1 \succeq L_2 \text{ if and only if}$$

$$U(L_1) := \sum_{n=1}^{N} p_n u(z_n) \geq U(L_2) := \sum_{n=1}^{N} p'_n u(z_n) \ .$$

Expected Utility
von Neumann-Morgenstern Utility (14)

Proof:

- Suppose that there is a best and a worst lottery. With a finite set of outcomes this can be easily shown by means of the independence axiom. In addition $\bar{L} \succ \underline{L}$.
- By the definition of $\bar{L}$ and $\underline{L}$ we get: $\bar{L} \succeq L_c \succeq \underline{L}$, $\bar{L} \succeq L_1 \succeq \underline{L}$ and $\bar{L} \succeq L_2 \succeq \underline{L}$.
- We have to show that (i) $u(z_n)$ exists and (ii) that for any compound lottery $L_c = \beta L_1 + (1-\beta)L_2$ we have $U(\beta L_1 + (1-\beta)L_2) = \beta U(L_1) + (1-\beta)U(L_2)$ (expected utility structure).

Proof:

- Step 1: By the independence Axiom vNM4 we get if $L_1 \succ L_2$ and $\alpha \in (0,1)$ then $L_1 \succ \alpha L_1 + (1-\alpha)L_2 \succ L_2$.

- This follows directly from the independence axiom.

$$L_1 \sim \alpha L_1 + (1-\alpha)L_1 \succ \alpha L_1 + (1-\alpha)L_2 \succ \alpha L_2 + (1-\alpha)L_2 = L_2$$

Proof:

- Step 2: Want to show that $\beta > \alpha$, if and only if $\beta \bar{L} + (1-\beta)\underline{L} \succ \alpha \bar{L} + (1-\alpha)\underline{L}$ (monotonicity):

- Define $\gamma = (\beta - \alpha)/(1-\alpha)$; the assumptions imply $\gamma \in [0,1]$.

Expected Utility
von Neumann-Morgenstern Utility (17)

Proof:

- Then

$$\begin{aligned}
\beta\bar{L} + (1-\beta)\underline{L} &= \gamma\bar{L} + (1-\gamma)(\alpha\bar{L} + (1-\alpha)\underline{L}) \\
&\succ \gamma(\alpha\bar{L} + (1-\alpha)\underline{L}) + (1-\gamma)(\alpha\bar{L} + (1-\alpha)\underline{L}) \\
&\sim \alpha\bar{L} + (1-\alpha)\underline{L}
\end{aligned}$$

Proof:

- Step 2: For the converse we have to show that $\beta\bar{L} + (1-\beta)\underline{L} \succ \alpha\bar{L} + (1-\alpha)\underline{L}$ results in $\beta > \alpha$. We show this by means of the contrapositive: If $\beta \not> \alpha$ then $\beta\bar{L} + (1-\beta)\underline{L} \not\succ \alpha\bar{L} + (1-\alpha)\underline{L}$.

- Thus assume $\beta \leq \alpha$, then $\alpha\bar{L} + (1-\alpha)\underline{L} \succeq \beta\bar{L} + (1-\beta)\underline{L}$ follows in the same way as above. If $\alpha = \beta$ indifference follows.

Expected Utility
von Neumann-Morgenstern Utility (19)

Proof:

- Step 3: There is a unique α_L such that $L \sim \alpha_L \bar{L} + (1 - \alpha_L)\underline{L}$.

- Existence follows from $\bar{L} \succ \underline{L}$ and the continuity axiom:

- Ad existence: define the sets $\{\alpha \in [0,1] | \alpha\bar{L} + (1-\alpha)\underline{L} \succeq L\}$ and $\{\alpha \in [0,1] | L \succeq \alpha\bar{L} + (1-\alpha)\underline{L}\}$. Both sets are closed. Any α belongs to at least one of these two sets. Both sets are nonempty. Their complements are open and disjoint. The set $[0,1]$ is connected $\Rightarrow$ there is at least one α belonging to both sets.

- Uniqueness follows directly from step 2.

- **Definition**: Let $\Omega \neq \emptyset$ be an arbitrary set. A class $\tau \subset 2^{\Omega}$ of subsets of Ω is called a **topology** on Ω if it has the three properties:
  - $\emptyset, \Omega \in \tau$
  - $A \cap B \in \tau$ for any two sets $A, B \in \tau$.
  - $\bigcup_{A \in \mathcal{F}} A \in \tau$ for any $\mathcal{F} \subset \tau$.

- The pair (Ω, τ) is called a **topological space**. The sets $A \in \tau$ are called **open sets**, and the sets $A \subset \Omega$ with $A^c \in \tau$ are called **closed sets**; A^c stands for complementary set.

Expected Utility
Excursion: Connected Sets

- Consider the family $\tau_{\mathbb{R}}$ of subsets of $\mathbb{R}$: $O \in \tau_{\mathbb{R}}$ if and only if for each $x \in O$, there is an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset O$. That is, elements of O are arbitrary unions of open intervals.

- Fact from Math: $\tau_{\mathbb{R}}$ forms a topology on $\mathbb{R}$. It is called **Euclidean topology**.

- We consider the closed interval $[0, 1]$ with the following topology: $A \subset [0, 1]$ is open if and only if there is an $O \in \tau_{\mathbb{R}}$ such that $A = O \cap [0, 1]$. This topology is **induced** by $\tau_{\mathbb{R}}$.

Expected Utility
Excursion: Connected Sets

- **Definition**: Let (X, τ) be a topological space. The space is said to be **connected**, if for any two non-empty closed subsets $A, B \subset X$, $A \cup B = X$ implies $A \cap B \neq \emptyset$.

- Fact from Math: $[0, 1]$ with by $\tau_{\mathbb{R}}$ the induced topology is connected.

Proof:

- Step 4: The function $U(L) = \alpha_L$ represents the preference relations $\succeq$.

- Consider $L_1, L_2 \in \mathbf{L}_S$: If $L_1 \succeq L_2$ then $\alpha_1 \geq \alpha_2$. If $\alpha_1 \geq \alpha_2$ then $L_1 \succeq L_2$ by steps 2-3.

- It remains to show that this utility function has expected utility form.

Expected Utility
von Neumann-Morgenstern Utility (21)

Proof:

- Step 5: $U(L)$ is has expected utility form.

- We show that the linear structure also holds for the compound lottery $L_c = \beta L_1 + (1-\beta)L_2$.

- By using the independence we get:

$$\begin{aligned}
\beta L_1 + (1-\beta)L_2 &\sim \beta(\alpha_1 \bar{L} + (1-\alpha_1)\underline{L}) + (1-\beta)L_2 \\
&\sim \beta(\alpha_1 \bar{L} + (1-\alpha_1)\underline{L}) + (1-\beta)(\alpha_2 \bar{L} + (1-\alpha_2)\underline{L}) \\
&\sim (\beta\alpha_1 + (1-\beta)\alpha_2)\bar{L} + (\beta(1-\alpha_1) + (1-\beta)(1-\alpha_2))\underline{L}
\end{aligned}$$

- By the rule developed in step 4, this shows that $U(L_c) = U(\beta L_1 + (1-\beta)L_2) = \beta U(L_1) + (1-\beta)U(L_2)$.

Expected Utility
von Neumann-Morgenstern Utility (22)

- **Proposition - von Neumann Morgenstern Expect Utility Function are unique up to Positive Affine Transformations**: [P 6.B.2] If $U(.)$ represents the preference ordering $\succeq$, then V represents the same preference ordering if and only if $V = \alpha + \beta U$, where $\beta > 0$.

Proof:

- Note that if $V(L) = \alpha + \beta U(L)$, $V(L)$ fulfills the expected utility property (see also MWG p. 174).

- We have to show that if U and V represent preferences, then V has to be an affine linear transformation of U.

- If U is constant on $\mathbf{L}_S$, then V has to be constant. Both functions can only differ by a constant α.

Expected Utility
von Neumann-Morgenstern Utility (24)

Proof:

- Alternatively, for any $L \in \mathbf{L}_S$ and $\bar{L} \succ \underline{L}$, we get

$$f_1 := \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})}$$

and

$$f_2 := \frac{V(L) - V(\underline{L})}{V(\bar{L}) - V(\underline{L})} \ .$$

- f_1 and f_2 are linear transformations of U and V that satisfy the expected utility property.

- $f_i(\underline{L}) = 0$ and $f_i(\bar{L}) = 1$, for $i = 1, 2$.

Expected Utility
von Neumann-Morgenstern Utility (25)

Proof:

- $L \sim \underline{L}$ then $f_1 = f_2 = 0$; if $L \sim \bar{L}$ then $f_1 = f_2 = 1$.
- By expected utility $U(L) = \gamma U(\bar{L}) + (1-\gamma)U(\underline{L})$ and $V(L) = \gamma V(\bar{L}) + (1-\gamma)V(\underline{L})$.
- If $\bar{L} \succ L \succ \underline{L}$ then there has to exist a unique γ, such that $\underline{L} \prec L \sim \gamma\bar{L} + (1-\gamma)\underline{L} \prec \bar{L}$. Therefore

$$\gamma = \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})} = \frac{V(L) - V(\underline{L})}{V(\bar{L}) - V(\underline{L})}$$

Proof:

- Then $V(L) = \alpha + \beta U(L)$ where

$$\alpha = V(\underline{L}) - U(\underline{L}) \frac{V(\bar{L}) - V(\underline{L})}{U(\bar{L}) - U(\underline{L})}$$

and

$$\beta = \frac{V(\bar{L}) - V(\underline{L})}{U(\bar{L}) - U(\underline{L})} \ .$$

Expected Utility
von Neumann-Morgenstern Utility (27)

- The concept of expected utility can be extended to a set of distributions $F(x)$ where the expectation of $u(x)$ exists, i.e. $\int_A u(x)dF(x) < \infty$, $z \in \mathbb{R}$ and $A \subset \mathbb{R}$.

- For technical details see e.g. Robert (1994), The Bayesian Choice and DeGroot, Optimal Statistical Decisions.

- Note that expected utility is a probability weighted combination of Bernoulli utility functions. I.e. the properties of the random variable z, described by the lottery $l(z)$, are separated from the attitudes towards risk.

Expected Utility
VNM Indifference Curves (1)

- Indifferences curves are straight lines; see Ritzberger, Figure 2.4, page 41.

- Consider a VNM utility function and two indifferent lotteries L_1 and L_2. It has to hold that $U(L_1) = U(L_2)$.

- By the expected utility theorem $U(\alpha L_1 + (1-\alpha)L_2) = \alpha U(L_1) + (1-\alpha)U(L_2)$.

- If $U(L_1) = U(L_2)$ then $U(\alpha L_1 + (1-\alpha)L_2) = U(L_1) = U(L_2)$ has to hold and the indifferent lotteries is linear combinations of L_1 and L_2.

Expected Utility
VNM Indifference Curves (2)

- Indifference curves are parallel; see Ritzberger, Figure 2.5, 2.6, page 42.

- Consider $L_1 \sim L_2$ and a further lottery $L_3 \succ L_1$ (w.l.g.).

- From $\beta L_1 + (1 - \beta) L_3$ and $\beta L_2 + (1 - \beta) L_3$ we have received two compound lotteries.

- By construction these lotteries are on a line parallel to the line connecting L_1 and L_2.

Expected Utility
VNM Indifference Curves (3)

- The independence axiom vNM4 implies that $\beta L_1 + (1-\beta)L_3 \sim \beta L_2 + (1-\beta)L_3$ for $\beta \in [0,1]$.

- Therefore the line connecting the points $\beta L_1 + (1-\beta)L_3$ and $\beta L_2 + (1-\beta)L_3$ is an indifference curve.

- The new indifference curve is a parallel shift of the old curve; by the linear structure of the expected utility function no other indifference curves are possible.

Expected Utility
Allais Paradoxon (1)

Lottery	0	1-10	11-99
p_z	1/100	10/100	89/100
L_a	500,000	500,000	500,000
L_b	0	2,500,000	500,000
M_a	500,000	500,000	0
M_b	0	2,500,000	0

Expected Utility
Allais Paradoxon (2)

- Most people prefer L_a to L_b and M_b to M_a.

- This is a contradiction to the independence axiom G5.

- Allais paradoxon in the Machina triangle, Gollier, Figure 1.2, page 8.

Expected Utility
Allais Paradoxon (3)

- Expected utility theory avoids problems of **time inconsistency**.
- Agents violating the independence axiom are subject to Dutch book outcomes (violate no money pump assumption).

Expected Utility
Allais Paradoxon (4)

- Three lotteries: $L_a \succ L_b$ and $L_a \succ L_c$.

- But $L_d = 0.5L_b + 0.5L_c \succ L_a$.

- Gambler is willing to pay some fee to replace L_a by L_d.

Expected Utility
Allais Paradoxon (5)

- After nature moves: L_b or L_c with L_d.
- Now the agents is once again willing to pay a positive amount for receiving L_a
- Gambler starting with L_a and holding at the end L_a has paid two fees!
- Dynamically inconsistent/Time inconsistent.
- Discuss Figure 1.3, Gollier, page 12.

Expected Utility
Risk Attitude (1)

- For the proof of the vNM-utility function we did not place any assumptions on the Bernoulli utility function $u(z)$.

- For applications often a Bernoulli utility function has to be specified.

- In the following we consider $z \in \mathbb{R}$ and $u'(z) > 0$; abbreviate **lotteries with money amounts** $l \in \mathbf{L}_S$.

- There are interesting interdependences between the Bernoulli utility function and an agent's attitude towards risk.

Expected Utility
Risk Attitude (2)

- Consider a nondegenerated lottery $l \in \mathbf{L}_S$ and a degenerated lottery $\tilde{l}$. Assume that $E_l(z) = \tilde{z}_l$ holds. I.e. the degenerated lottery $\tilde{l}$ pays the expectation $\tilde{z}_l$ of l for sure.
- **Definition - Risk Aversion**: A consumer is risk averse if for any lottery l, $\tilde{z}_l$ is at least as good as l. A consumer is strictly risk averse if for any lottery l, $\tilde{z}$ is strictly preferred to l, whenever l is non-degenerate.
- **Definition - Risk Neutrality**: A consumer is risk neutral if $\tilde{z}_l \sim l$ for all l.
- **Definition - Risk Loving**: A consumer is risk loving if for any lottery l, $\tilde{z}_l$ is at most as good as l. A consumer is strictly risk loving if for any lottery l, l is strictly preferred to $\tilde{z}_l$, whenever l is non-degenerate.

Expected Utility
Risk Attitude (3)

- By the definition of risk aversion, we see that the utility function $u(.)$ has to satisfy for any non-degenerate distribution F,
  $u(E(z)) = u(\int z dF(z)) \geq E(u(z)) = \int u(z) dF(z)$.
- If $u(z)$ is a concave function and z is distributed according to $F(z)$ (such that the expectations exist), then

$$\int u(z) dF(z) \leq u(\int z dF(z))$$

  Jensen's inequality. In addition, if $\int u(z) dF(z) \leq u(\int z dF(z))$ holds for any distribution F, then $u(z)$ is concave.
- For sums this implies:

$$\sum p_z u(z) \leq u(\sum p_z z) \ .$$

  For strictly concave function, $<$ has to hold whenever F is nondegenerate, for convex functions we get $\geq$; for strictly convex functions $>$ whenever F is non-degenerate.

Expected Utility
Risk Attitude (4)

- For a lottery l where $E(u(z)) < \infty$ and $E(z) < \infty$ we can calculate the amount C where a consumer is indifferent between receiving C for sure and the lottery l. I.e. $l \sim C$ and $E(u(z)) = u(C)$ hold.

- In addition we are able to calculate the maximum amount π an agent is willing to pay for receiving the fixed amount $E(z)$ for sure instead of the lottery l. I.e. $l \sim E(z) - \pi$ or $E(u(z)) = u(E(z) - \pi)$.

Expected Utility
Risk Attitude (5)

- **Definition - Certainty Equivalent** [D 6.C.2]: The fixed amount C where a consumer is indifferent between C an a gamble l is called certainty equivalent.

- **Definition - Risk Premium**: The maximum amount π a consumer is willing to pay to exchange the gamble l for a sure state with outcome $E(z)$ is called risk premium.

- Note that C and π depend on the properties of the random variable (described by l) and the attitude towards risk (described by u).

Expected Utility
Risk Attitude (6)

- **Remark**: the same analysis can also be performed with risk neutral and risk loving agents.

- **Remark**: MWG defines a probability premium, which is abbreviated by π in the textbook. Given a degenerated lottery and some $\varepsilon > 0$. The **probability-premium** π^R is defined as $u(\tilde{l}_z) = (\frac{1}{2} + \pi^R)u(z + \varepsilon) + (\frac{1}{2} - \pi^R)u(z - \varepsilon)$. I.e. mean-preserving spreads are considered here.

Expected Utility
Risk Attitude (7)

- **Proposition - Risk Aversion and Bernoulli Utility**: Consider an expected utility maximizer with Bernoulli utility function $u(.)$. The following statements are equivalent:
  - The agent is risk averse.
  - $u(.)$ is a (strictly) concave function.
  - $C \leq E(z)$. ($<$ with strict version)
  - $\pi \geq 0$. ($>$ with strict version)

Expected Utility
Risk Attitude (8)

Proof: (sketch)

- By the definition of risk aversion: for a lottery l where $E(z) = z_{\tilde{l}}$, a risk avers agent $\tilde{l} \succeq l$.
- I.e. $E(u(z)) \leq u(z_{\tilde{l}}) = u(E(z))$ for a VNM utility maximizer.
- (ii) follows from Jensen's inequality.
- (iii) If $u(.)$ is (strictly) concave then $E(u(z)) = u(C) \leq u(E(z))$ can only be matched with $C \leq E(z)$.
- (iv) With a strictly concave $u(.)$, $E(u(z)) = u(E(z) - \pi) \leq u(E(z))$ can only be matched with $\pi \geq 0$.

Expected Utility
Arrow Pratt Coefficients (1)

- Using simply the second derivative $u''(z)$ of the Bernoulli utility function, causes problems with affine linear transformations.

- **Definition - Arrow-Pratt Coefficient of Absolute Risk Aversion**: [D 6.C.3] Given a twice differentiable Bernoulli utility function $u(.)$, the coefficient of absolute risk aversion is defined by $A(z) = -u''(z)/u'(z)$.

- **Definition - Arrow-Pratt Coefficient of Relative Risk Aversion**: [D 6.C.5] Given a twice differentiable Bernoulli utility function $u(.)$, the coefficient of relative risk aversion is defined by $R(z) = -zu''(z)/u'(z)$.

Expected Utility
Comparative Analysis (1)

- Consider two agents with Bernoulli utility functions u_1 and u_2. We want to compare their attitudes towards risk.

  Definition - More Risk Averse: Agent 1 is more risk averse than agent 2: Whenever agent 1 finds a lottery F at least good as a riskless outcome $\tilde{x}$, then agent 2 finds F at least good as $\tilde{x}$. I.e. if $F \succeq_1 \tilde{L}_{\tilde{x}}$ then $F \succeq_2 \tilde{L}_{\tilde{x}}$.

  In terms of a VNM-ultility maximizer: If
  $\mathbb{E}_F(u_1(z)) = \int u_1(z)dF(z) \geq u_1(\tilde{x})$ then
  $\mathbb{E}_F(u_2(z)) = \int u_2(z)dF(z) \geq u_2(\tilde{x})$ for any $F(.)$ and $\tilde{x}$.

Expected Utility
Comparative Analysis (2)

- Define a function $\phi(x) = u_1(u_2^{-1}(x))$. Since $u_2(.)$ is an increasing function this expression is well defined. We, in addition, assume that the first and the second derivatives exist.

- By construction with $x = u_2(z)$ we get:
  $\phi(x) = u_1(u_2^{-1}(x)) = u_1(u_2^{-1}(u_2(z))) = u_1(z)$. I.e. $\phi(x)$ transforms u_2 into u_1, such that $u_1(z) = \phi(u_2(z))$.

- In the following we assume that u_i and ϕ are differentiable. In the following theorem we shall observe that $\phi' > 0$ for u_1' and $u_2' > 0$.

Expected Utility
Comparative Analysis (3)

- **Proposition - More Risk Averse Agents** [P 6.C.2]: Assume that the first and second derivatives of the Bernoulli utility functions u_1 and u_2 exist ($u' > 0$ and $u'' < 0$). Then the following statements are equivalent:
  - Agent 1 is (strictly) more risk averse than agent 2.
  - u_1 is a (strictly) concave transformation of u_2 (that is, there exists a (strictly) concave ϕ such that $u_1(.) = \phi(u_2(.))$)
  - $A_1(z) \geq A_2(z)$ ($>$ for strict) for all z.
  - $C_1 \leq C_2$ and $\pi_1 \geq \pi_2$; ($<>$ for strict).

Expected Utility
Comparative Analysis (4)

Proof:

- Step 1: (i) follows from (ii): We have to show that if ϕ is concave, then if $\mathbb{E}_F(u_1(z)) = \int u_1(z)dF(z) \geq u_1(\tilde{x}) \Rightarrow$ $\mathbb{E}_F(u_2(z)) = \int u_2(z)dF(z) \geq u_2(\tilde{x})$ has to follow.
- Suppose that for some lottery F the inequality $\mathbb{E}_F(u_1(z)) = \int u_1(z)dF(z) \geq u_1(\tilde{x})$ holds. This implies $\mathbb{E}_F(u_1(z)) = \int u_1(z)dF(z) \geq u_1(\tilde{x}) = \phi(u_2(\tilde{x}))$.
- By means of Jensen's inequality we get for a concave $\phi(.)$; (with strict concave we get $<$) $\mathbb{E}(u_1(z)) = \mathbb{E}(\phi(u_2(z)) \leq \phi(\mathbb{E}(u_2(z)))$.
- Then $\phi(\mathbb{E}(u_2(z))) \geq \mathbb{E}(u_1(z))$ and $\mathbb{E}(u_1(z)) \geq u_1(\tilde{x}) = \phi(u_2(\tilde{x}))$ implies $\phi(\mathbb{E}(u_2(z))) \geq \phi(u_2(\tilde{x}))$.
- Since ϕ is increasing this implies $\mathbb{E}(u_2(z)) \geq u_2(\tilde{x})$.

Expected Utility
Comparative Analysis (5)

Proof:

- (ii) follows from (i): Suppose that $\mathbb{E}_F(u_1(z)) = \int u_1(z)dF(z) \geq u_1(\tilde{x}) \Rightarrow$ $\mathbb{E}_F(u_2(z)) = \int u_2(z)dF(z) \geq u_2(\tilde{x})$ for any $F(.)$ and $\tilde{x}$ holds and ϕ is not concave.

- Then $\mathbb{E}_F(u_1(z)) = u_1(C_{F1})$ has to hold as well with $\tilde{x} = C_{F1}$. This implies $\mathbb{E}_F(u_1(z)) = \mathbb{E}_F(\phi(u_2(z))) = \phi(u_2(C_{F1}))$ for lottery F.

- Since ϕ is not concave, there exits a lottery where $\phi(\mathbb{E}_F(u_2(z))) < \mathbb{E}_F(\phi(u_2(z))) = \phi(u_2(C_{F1}))$. This yields $\mathbb{E}_F(u_2(z)) < u_2(C_{F1})$. Contradiction!

Expected Utility
Comparative Analysis (6)

Proof:

- Step 2 (iii)$\sim$ (ii): By the definition of ϕ and our assumptions we get
$$u_1'(z) = \frac{d\phi((u_2(z)))}{dz} = \phi'(u_2(z))u_2'(z) \ .$$
(since $u_1', u_2' > 0 \Rightarrow \phi' > 0$) and
$$u_1''(z) = \phi'(u_2(z))u_2''(z) + \phi''(u_2(z))(u_2'(z))^2 \ .$$

Expected Utility
Comparative Analysis (7)

Proof:

- Divide both sides by $-u_1'(z) < 0$ and using $u_1'(z) = ...$ yields:

$$-\frac{u_1''(z)}{u_1'(z)} = A_1(z) = A_2(z) - \frac{\phi''(u_2(z))}{\phi'(u_2(z))} u_2'(z) \ .$$

- Since $A_1, A_2 > 0$ due to risk aversion, $\phi' > 0$ and $\phi'' \leq 0$ ($<$) due to its concave shape we get $A_1(z) \geq A_2(z)$ ($>$) for all z.

Expected Utility
Comparative Analysis (8)

Proof:

- Step 3 (iv)$\sim$ (ii): Jensen's inequality yields (with strictly concave ϕ)

  $$u_1(C_1) = \mathbb{E}(u_1(z)) = \mathbb{E}(\phi(u_2(z)) < \phi(\mathbb{E}(u_2(z))) = \phi(u_2(C_2)) = u_1(C_2)$$

- Since $u_1' > 0$ we get $C_1 < C_2$.

- $\pi_1 > \pi_2$ works in the same way.

- The above considerations also work in both directions, therefore (ii) and (iv) are equivalent.

Expected Utility
Comparative Analysis (9)

Proof:

- Step 4 (vi)$\sim$ (ii): Jensen's inequality yields (with strictly concave ϕ)

$$u_1(\mathbb{E}(z)-\pi_1) = \mathbb{E}(u_1(z)) = \mathbb{E}(\phi(u_2(z)) < \phi(\mathbb{E}(u_2(z))) = \phi(u_2(\mathbb{E}(z)-\pi_2)) = u_1(\mathbb{E}(z)-\pi_2)$$

- Since $u_1' > 0$ we get $\pi_1 > \pi_2$.

Expected Utility
Stochastic Dominance (1)

- In an application, do we have to specify the Bernoulli utility function?
- Are there some lotteries (distributions) such that $F(z)$ is (strictly) preferred to $G(z)$?
- E.g. if $X(\omega) > Y(\omega)$ $a.s.$?
- YES $\Rightarrow$ Concept of stochastic dominance.
- MWG, Figure 6.D.1., page 196.

Expected Utility
Stochastic Dominance (2)

- **Definition - First Order Stochastic Dominance**: [D 6.D.1] A distribution $F(z)$ first order dominates the distribution $G(z)$ if for every nondecreasing function $u : \mathbb{R} \to \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} u(z)dF(z) \geq \int_{-\infty}^{\infty} u(z)dG(z).$$

- **Definition - Second Order Stochastic Dominance**: [D 6.D.2] A distribution $F(z)$ second order dominates the distribution $G(z)$ if $\mathbb{E}_F(z) = \mathbb{E}_G(z)$ and for every nondecreasing concave function $u : \mathbb{R}_+ \to \mathbb{R}$ the inequality $\int_0^\infty u(z)dF(z) \geq \int_0^\infty u(z)dG(z)$ holds.

Expected Utility
Stochastic Dominance (3)

- **Proposition - First Order Stochastic Dominance**: [P 6.D.1] $F(z)$ first order dominates the distribution $G(z)$ if and only if $F(z) \leq G(z)$.

- **Proposition - Second Order Stochastic Dominance**: [D 6.D.2] $F(z)$ second order dominates the distribution $G(z)$ if and only if

$$\int_0^{\bar{z}} F(z)dz \leq \int_0^{\bar{z}} G(z)dz \quad \text{for all } \bar{z} \text{ in } \mathbb{R}_+ \ .$$

- **Remark**: I.e. if we can show stochastic dominance we do not have to specify any Bernoulli utility function!

Expected Utility
Stochastic Dominance (4)

Proof:

- Assume that u is differentiable and $u' \geq 0$
- Step 1: First order, if part: If $F(z) \leq G(z)$ integration by parts yields:

$$\int_{-\infty}^{\infty} u(z)dF(z) - \int_{-\infty}^{\infty} u(z)dG(z) = \int_{-\infty}^{\infty} u(z)F'(z)dz - \int_{-\infty}^{\infty} u(z)G'(z)dz$$

$$= \quad u(z)(F(z) - G(z))|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u'(z)(F(z) - G(z))dz$$

$$= \quad -\int_{-\infty}^{\infty} u'(z)(F(z) - G(z))dz \geq 0 \; .$$

- The above inequality holds since the terms inside the integral $(F(z) - G(z)) \leq 0$. In addition, $lim_{t\to\infty} F(t) = 1$ and $lim_{t\to-\infty} F(t) = 0$ and likewise for $G(.)$.

Expected Utility
Stochastic Dominance (5)

Proof:

- Step 2: First order, only if part: If FOSD then $F(z) \leq G(z)$ holds. Proof by means of contradiction.
- Assume there is a $\bar{z} \in \mathbb{R}$ such that $F(\bar{z}) > G(\bar{z})$. $\bar{z} > -\infty$ by construction. Set $u(z) = 0$ for $z \leq \bar{z}$ and $u(z) = 1$ for $z > \bar{z}$. Here we get

$$\begin{aligned}
& \int_{-\infty}^{\infty} u(z) dF(z) - \int_{-\infty}^{\infty} u(z) dG(z) \\
= & \int_{-\infty}^{\infty} u(z) F'(z) dz - \int_{-\infty}^{\infty} u(z) G'(z) dz \\
= & \int_{\bar{z}}^{\infty} F'(z) dz - \int_{\bar{z}}^{\infty} G'(z) dz \\
= & (1 - F(\bar{z})) - (1 - G(\bar{z})) = -F(\bar{z}) + G(\bar{z}) < 0
\end{aligned}$$

Expected Utility
Stochastic Dominance (6)

Proof:

- Second Order SD: Assume that u is twice continuously differentiable, such that $u''(z) \leq 0$, w.l.g. $u(0) = 0$.
- Remark: The equality of means implies:

$$\begin{aligned}
0 &= \int_0^\infty z dF(z) - \int_0^\infty z dG(z) \\
&= \int_0^\infty z F'(z) dz - \int_0^\infty z G'(z) dz \\
&= z(F(z) - G(z))|_0^\infty - \int_0^\infty (F(z) - G(z)) dz \\
&= -\int_0^\infty (F(z) - G(z)) dz \ .
\end{aligned}$$

Expected Utility
Stochastic Dominance (7)

Proof:

- Step 3: Second order, if part: Integration by parts yields:

$$
\begin{aligned}
& \int_0^\infty u(z)dF(z) - \int_0^\infty u(z)dG(z) \\
= \quad & u(z)(F(z)-G(z))|_0^\infty - \int_0^\infty u'(z)(F(z)-G(z))dz \\
= \quad & -\int_0^\infty u'(z)(F(z)-G(z))dz \\
= \quad & -u'(z)\int_0^z (F(x)-G(x))dx|_0^\infty - \int_0^\infty -u''(z)\left(\int_0^z (F(x)-G(x))dx\right)dz \\
= \quad & \int_0^\infty u''(z)\left(\int_0^z (F(x)-G(x))dx\right)dz \geq 0
\end{aligned}
$$

- Note that $u'' \leq 0$ by assumption.

Expected Utility
Stochastic Dominance (8)

Proof:

- Step 4: Second order, only if part: Consider a $\bar{z}$ such that $u(z) = \bar{z}$ for all $z > \bar{z}$ and $u(z) = z$ for all $z \leq \bar{z}$. This yields:

$$
\begin{aligned}
& \int_0^{\infty} u(z)dF(z) - \int_0^{\infty} u(z)dG(z) \\
= \quad & \int_0^{\bar{z}} zdF(z) - \int_0^{\bar{z}} zdG(z) + \bar{z}\left((1 - F(\bar{z})) - (1 - G(\bar{z}))\right) \\
= \quad & z\left(F(z) - G(z)\right)|_0^{\bar{z}} - \int_0^{\bar{z}} \left(F(z) - G(z)\right) dz - \bar{z}\left(F(\bar{z}) - G(\bar{z})\right) \\
= \quad & -\int_0^{\bar{z}} \left(F(z) - G(z)\right) dz \geq 0 \ .
\end{aligned}
$$

- **Definiton - Monotone Likelihood Ratio Property**: The distributions $F(z)$ and $G(z)$ fulfill, the monotone likelihood rate property if $G(z)/F(z)$ is non-increasing in z.

- For $x \to \infty$ $G(z)/F(z) = 1$ has to hold. This and the fact that $G(z)/F(z)$ is non-increasing implies $G(z)/F(z) \geq 1$ for all z.

- **Proposition - First Order Stochastic Dominance follows from MLP**: MLP results in $F(z) \leq G(z)$.

Expected Utility
Arrow-Pratt Approximation (1)

- By means of the Arrow-Pratt approximation we can express the risk premium π in terms of the Arrow-Pratt measures of risk.

- Assume that $z = w + kx$, where w is a fixed constant (e.g. wealth), x is a mean zero random variable and $k \geq 0$. By this assumption the variance of z is given by $\mathbb{V}(z) = k^2\mathbb{V}(x) = k^2\mathbb{E}(x^2)$.

- **Proposition - Arrow-Pratt Risk Premium with respect to Additive risk**: If risk is additive, i.e. $z = w + kx$, then the risk premium π is approximately equal to $0.5A(w)\mathbb{V}(z)$.

Expected Utility
Arrow-Pratt Approximation (2)

Proof:

- By the definition of the risk premium we have $\mathbb{E}(u(z)) = \mathbb{E}(u(w + kx)) = u(w - \pi(k))$.

- For $k = 0$ we get $\pi(k) = 0$. For risk averse agents $d\pi(k)/dk \geq 0$.

- Use the definition of the risk premium and take the first derivate with respect to k on both sides:

$$\mathbb{E}(xu'(w + kx)) = -\pi'(k)u'(w - \pi(k)) \ .$$

Expected Utility
Arrow-Pratt Approximation (3)

Proof:

- For the left hand side we get at $k = 0$:
  $\mathbb{E}(xu'(w + 0x)) = u'(w)\mathbb{E}(x) = 0$ since $\mathbb{E}(x) = 0$ by assumption.

- Matching LHS with RHS results in $\pi'(k) = 0$ at $k = 0$, while $u'(.) > 0$ by assumption.

Expected Utility
Arrow-Pratt Approximation (4)

Proof:

- Taking the second derivative with respect to k yields:

$$\mathbb{E}(x^2 u''(w+kx)) = (\pi'(k))^2 u''(w-\pi(k)) - \pi''(k)u'(w-\pi(k))$$

- At $k = 0$ this results in (note that $\pi'(0) = 0$):

$$\pi''(0) = -\frac{u''(w)}{u'(w)}\mathbb{E}(x^2) = A(w)\mathbb{E}(x^2)$$

Expected Utility
Arrow-Pratt Approximation (5)

- A second order Taylor expansion of $\pi(k)$ around $k = 0$ results in

$$\pi(k) \approx \pi(0) + \pi'(0)k + \frac{\pi''(0)}{2}k^2$$

- Thus

$$\pi(k) \approx 0.5A(w)\mathbb{E}(x^2)k^2$$

- Since $\mathbb{E}(x) = 0$ by assumption, the risk premium is proportional to the variance of x, that is $\mathbb{V}(z) = k^2\mathbb{E}(x^2)$.

Expected Utility
Arrow-Pratt Approximation (6)

- For multiplicative risk we can proceed as follows: $z = w(1 + kx)$ where $\mathbb{E}(x) = 0$.

- Proceeding the same way results in:

$$\frac{\pi(k)}{w} \approx -\frac{wu''(w)}{u'(w)}k^2\mathbb{E}(x^2) = 0.5R(w)\mathbb{E}(x^2)k^2$$

- **Proposition - Arrow-Pratt Relative Risk Premium with respect to Multiplicative risk**: If risk is multiplicative, i.e. $z = w(1 + kx)$, then the relative risk premium π/w is approximately equal to $0.5R(w)k^2\mathbb{V}(x)$.

- Interpretation: Risk premium per monetary unit of wealth.

Expected Utility
Decreasing Absolute Risk Aversion (1)

- It is widely believed that the more wealthy an agent, the smaller his/her willingness to pay to escape a given additive risk.

- **Definition - Decreasing Absolute Risk Aversion**[D 6.C.4]: The Bernoulli utility function for money exhibits decreasing absolute risk aversion if the Arrow-Pratt coefficient of absolute risk aversion $-\frac{u''(.)}{u'(.)}$ is a decreasing function of wealth w.

Expected Utility
Decreasing Absolute Risk Aversion (2)

- **Proposition - Decreasing Absolute Risk Aversion**: [P 6.C.3]
  The following statements are equivalent
  - The risk premium is a decreasing function in wealth w.
  - Absolute risk aversion $A(w)$ is decreasing in wealth.
  - $-u'(z)$ is a concave transformation of u. I.e. u' is sufficiently convex.

Expected Utility
Decreasing Absolute Risk Aversion (3)

Proof: (sketch)

- Step 1, $(i) \sim (iii)$: Consider additive risk and the definition of the risk premium. Treat π as a function of wealth:

$$\mathbb{E}(u(w + kx)) = u(w - \pi(w)) \ .$$

- Taking the first derivative yields:

$$\mathbb{E}(1u'(w + kx)) = (1 - \pi'(w))u'(w - \pi(w)) \ .$$

Expected Utility
Decreasing Absolute Risk Aversion (4)

Proof: (sketch)

- This yields:

$$\pi'(w) = -\frac{\mathbb{E}(u'(w+kx)) - u'(w-\pi(w))}{u'(w-\pi(w))} \; .$$

- $\pi'(w)$ decreases if $\mathbb{E}(u'(w+kx)) - u'(w-\pi(w)) \geq 0$.

- This is equivalent to $\mathbb{E}(-u'(w+kx)) \leq -u'(w-\pi(w))$.

- Note that we have proven that if $\mathbb{E}(u_2(z)) = u_2(z-\pi_2)$ then $\mathbb{E}(u_1(z)) \leq u_1(z-\pi_2)$ if agent 1 were more risk averse.

Expected Utility
Decreasing Absolute Risk Aversion (5)

Proof: (sketch)

- Here we have the same mathematical structure (see slides on Comparative Analysis): set $z = w + kx$, $u_1 = -u'$ and $u_2 = u$.

- $\Rightarrow -u'$ is more concave than u such that $-u'$ is a concave transformation of u.

Expected Utility
Decreasing Absolute Risk Aversion (6)

Proof: (sketch)

- Step 2, $(iii) \sim (ii)$: Next define $P(w) := -\frac{u'''}{u''}$ which is often called **degree of absolute prudence**.
- From our former theorems we get: $P(w) \geq A(w)$ has to be fulfilled (see A_1 and A_2).
- Take the first derivative of the Arrow-Pratt measure yields:

$$\begin{aligned}
A'(w) &= -\frac{1}{(u'(w))^2}(u'''(w)u'(w) - (u''(w))^2) \\
&= -\frac{u''(w)}{(u'(w))}(u'''(w)/u''(w) - u''(w)/u'(w)) \\
&= \frac{u''(w)}{(u'(w))}(P(w) - A(w))
\end{aligned}$$

Expected Utility
Decreasing Absolute Risk Aversion (7)

Proof: (sketch)

- A decreases in wealth if $A'(w) \leq 0$.

- We get $A'(w) \leq 0$ if $P(w) \geq A(w)$.

Expected Utility
HARA Utility (1)

- **Definition - Harmonic Absolute Risk Aversion**: A Bernoulli utility function exhibits HARA if its **absolute risk tolerance** ($=$ inverse of absolute risk aversion) $T(z) := 1/A(z)$ is linear in wealth z.

- I.e. $T(z) = -u'(z)/u''(z)$ is linear in z

- These functions have the form $u(z) = \zeta\left(\eta + z/\gamma\right)^{1-\gamma}$.

- Given the domain of z, $\eta + z/\gamma > 0$ has to hold.

- Taking derivatives results in:

$$
\begin{aligned}
u'(z) &= \zeta \frac{1-\gamma}{\gamma} (\eta + z/\gamma)^{-\gamma} \\
u''(z) &= -\zeta \frac{1-\gamma}{\gamma} (\eta + z/\gamma)^{-\gamma-1} \\
u'''(z) &= \zeta \frac{(1-\gamma)(\gamma+1)}{\gamma^2} (\eta + z/\gamma)^{-\gamma-2}
\end{aligned}
$$

Expected Utility
HARA Utility (3)

- Risk aversion: $A(z) = (\eta + z/\gamma)^{-1}$
- Risk Tolerance is linear in z: $T(z) = \eta + z/\gamma$
- Absolute Prudence: $P(z) = \frac{\gamma+1}{\gamma}(\eta + z/\gamma)^{-1}$
- Relative Risk Aversion: $R(z) = z\,(\eta + z/\gamma)^{-1}$

Expected Utility
HARA Utility (4)

- With $\eta = 0$, $R(z) = \gamma$: **Constant Relative Risk Aversion Utility Function**: $u(z) = \log(z)$ for $\gamma = 1$ and $u(z) = \frac{z^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$.

- This function exhibits DARA; $A'(z) = -\gamma^2/z^2 < 0$.

Expected Utility
HARA Utility (5)

- With $\gamma \rightarrow \infty$: **Constant Absolute Risk Aversion Utility Function**: $A(z) = 1/\eta$.

- Since $u''(z) = Au'(z)$ we get $u(z) = -\exp(-Az)/A$.

- This function exhibits increasing relative risk aversion.

Expected Utility
HARA Utility (6)

- With $\gamma = -1$: **Quadratic Utility Function**:
- This functions requires $z < \eta$, since it is decreasing over η.
- Increasing absolute risk aversion.

Expected Utility
State Dependent Utility (1)

- With von Neumann Morgenstern utility theory only the consequences and their corresponding probabilities matter.

- I.e. the underlying cause of the consequence does not play any role.

- If the cause is one's state of health this assumption is unlikely to be fulfilled.

- Example car insurance: Consider fair full cover insurance. Under VNM utility $U(l) = pu(w-P) + (1-p)u(w-P)$, etc. If however it plays a role whether we have a wealth of $w-P$ in the case of no accident or getting compensated by the insurance company such the wealth is $w-P$, the agent's preferences depend on the states $accident$ and $no\ accident$.

Expected Utility
State Dependent Utility (2)

- With VNM utility theory we have considered the set of simple lotteries L_S over the set of consequences Z. Each lottery l_i corresponds to a probability distribution on Z.
- Assume that Ω has finite states. Define a random variable f mapping from Ω into L_S. Then $f(\omega) = l_\omega$ for all ω of Ω. I.e. f assigns a simple lottery to each state ω.
- If the probabilities of the states are given by $\pi(\omega)$, we arrive at the compound lotteries $l_{SDU} = \sum \pi(\omega) l_\omega$.
- I.e. we have calculated probabilities of compound lotteries.

Expected Utility
State Dependent Utility (3)

- The set of l_{SDU} will be called L_{SDU}. Such lotteries are also called **horse lotteries**.

- Note that also convex combinations of l_{SDU} are $\in L_{SDU}$.

- **Definition - Extended Independence Axiom**: The preference relation $\succeq$ satisfies extended independence if for all $l^1_{SDU}, l^2_{SDU}, l_{SDU} \in L_{SDU}$ and $\alpha \in (0,1)$ we have $l^1_{SDU} \succeq l_{SDU}$ if and only if $\alpha l^1_{SDU} + (1-\alpha) l^2_{SDU} \succeq \alpha l_{SDU} + (1-\alpha) l^2_{SDU}$.

Expected Utility
State Dependent Utility (4)

- **Proposition - Extended Expected Utility/State Dependent Utility**: Suppose that Ω is finite and the preference relation $\succeq$ satisfies continuity and in independence on L_{SDU}. Then there exists a real valued function $u : Z \times \Omega \rightarrow \mathbb{R}$ such that

$$l^1_{SDU} \succeq l^2_{SDU}$$

if and only if

$$\sum_{\omega \in \Omega} \pi(\omega) \sum_{z \in supp(l^1_{SDU}(\omega))} p_{l1}(z|\omega) u(z, \omega) \geq$$

$$\sum_{\omega \in \Omega} \pi(\omega) \sum_{z \in supp(l^2_{SDU}(\omega))} p_{l2}(z|\omega) u(z, \omega) \ .$$

Expected Utility
State Dependent Utility (4)

- u is unique up to positive linear transformations.
- Proof: see Ritzberger, page 73.
- If only consequences matter such that $u(z, \omega) = u(z)$ then state dependent utility is equal to VNM utility.

Competitive Markets
Outline

- Partial equilibrium analysis
- Perfect Competition
- Entry and perfect competition

MWG: Chapter 10 A, B, C, F

Competitive Markets
Pareto Optimality (1)

- Consider I consumers, indexed $i = 1, \ldots, I$. $X_i \subset \mathbb{R}^L_+$ are the consumption sets. Each consumer chooses a consumption bundle x_i, the utility is given by $u_i(x_i)$.

- J firms, indexed $j = 1, \ldots, J$. The production possibility sets are $Y_j \in \mathbb{R}^L$. The production vectors are y_j.

- L goods, indexed $l = 1, \ldots, L$.

- Total endowments of good l is $w_l \geq 0$. The total net amount of goods available is $w_l + \sum_j y_{lj}$, $l = 1, \ldots, L$.

- We assume that the initial endowments and technological possibilities (i.e. the firms) are owned by consumers. Shares θ_{ij}, where $\sum_{i=1}^{I} \theta_{ij} = 1$ for all $j = 1, \ldots, J$.

- **Remark**: Often the endowments are abbreviated by e_l. Here we stick to MWG and use w_l.

- **Definition - Economic Allocation** [D 10.B.1]: An economic allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \ldots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \ldots, J$. The allocation is **feasible** if

$$\sum_{i=1}^{I} x_{li} \leq w_l + \sum_{j=1}^{J} y_{lj} \text{ for } l = 1, \ldots, L.$$

Competitive Markets
Pareto Optimality (3)

- **Definition - Pareto Optimality** [D 10.B.2]: A feasible allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is Pareto optimal (efficient) if there is no other feasible allocation $(x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)$ such that $u_i(x'_i) \geq u_i(x_i)$ for all $i = 1, \ldots, I$ and $u_i(x'_i) > u_i(x_i)$ for some i.

- **Definition - Utility Possibility Set**: "The set of attainable utility levels".

  $U = \{(u_1, \ldots, u_I) \in \mathbb{R}^I | \exists$ feasible allocation (x, y): $u_i \leq u_i(x_i)$ for $i = 1, \ldots, I\}$

- Pareto efficient allocations are on the north-east boundary of this set. See Figure 10.B.1.

Competitive Markets
Competitive Equilibria (1)

- **Definition - Competitive Economy**
  - Suppose that consumer i initially owns w_{li}, where $w_l = \sum_{i=1}^{I} w_{li}$ for $l = 1, \ldots, L$, $w_i = (w_{i1}, \ldots, w_{iL})$.
  - Consumers i owns the shares $\theta_i = (\theta_{i1}, \ldots, \theta_{ij}, \ldots, \theta_{iJ})$, where $\sum_{i=1}^{I} \theta_{ij} = 1$ for $j = 1, \ldots, J$.
  - Markets exist for all L goods and all firms are price takers; the prices are $p = (p_1, \ldots, p_L)$.

Competitive Markets
Competitive Equilibria (2)

- **Definition - Walrasian Equilibrium** [D 10.B.3] The allocation (x, y) and the price vector $p \in \mathbb{R}^L$ constitute a competitive (Walrasian) equilibibirium if the following conditions are met:

Competitive Markets
Competitive Equilibria (3)

- Profit maximization: each firm j solves $\max_{y_j \in Y_j} p \cdot y_j$ where $y_j \in Y_j$.

- Utility maximization: each consumer i solves

$$\max_{x_i \in X_i} u(x_i) \text{ s.t. } p \cdot x_i \leq p \cdot w_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j).$$

- Market clearing: For each good $l = 1, \ldots, L$:

$$\sum_{i=1}^{I} x_{li} = w_l + \sum_{j=1}^{J} y_{li}.$$

Competitive Markets
Competitive Equilibria (3)

- **Proposition** [P 10.B.1]: If the allocation (x, y) and the price vector $p \gg 0$ satisfy the market clearing condition for all goods $l \neq k$ and if every consumer's budget constraint is satisfied with equality ($p \cdot x_i = p \cdot w_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j)$), then the market for good k also clears.

Competitive Markets
Competitive Equilibria (4)

Proof:

- $p \cdot x_i = p \cdot w_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j)$ holds for all $i = 1, \ldots, I$.

- This yields $\sum_i \sum_l p_l x_{li} = \sum_i (p \cdot w_i + \sum_{j=1}^{J} \theta_{ij} y_j)$.

- Rearranging terms results in (i):

$$\sum_l p_l \sum_i x_{li} = \sum_l p_l x_l = \sum_l p_l \sum_i (w_{li} + \sum_{j=1}^{J} \theta_{ij} y_{lj}) = \sum_l p_l (w_l + y_l)$$

  for all l.

Proof:

- Markets clear for markets $l \neq k$ by assumption such that (ii):

$$p_l x_l = p_l(w_l + y_l)$$

for $l \neq k$.

- Combining (i) and (ii) results in

$$p_k x_k = p_k(w_k + y_k).$$

Competitive Markets
Partial Equilibrium (1)

- Marshallian partial equilibrium analysis investigates the market for one good (or several goods).

- Argument I: when the expenditure for the good is small, only a small fraction of wealth will be spent on this good, such that the wealth effect is small.

- Argument II: Due to small size of the market (and similarly dispersed substitution effects), a change in the price of the good considered has a neglectable impact on the other prices.

- Consider the other goods as a single composite commodity, which we call **numeraire** (see Hicksian composite commodity).

Competitive Markets Partial Equilibrium (2)

- **Two good quasilinear economy**: good l and the numeraire, x_i is the consumption of consumer i of the second good, m_i is the consumption of the numeraire.

- Each consumer has quasilinear utility: $u_i(m_i, x_i) = m_i + \phi_i(x_i)$, $i = 1, \ldots, I$. To avoid any boundary problems $(m_i, x_i) \in \mathrm{R} \times \mathbb{R}_+$. ϕ_i is bounded and twice differentiable, with $\phi_i' > 0$, $\phi_i'' < 0$ for $x_i \geq 0$ and $\phi_i(0) = 0$.

- Good l is the good of the market under study, m_i stands for the rest. We already know that there are no wealth effects for non-numeraire goods, i.e. there are no wealth effects for x_i. p is the price of the good considered, the price of the numeraire good should be one.

Competitive Markets
Partial Equilibrium (3)

- Each firm $j = 1, \ldots, J$ uses m as an input to produce good l; z_j is the amount imputed into the production process. $q_j \geq 0$ is the amount produced by firm j. $z_j = c_j(q_j)$ is the cost to produce q_j units of good l.

- The production possibility set is given by $Y_j = \{(-z_j, q_j) | q_j, z_j \geq c_j(q_j)\}$, $c(.)$ is twice differentiable with $c' > 0$ and $c'' \geq 0$ for all $q_j \geq 0$.

- The initial endowments of good l are zero, while the consumers own w_{mi} of the numeraire, $w_m = \sum_{i=1}^{I} w_{mi}$.

Competitive Markets
Partial Equilibrium (4)

- Goal: try to find competitive equilibrium/equilibria.

- Firms maximize profits: $\max_{q_j \geq 0} pq_j - c_j(q_j)$

- First order condition:

$$p \leq c_j'(q_j) \text{ with equality if } q_j > 0.$$

- Second order condition is assumed to be satisfied.

Competitive Markets
Partial Equilibrium (5)

- Consumers maximize utility: $\max_{m_i, x_i} m_i + \phi(x_i)$ s.t. $m_i + px_i \leq w_{mi} + \sum_{j=1}^{J} \theta_{ij}(pq_j - c_j(q_j))$.

- Given our assumptions on $u_i(.)$ the budget constraint has to hold with equality.

Competitive Markets
Partial Equilibrium (6)

- Either by a plug in of m_i from the budget constraint in the utility function or by applying Kuhn/Tucker we get

$$\phi'(x_i) \le p \text{ with equality for } x_i > 0.$$

- Then $m_i = w_{mi} + \sum_{j=1}^{J} \theta_{ij}(pq_j - c_j(q_j)) - px_i$, input $z_j = c_j(q_j)$.

Competitive Markets
Partial Equilibrium (7)

- For the two good economy it is sufficient to check whether the market for good l clears.

- Therefore the allocation (x, q) and the price p are a competitive equilibrium if

$$
\begin{aligned}
p &\leq c'(q_j) \text{ with equality for } q_j > 0 \\
\phi'(x_i) &\leq p \text{ with equality for } x_i > 0 \\
& \sum_{i=1}^{I} x_i = \sum_{j=1}^{J} q_j \ .
\end{aligned}
$$

- $J + I + 1$ equations for the same number of unknowns x, y and p.

Competitive Markets
Partial Equilibrium (8)

- As long as $\max_i \phi_i'(0) > \min_j c_j'(0)$ aggregate consumption and production have to be strictly positive.

- In this setting the equilibrium outcome is independent of the distribution of the endowments w_{mi} and the shares θ_{ij} (ownership structure). This is a result of the quasilinear economy.

- Aggregate demand $x(p) = \sum x_i(p)$. By the assumptions of ϕ, the Walrasian demand correspondence is single valued, continuous and downward sloping (at any $p < \max \phi_i'(0)$). Since preferences are quasilinear $x(p)$ does not depend on wealth.

Competitive Markets
Partial Equilibrium (9)

- Aggregate supply $q(p) = \sum_{j=1}^{J} q_j(p)$ is continuous and non-decreasing for all $p > 0$. It is strictly increasing for all $p > \min c_j'(0)$. Then we get the equilibrium price p by intersecting market demand and supply. See Figures page 320.

- An equilibrium price need not exist.

- If $c_j(.)$ is linear, then the firms' production levels are not uniquely determined.

- In the above model we have assumed that the preferences and the technologies are convex. If we deviate from this assumption a competitive equilibrium need not exist (see e.g. Figure 10.C.8, page 324).

Competitive Markets
Free Entry (1)

- Until now J has been fixed. In the following we shall consider "long run behavior".

- Assume that an infinite number of firms has potential access to a production technology to produce good l. The cost function is $c(q)$ where $c(0) = 0$ is assumed.

- $x(p)$ denotes the aggregate demand function, $P(.)$ is its inverse.

- Assumption: the identical active firms produce the same level of output q. With J active firms ($q_j > 0$) the total output is $Q = qJ$.

Competitive Markets
Free Entry (2)

- **Remark:** J is a non-negative integer. We assume that firms are sufficiently small such that the fact that J is an integer can be neglected.

- J is endogenously determined by market entry and exit. Exit if the profit is smaller zero, entry with positive profit.

- In equilibrium the profits have to be zero.

Competitive Markets
Free Entry (3)

- **Definition - Long-run Competitive Equilibrium** [D 10.F.1]: Given the aggregate demand function $x(p)$ and a cost function $c(q)$ for each potentially active firm having $c(0) = 0$, a triple (p, q, J) is a long-run competitive equilibrium if
  - Profit maximization: q solves $\max_{q \geq 0} pq - c(q)$.
  - Market clearing: $x(p) = Jq$.
  - Free entry condition: $pq - c(q) = 0$.

Competitive Markets
Free Entry (4)

- First order condition: $p \le c'(q)$ with equality if $q > 0$.

- **Supply correspondence** $q(p)$: For any firm $q(p)$ solves the profit maximization problem - see Chapter 5.

- **Long-run Aggregate Supply Correspondence** $Q(p) = Jq(p)$:
  (i) $Q(p) \to \infty$ if $\pi(p) > 0$, (ii)
  $Q(p) = \{Q \ge 0 | Q = Jq$ for some integer $J \ge 0$ and $q \in q(p)\}$ if $\pi(p) = 0$. J is the number of active firms.

Competitive Markets
Free Entry (5)

- Market clearing demands for $Q = x(p)$, $Q \in Q(p)$.

- Free entry condition and the aggregate supply correspondence result in $pq - c(q) = 0$.

- $\Rightarrow p$ is a competitive equilibrium if $x(p) \in Q(p)$.

Competitive Markets
Free Entry (6)

- Consider constant returns to scale such that $c(q) = cq$, $c > 0$.
- Assume that $x(c) > 0$.
- The first order condition for the firms result in $p \leq c$. Since $x(c) > 0$, we must have $(p - c)q = 0$ in an equilibrium. Then $p = c$.
- J and q are not determined, but $Jq = x(c)$.
- $Q(p) = \infty$ if $p > c$, $Q(p) \in [0, \infty)$ if $p = c$ and 0 if $p < c$.

Competitive Markets
Free Entry (7)

- Suppose that $c(q)$ is increasing and strictly convex.

- Assume that $x(c'(0)) > 0$.

- If $p > c'(0)$, then $\pi(p) > 0$ such that $Q(p)$ becomes infinite. If $p \leq c'(0)$ the long run supply is zero, while $x(p) > 0$. In words, in the first case $p = MC(q) > AC(q)$, such that $J \to \infty$. In the second case the first order condition yields $q = 0$.

- Here no equilibrium exists. See Figure 10.F.2, page 337.

- Suppose that there exists a unique $q > 0$ where average cost is minimized (the output level is also called efficient scale).

- Assume that $c = c(q)/q$ is minimized at $\bar{q}$. Here we get $c'(\bar{q}) = c(\bar{q})/\bar{q} = \bar{c}$. Assume that $x(\bar{c}) > 0$.

- If $p > c'(\bar{q}) = c(\bar{q})/\bar{q}$ this cannot be an equilibrium since $\pi(p) > 0$ and $Q(p)$ becomes infinite.

- For $p < c'(\bar{q}) = c(\bar{q})/\bar{q}$, $\pi(p) < 0$ and $Q(p) = 0$ which is not compatible with $x(p) \geq x(\bar{c}) > 0$.

Competitive Markets
Free Entry (9)

- If $p = c'(\bar{q}) = c(\bar{q})/\bar{q}$, then $\pi(p) = 0$ and $Q(p) = J\bar{q} = x(\bar{c})$.

- Thus $Q(p) = \infty$ if $p > \bar{c}$, $Q(p) = 0$ if $p < \bar{c}$ and $\{Q \geq 0 | Q = J\bar{q}$ for some integer $J \geq 0\}$.

- With $x(p) \in Q(p)$ we get an equilibrium.

- See Figure 10.F.4

- With quasilinear preferences the boundary of the utility possibility set is a hyperplane.

- Unit to unit transfer of utility is possible. With these preferences

$$\{(u_1, \ldots, u_I) | \sum_{i=1}^{I} u_i \leq \sum_{i=1}^{I} \phi_i(x_i) + w_m - \sum_{j=1}^{J} c_j(q_j)\}$$

- The boundary of this set is a hyperplane with normal vector $(1, \ldots, 1)$.

- **Marshallian surplus**:

$$S = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$

Maximizing S given the constraint $\sum_i x_i - \sum_j q_j = 0$ yields the same outcome as the allocation attained with the competitive economy. The Lagrange multiplier in this constraint optimization problem is equal to p.

Competitive Markets
Some further results (3)

- Proposition - **First Fundamental Theorem of Welfare Economics** [10.D.1] If the price p and the allocation (x, y) constitute a competitive equilibrium, the this allocation is Pareto optimal.

- Proposition - **Second Fundamental Theorem of Welfare Economics** [10.D.2] For any Pareto optimal levels of utility $(u_1, \ldots, u_I)$ there are transfers of the numeraire commodity $(T_1, \ldots, T_I)$ satisfying $\sum_{i=1}^{I} T_i = 0$, such that a competitive equilibrium reached from the endowments $(w_{m1} + T_1, \ldots, w_{mI} + T_I)$ yields precisely the utilities $(u_1, \ldots, u_I)$.

General Equilibrium
Outline

- Motivation and main questions to be investigated:
  - Does a competitive economy result in a Pareto efficient allocation?
  - Can any Pareto efficient allocation be obtained by means of a price system in a competitive economy?
- Edgeworth Box
- Robinson Crusoe economies
- Small open economies and trade
- General vs. partial equilibrium

MWG, Chapter 15

General Equilibrium
Motivation (1)

- Consider the economy as a closed and interrelated system.
- With the partial equilibrium approach these interrelations are mainly ignored.
- The exogenous variables in general equilibrium theory are reduced to a small number of physical realities (number of agents, technologies available, preferences of the agents, endowments of various agents).

General Equilibrium
Motivation (2)

- First we consider:
  - A pure exchange economy: no production is possible, commodities are ultimately consumed, the individuals are permitted to trade the commodities among themselves. With two consumers and two goods this can be represented in the **Edgeworth box**.
  - **One consumer - one firm economy**, to get a first impression on the impacts of production.

General Equilibrium
Motivation (3)

- Consider I consumers, indexed $i = 1, \ldots, I$. $X_i \subset \mathbb{R}^L$ are the consumption sets. Each consumer chooses a consumption bundle x_i, the utility is given by $u_i(x_i)$. The preferences are $\succeq_i$.

- J firms, indexed $j = 1, \ldots, J$. The production possibility sets are $Y_j \in \mathbb{R}^L$. The production vectors are y_j.

- L goods, indexed $\ell = 1, \ldots, L$.

General Equilibrium
Motivation (4)

- Total endowments of good ℓ is $\omega_\ell \geq 0$. The total net amount of good ℓ available is $\omega_\ell + \sum_j y_{\ell j}$, $\ell = 1, \ldots, L$.

- We assume that the initial endowments and technological possibilities (i.e. the firms) are owned by consumers. Shares θ_{ij}, where $\sum_{i=1}^{I} \theta_{ij} = 1$ for all $j = 1, \ldots, J$.

- The wealth of consumer i is $\mathtt{w}_i(p) = p \cdot \omega_i$.

- **Remark**: often the endowments are abbreviated by e_ℓ. Here we stick to MWG and use ω_ℓ.

- **Definition - Economic Allocation** [D 10.B.1]: An economic allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \ldots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \ldots, J$. The allocation is **feasible** if

$$\sum_{i=1}^{I} x_{\ell i} \leq \omega_\ell + \sum_{j=1}^{J} y_{\ell j} \text{ for } \ell = 1, \ldots, L.$$

- **Definition - Competitive Economy**
  - Suppose that consumer i initially owns $\omega_{\ell i}$, where $\omega_\ell = \sum_{i=1}^{I} \omega_{\ell i}$ for $\ell = 1, \ldots, L$, $\omega_i = (\omega_{i1}, \ldots, \omega_{iL})$.
  - Consumer i owns the shares $\theta_i = (\theta_{i1}, \ldots, \theta_{ij}, \ldots, \theta_{iJ})$, where $\sum_{i=1}^{I} \theta_{ij} = 1$ for $j = 1, \ldots, J$.
  - Markets exist for all L goods and all firms are price takers; the prices are $p = (p_1, \ldots, p_L)$.

General Equilibrium
Motivation (7)

- **Definition - Walrasian/Competitive Equilibrium** [D 10.B.3] The allocation (x, y) and the price vector $p \in \mathbb{R}^L$ constitute a competitive (Walrasian) equilibirium if the following conditions are met:
  - Profit maximization: each firm j solves $\max_{y_j \in Y_j} p \cdot y_j$ where $y_j \in Y_j$.
  - Utility maximization: each consumer i solves

$$\max_{x_i \in X_i} u(x_i) \text{ s.t. } p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p \cdot y_j).$$

  - Market clearing: For each good $\ell = 1, \ldots, L$:

$$\sum_{i=1}^{I} x_{\ell i} = \omega_\ell + \sum_{j=1}^{J} y_{\ell j}.$$

General Equilibrium
Motivation (8)

- **Definition - Pareto Optimality** [D 10.B.2]: A feasible allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is Pareto optimal (efficient) if there is no other feasible allocation $(x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)$ such that $u_i(x'_i) \geq u_i(x_i)$ for all $i = 1, \ldots, I$ and $u_i(x'_i) > u_i(x_i)$ for some i.

- **Definition - Utility Possibility Set**: "The set of attainable utility levels".

$U = \{(u_1, \ldots, u_I) \in \mathbb{R}^I | \exists$ feasible allocation (x, y): $u_i \leq u_i(x_i)$ for $i = 1, \ldots, I\}$

- Pareto efficient allocations are on the north-east boundary of this set. See MWG, Figure 10.B.1.

Edgeworth Box (1)

- We consider a pure exchange economy.

- Consumers posses initial endowments of commodities. Economic activity consists of trading and consumption.

- Now we restrict to a two good - two consumer exchange economy. Then, $L = 2$, $X_1 = X_2 = \mathbb{R}^2_+$, $Y_1 = Y_2 = -\mathbb{R}^2_+$ (the free disposal technology). i is the index of the consumer, ℓ the index of our goods.

- $x_i = (x_{1i}, x_{2i}) \in X_i$. $\succeq_i$ are the preferences of consumer i.

- The initial endowments are $\omega_{\ell i} \geq 0$. The endowment vector of consumer i is $\omega_i = (\omega_{1i}, \omega_{2i})$. The total endowments of good ℓ are $\bar{\omega}_\ell = \omega_{\ell 1} + \omega_{\ell 2}$. We assume that $\bar{\omega}_\ell > 0$ for $\ell = 1, 2$.

Edgeworth Box (2)

- From the above Definition [D 10.B.1] it follows that an economic allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \ldots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \ldots, J$. It is feasible if

$$\sum_{i=1}^{I} x_{\ell i} \leq \bar{\omega}_\ell + \sum_{j=1}^{J} y_{\ell j} \text{ for } \ell = 1, \ldots, L.$$

- For the Edgeworth Box an allocation is some consumption vector $x = (x_{11}, x_{21}, x_{21}, x_{22}) \in \mathbb{R}^4_+$.

- An allocation is feasible if $x_{\ell 1} + x_{\ell 2} \leq \bar{\omega}_\ell$ for $\ell = 1, 2$.

Edgeworth Box (3)

- **Definition - Nonwasteful allocation**: If $x_{\ell 1} + x_{\ell 2} = \bar{\omega}_\ell$ for $\ell = 1, 2$, then the allocation is called **nonwasteful**.

- Nonwasteful allocations can be described by means of an Edgeworth box.

- See MWG, Figure 15.B.1.

- For a given price vector $p = (p_1, p_2)$ the **budget line** intersects the initial endowment point $w_i = (w_{1i}, w_{2i})$. The slope is $-\frac{p_1}{p_2}$. Note that only the relative price $-\frac{p_1}{p_2}$ matters, with $-\frac{\lambda p_1}{\lambda p_2}$, $\lambda \in \mathbb{R}_{++}$, we get the same Edgeworth box with the same budget sets.

- See MWG, Figure 15.B.2.

Edgeworth Box (4)

- Next we assume that the preferences of both consumers are strongly monotone and strictly convex.

- For each price p consumer i obtains the budget set $B_i(p)$. By solving the utility maximization problem

$$max_{x_{1i},x_{2i}}\ u(x_i) \ \ s.t. \ \ p \cdot x_i \leq \mathtt{w}_i(p)$$

  we obtain the optimal quantities $x_{1i}(p), x_{2i}(p)$. By collecting $x_{1i}(p), x_{2i}(p)$ for different p, we obtain the mapping $OC_i : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $p \mapsto (x_{1i}(p), x_{2i}(p))$. This mapping is called **offer curve**.

- By the assumptions on the preferences the solution of the UMP is unique, hence here we obtain a function.

Edgeworth Box (5)

- The consumer's offer curve lies within the upper contour set of ω_i.
- See MWG, Figures 15.B.3.-15.B.5.

Edgeworth Box (6)

- **Definition** [D 15.B.1] A Walrasian/Competivie Equilibrium for an Edgeworth box economy is a price vector p^* and a feasible allocation $x^* = (x_1^*, x_2^*)$ in the Edgeworth box such that for $i = 1, 2$,

$$x_i^* \succeq_i x_i' \text{ for all } x_i' \in B_i(p^*).$$

- At any equilibrium the offer curves intersect.

- Consumer's demand is homogeneous of degree zero in p, i.e. only the relative price matters.

- See MWG, Figures 15.B.7 and 15.B.8.

Edgeworth Box (7)

- A Walrasian equilibrium need not be unique.

- See MWG, Figure 15.B.9.

- This could already happen with quasilinear preferences, where the preferences are such that different numeraire goods are used.

- MWG, Chapter 10 constructs a model where all agents have quasilinear preferences with respect to the same numeraire good.

Edgeworth Box (8)

- Recall: **Definition - Quasilinear Preferences**: A monotone preference relation $\succeq$ on $X = (-\infty, \infty) \times \mathbb{R}^{L-1}$ is quasilinear with respect to commodity one (the numeraire good) if : (i) all indifference sets are parallel displacements of each other along the axis of commodity one. I.e. $x \sim y$ then $x + \alpha e_1 \sim y + \alpha e_1$ and $e_1 = (1, 0, \ldots)$. (ii) Good one is desirable: $x + \alpha e_1 \succ x$ for all $\alpha > 0$. [D 3.B.7]

- A Walrasian equilibrium need not exist.

- This happens e.g. if (i) one consumer only desires only one good or (ii) preferences are non-convex.

- See MWG, Figure 15.B.10.

Edgeworth Box (9)

- **Definition - Pareto Optimality** [D 15.B.2]: A feasible allocation x in the Edgeworth box is Pareto optimal (or Pareto efficient) if there is no other allocation x' in the Edgeworth box with $x_i' \succeq_i x_i$ for $i = 1, 2$ and $x_i' \succ_i x_i$ for some i. The set of all Pareto optimal allocations is called **Pareto set**. The **contract curve** is the part of the Pareto set where both consumers do at least as well as at their initial endowments.

- See MWG, Figures 15.B.11 and 15.B.12.

Edgeworth Box (10)

- We observe in the Edgeworth box that "every Walrasian equilibrium allocation x^* belongs to the Pareto set". This corresponds to the first theorem of welfare economics. Figure 15.B.7(a) and 15.B.8, MWG, page 520.

- Regarding the second theorem: a planner can (under convexity assumptions, see MWG, Chapter 16) achieve any desired Pareto efficient allocation.

- Hence we define:

Edgeworth Box (11)

- **Definition - Equilibrium with Transfers** [D 15.B.3]: An allocation x in the Edgeworth box is supportable as an equilibrium with transfers, if there is a price system p^* and wealth transfers T_1 and T_2 satisfying $T_1 + T_2 = 0$, such that for each consumer i we have

$$x_i^* \succeq x_i' \text{ for all } x_i' \in \mathbb{R}_+^2 \text{ such that } p^* \cdot x_i' \leq p^* \cdot \omega_i + T_i.$$

- In the Edworth box we observe that with continuous, strongly monotone and strictly convex preferences any Pareto optimal allocation is supportable.

- See MWG, Figure 15.B.13.

- See MWG, Figure 15.B.14 - to observe how the second theorem fails with non-convex preferences.

One-Consumer, One-Producer (1)

- We introduce production in the most simple way.

- There are two price taking agents, a single consumer and a single firm.

- There are two goods, labor (or leisure) of the consumer and the consumption good produced by the firm.

- The preferences $\succeq$ defined over leisure x_1 and the consumption good x_2 are continuous, strongly monotone and strictly convex. The initial endowment consists of $\bar{L}$ units of leisure and no endowment of the consumption good.

One-Consumer, One-Producer (2)

- The firm uses labor to produce the consumption good under the increasing and strictly concave production function $q = f(z)$, where z is labor input and q the amount of x_2 produced.

- The firm maximizes its profit:

$$\max_{z \geq 0} pf(z) - wz$$

given the prices (p, w). This optimization problem results in the optimal labor demand $z(p, w)$ and output $q(p, w)$. The profit is $\pi(p, w)$.

One-Consumer, One-Producer (3)

- The consumer maximizes the utility function $u(x_1, x_2)$:

$$\max_{x_1, x_2 \geq 0} u(x_1, x_2) \quad s.t.\ px_2 \leq w(\bar{L} - x_1) + \pi(p, w).$$

  This results in the Walrasian demand $x_1(p, w)$ and $x_2(p, w)$. Labor supply corresponds to $\bar{L} - x_1(p, w)$.

- See MWG, Figure 15.C.1 on these optimization problems.

One-Consumer, One-Producer (4)

- Walrasian equilibrium is attained at a pair (p^*, w^*) where

$$x_2(p^*, w^*) = q(p^*, w^*) \text{ and } z(p^*, w^*) = \bar{L} - x_1(p^*, w^*).$$

- See MWG, Figure 15.C.1 on these optimization problems. See MWG, Figure 15.C.2 for an equilibrium.

One-Consumer, One-Producer (5)

- Remark: A particular consumption-leisure combination can arise in a competitive equilibrium if and only if it maximizes the consumer's utility subject to the technological and endowment constraints.

- $\Rightarrow$ A Walrasian equilibrium allocation is the same as if a social planner would maximize the consumer's utility given the technological constraints of the economy. A Walrasian equilibrium is Pareto optimal.

One-Consumer, One-Producer (6)

- Remark on Non-convexity: Suppose the the production set is not convex, then we can construct examples where the price system does not support the allocation x^*.
- See MWG, Figure 15.C.3 (a).

General vs. Partial Equilibrium (1)

- Bradford's (1978) example on taxation:

- Consider an economy with N large towns. Each town has a single price taking firm producing a consumption good by means of a strictly concave production function $f(z)$. The consumption good is identical.

- The overall economy has M units of labor, inelastically supplied. Utility is derived from consuming the output.

- Workers are free to move to another town. Hence the equilibrium wage must be the same, i.e. $w_1, \dots, w_N = \bar{w}$.

- Without loss of generality the price of the output is normalized, i.e. $p = 1$.

General vs. Partial Equilibrium (2)

- By the symmetric construction of the model we get: each firm hires M/N works, the output of each firm is $f(M/N)$.

- Due to price taking we get $\bar{w} = f' = \frac{\partial f(M/N)}{\partial (M/N)}$.

- The equilibrium profits are: $f(M/N) - \frac{\partial f(M/N)}{\partial (M/N)}(M/N)$.

General vs. Partial Equilibrium (3)

- Suppose that town 1 levies a tax on labor, the tax rate is $t > 0$.
- Given the wage w_1 and the tax rate t we arrive at a labor demand z_1, which is implicitly given by $f'(z_1) = t + w_1$.

General vs. Partial Equilibrium (4)

- Partial equilibrium argument: N is large, in impact on the other wage rates can be neglected. Hence $\bar{w}$ remains the same.

- Since labor moves freely, we get $w_1 = \bar{w}$. The supply correspondence is 0 at $w_1 < \bar{w}$ and ∞ at $w_1 > \bar{w}$. It is $[0, \infty]$ at $w_1 = \bar{w}$.

- Then $f'(z_1) = t + \bar{w}$. z_1 falls by our assumptions on $f(.)$, labor moves to other towns.

- The incomes of the workers and the profits in towns $2, \ldots, N$ remain the same. The profit of firm 1 decreases, the firms completely bear the tax burden.

General vs. Partial Equilibrium (5)

- General equilibrium argument: Since labor moves freely, $w_1, \ldots, w_N = w$ still has to hold. All M units of labor are employed by the structure of $f(.)$.

- $w(t)$ denotes the equilibrium wage rate when the tax rate is t. By symmetry $z_2(t) = \cdots = z_N(t) = z(t)$. $z_1(t)$ is the labor demand in town 1.

- Then equilibrium demands for:

$$z_1(t) + (N-1)z(t) = M \ , \ f'(z(t)) = w(t) \ , \ \ f'(z_1(t)) = w(t) + t.$$

General vs. Partial Equilibrium (6)

- Next, $f'(z_1(t)) = w(t) + t = f'(M - (N-1)z(t)) = w(t) + t$. By taking the first derivative w.r.t. to t and evaluating at $t = 0$ (where $z_1(0) = z(0) = M/N$) yields

$$f^{''}(M/N)[-(N-1)]z'(0) = w'(0) + 1.$$

General vs. Partial Equilibrium (7)

- The derivative of $f'(z(t)) = w(t)$ w.r.t. to t yields $f''(M/N)z'(0) = w(0)$ such that

$$w'(0) = -\frac{1}{N}.$$

- Hence, the wage rates in all towns decrease due to the tax in town 1. Only if N goes to infinity this effect becomes zero.

- In addition, when we consider the profits of the firms, we observe:

$$\pi'(\bar{w})(w'(0) + 1) + (N-1)\pi'(\bar{w})w'(0) = \pi'(\bar{w})\left(-\frac{N-1}{N} + \frac{N-1}{N}\right) = 0.$$

Hence, aggregate profit remains constant. The complete burden is attributed to the workers.

General vs. Partial Equilibrium (8)

- For N large the partial equilibrium approximation regarding prices and quantities is correct. However, the distributional effects remain wrong.

- The derivative of $f'(z(t)) = w(t)$ w.r.t. to t yields $f''(M/N)z'(0) = w(0)$ such that

$$w'(0) = -\frac{1}{N}.$$

- Hence, the wage rates in all towns decrease due to the tax in town 1. Only if N goes to infinity this effect becomes zero.

General vs. Partial Equilibrium (9)

- In addition, when we consider the profits of the firms, we observe:

$$\pi'(\bar{w})(w'(0)+1)+(N-1)\pi'(\bar{w})w'(0)=\pi'(\bar{w})\left(-\frac{N-1}{N}+\frac{N-1}{N}\right)=0.$$

- Hence, aggregate profit remains constant. The complete burden is attributed to the workers.

- For N large the partial equilibrium approximation regarding prices and quantities is correct. However, the distributional effects remain wrong.

- First Fundamental Theorem of Welfare Economics

- Second Fundamental Theorem of Welfare Economics

  MWG, Chapter 16

Notation (1)

- Consider I consumers, indexed $i = 1, \ldots, I$. $X_i \subset \mathbb{R}^L$ are the consumption sets. The preferences are $\succeq_i$. $\succeq_i$ is complete and transitive (rationale consumers).

- J firms, indexed $j = 1, \ldots, J$. The production possibility sets are $Y_j \in \mathbb{R}^L$. Y_j is non-empty and closed. The production vectors are y_j.

- L goods, indexed $\ell = 1, \ldots, L$.

Notation (2)

- The initial endowment of good ℓ is $\bar{\omega}_\ell \in \mathbb{R}^L$. The total endowments are $\bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_L) \in \mathbb{R}^L$.
- Basis data of the economy: $([X_i, \succeq_i]_{i=1}^I, [Y_j]_{j=1}^J, \bar{\omega})$.
- The wealth of consumer i is $\mathtt{w}_i(p) = p \cdot \omega_i$.

Notation (3)

- **Definition - Economic Allocation** [D 16.B.1]: An economic allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \ldots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \ldots, J$. The allocation is **feasible** if

$$\sum_{i=1}^{I} x_{\ell i} = \bar{\omega}_\ell + \sum_{j=1}^{J} y_{\ell j} \text{ for } \ell = 1, \ldots, L.$$

This is $\sum_{i=1}^{I} x_i = om\bar{e}ga_\ell + \sum_{j=1}^{J} y_j$. We denote the set of feasible allocations by

$$A := \{(x, y) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J : \sum_{i=1}^{I} x_i = \bar{\omega}_\ell + \sum_{j=1}^{J} y_j\} \subset \mathbb{R}^{L(I+J)}.$$

Notation (4)

- **Definition - Pareto Optimality** [D 16.B.2]: A feasible allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is Pareto optimal (efficient) if there is no other feasible allocation $(x', y') \in A$ that Pareto dominates it. This is, if there is no feasible allocation (x', y') such that $x'_i \succeq_i x_i$ for all $i = 1, \ldots, I$ and and $x'_i \succ_i x_i$ for some i.

Notation (5)

- Suppose that consumer i initially owns $\omega_{\ell i}$, where $\bar{\omega}_\ell = \sum_{i=1}^{I} \omega_{\ell i}$ for $\ell = 1, \ldots, L$, $\omega_i = (\omega_{i1}, \ldots, \omega_{iL})$.

- Consumer i owns the shares $\theta_i = (\theta_{i1}, \ldots, \theta_{ij}, \ldots, \theta_{iJ})$, where $\sum_{i=1}^{I} \theta_{ij} = 1$ for $j = 1, \ldots, J$.

- Markets exist for all L goods and all firms are price takers; the prices are $p = (p_1, \ldots, p_L)$.

Notation (6)

- **Definition** [D 16.B.3] (Walrasian/Competitive Equilibrium)
  - Given a private ownership economy by $([X_i, \succeq_i]_{i=1}^I, [Y_j]_{j=1}^J, \bar{\omega}, \theta)$. An allocation (x^*, y^*) and the price vector $p \in \mathbb{R}^L$ constitute a **competitive (Walrasian) equilibrium** if the following conditions are met:
    * Profit maximization: For each firm j, y_j^* solves the profit maximization problem, i.e.
    $$p \cdot y_j \leq p \cdot y_j^* \text{ for all } y_j \in Y_j.$$
    * Preference maximization: For each consumer i, x_i^* is maximal for $\succeq_i$ in the budget set
    $$\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^J \theta_{ij} p \cdot y_j^*\}.$$
    * Market clearing: For each good $\ell = 1, \ldots, L$:
    $$\sum_{i=1}^I x_{\ell i}^* = \bar{\omega}_\ell + \sum_{j=1}^J y_{\ell i}^* \text{ or } \sum_{i=1}^I x_i^* = \bar{\omega} + \sum_{j=1}^J y_j^*.$$

Notation (7)

- **Definition** [D 16.B.4] (Price Equilibrium with Transfers)
  - Given a private ownership economy by $([X_i, \succeq_i]_{i=1}^I, [Y_j]_{j=1}^J, \bar{\omega}, \theta)$. An allocation (x^*, y^*) and the price vector $p \in \mathbb{R}^L$ constitute a **price equilibrium with transfers** if there is an assignment of wealth levels $(\mathtt{w}_1, \ldots, \mathtt{w}_I)$ with $\sum_{i=1}^I \mathtt{w}_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that
    * For each firm j, y_j^* solves the profit maximization problem, i.e.

$$p \cdot y_j \le p \cdot y_j^* \text{ for all } y_j \in Y_j.$$

    * For each consumer i, x_i^* is maximal for $\succeq_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \le p \cdot \mathtt{w}_i\}.$$

    * Market clearing: $\sum_{i=1}^I x_i^* = \bar{\omega} + \sum_{j=1}^J y_j^*$.

First Fundamental Theorem of Welfare Economics (1)

- **Proposition** [16.C.1] (First Fundamental Theorem of Welfare Economics)
  - If the preference relations $\succeq_i$ are locally nonsatiated and if (x^*, y^*, p) is a price equilibrium with transfers, then the allocation x^*, y^* is Pareto optimal. In particular, any Walrasian equilibrium is Pareto optimal.
- Proof: See MWG page 549.

First Fundamental Theorem of Welfare Economics (2)

- The First Fundamental Theorem of Welfare Economics is on Pareto optimality.

- Recall - Local Nonsatiation: For all $x \in X$ and for all $\varepsilon > 0$ there exists some $y \in X$ such that $||x - y|| \leq \varepsilon$ and $y \succ x$. [D 3.B.3]

- Note that markets are complete and price taking is assumed.

Second Fundamental Theorem of Welfare Economics (1)

- First theorem: Given some assumptions and a price equilibrium with transfers $\Rightarrow$ Pareto.
- Consider a competitive economy with transfers. Given some Pareto efficient allocation (x, y). Does there exist a price system p which supports this Pareto efficient allocation?
- Problem I: Convexity - see MWG, Figure 15.C.3 (a).
- Problem II: Minimum wealth problem - see MWG, Figure 15.B.10 (a). Let $x_i \in \mathbb{R}^2$ and consider the sequence $p_n = (1/n, 1)$, $n = 1, 2, \ldots$. Let $w_i = (1, 0)$. Then $p_n \cdot w_i \to 0$. This is the minimum wealth you can have on $X = \mathbb{R}^2$. In the limit $p = (0, 1)$ and the budget set is the horizontal axis intersected with $\mathbb{R}^2_+$. So everything of x_1 costs nothing. The budget correspondence is not continuous so Berge's maximum theorem does not apply.
- First investigate convexity. To do this we consider the concept of a quasi-equilibrium.

Second Fundamental Theorem of Welfare Economics (2)

- **Definition** [16.D.1] (Price Quasi-equilibrium with Transfers)
  - Given a private ownership economy by $([X_i, \succeq_i]_{i=1}^{I}, [Y_j]_{j=1}^{J}, \bar{\omega})$. An allocation (x^*, y^*) and the price vector $p \neq 0$ constitute a **price quasi-equilibrium with transfers** if there is an assignment of wealth levels $(\mathtt{w}_1, \ldots, \mathtt{w}_I)$ with $\sum \mathtt{w}_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that
    * For each firm j, y_j^* solves the profit maximization problem, i.e.
    $$p \cdot y_j \leq p \cdot y_j^* \text{ for all } y_j \in Y_j.$$
    * For each consumer i: If $x_i \succ_i x_i^*$, then $p \cdot x_i \geq \mathtt{w}_i$.
    * Market clearing: $\sum_{i=1}^{I} x_i^* = \bar{\omega} + \sum_{j=1}^{J} y_j^*$.

Second Fundamental Theorem of Welfare Economics (3)

- With local nonsatiation the second condition becomes: If $x_i \succ x_i^*$ then $p \cdot x_i \geq p \cdot x_i^*$.

- I.e. with local non-satiation, x_i^* minimizes the expenditures given $\{x_i : \ x_i \succeq x_i^*\}$.

Second Fundamental Theorem of Welfare Economics (4)

- **Proposition** [16.D.1] (Second Fundamental Theorem of Welfare Economics)
  - Consider an economy specified by $([X_i, \succeq_i]_{i=1}^I, [Y_j]_{j=1}^J, \bar{\omega})$, and suppose that every Y_j is convex and every preference relation $\succeq_i$ is convex (the set $\{x_i \in X_i : \; x_i' \succeq_i x_i\}$ is convex for every $x_i \in X_i$) and locally non-satiated.
    Then for every Pareto optimal allocation (x^*, y^*) there exists a price vector $p \neq 0$ such that (x^*, y^*, p) is a price quasi-equilibrium with transfers.

- Proof: See MWG, page 553.

Second Fundamental Theorem of Welfare Economics (5)

- When is a price quasi-equilibrium with transfers a price equilibrium with transfers?

- The example considered in MWG, Figure 15.B.10 (a) and on page 554, is a quasi-equilibrium but not an equilibrium.

- In this example the wealth of consumer 1 is zero (hence, zero wealth problem).

- We need a sufficient condition under which which "$x_i \succ x_i^* \Rightarrow p \cdot x_i \geq \mathtt{w}_i$" implies "$x_i \succ x_i^* \Rightarrow p \cdot x_i > \mathtt{w}_i$".

Second Fundamental Theorem of Welfare Economics (6)

- **Proposition** [16.D.2]
  - Assume that X_i is convex and $\succeq_i$ is continuous. Suppose also that the consumption vector $x_i^* \in X_i$, the price vector p and the wealth level $\mathtt{w}_i$ are such that $x_i \succ_i x_i^*$ implies $p \cdot x_i \geq \mathtt{w}_i$. Then, if there is a consumption vector $x_i' \in X_i$ such that $p \cdot x_i' < \mathtt{w}_i$ [a cheaper consumption for $(p, \mathtt{w}_i)$], it follows that $x_i \succ x_i^*$ implies $p \cdot x_i > \mathtt{w}_i$.
- Proof: See MWG page 555. See also MWG, Figure 16.D.3 (right).

Second Fundamental Theorem of Welfare Economics (7)

- **Proposition** [16.D.3]
  - Suppose that for every $i = 1, \ldots, L$, X_i is convex and $\succeq_i$ is continuous. Then, any price quasi-equilibrium with transfers that has $(\mathtt{w}_1, \ldots, \mathtt{w}_L) \gg 0$ is a price equilibrium with transfers.

- Proof: See MWG page 556.

Quasiconcave Functions
Motivation (1)

- Motivation: Sufficient conditions for maximum for Kuhn Tucker problem: Suppose that there are no nonlinear equality constraints and each inequality constraint is given by a quasiconvex function. Suppose that the objective function satisfies $\nabla f(x)(x' - x) > 0$ for any x and x' with $f(x') > f(x)$. If x^* satisfied the Kuhn-Tucker conditions, then x^* is a global maximizer. (see Mas-Colell, Theorem [M.K.3]).

- Jehle, Reny: Chapter A 1.4, 1.4.

- Mas-Colell, Chapter M.C

Quasiconcave Functions
Concave Functions (1)

- Consider a convex subset A of $\mathbb{R}^n$.

- **Definition - Concave Function**: A function $f : A \to \mathbb{R}$ is concave if

$$f(\nu x' + (1-\nu)x) \geq \nu f(x') + (1-\nu) f(x)\ ,\ \ \nu \in [0,1].$$

If strict $>$ holds then f is strictly concave; $\nu \in (0,1)$ and $x \neq x'$.

This last equation can be rewritten with $z = x' - x$ and $\alpha = \nu$:

$$f(x + \alpha z) \geq \alpha f(x') + (1-\alpha) f(x)\ .$$

- If f is (strictly) concave then $-f$ is (strictly) convex.

Quasiconcave Functions
Concave Functions (2)

- **Theorem - Tangents and Concave Functions**: If f is continuously differentiable and concave, then $f(x') \leq f(x) + \nabla f(x)) \cdot (x' - x)$ (and vice versa). $<$ holds if f is strict concave for all $x \neq x'$. [Theorem M.C.1]

- For the univariate case this implies that the tangent line is above the function graph of $f(x)$; strictly for $x' \neq x$ with strict concave functions.

Proof:

- $\Rightarrow$: For $\alpha \in (0,1]$ the definition of a concave function implies:

$$f(x') = f(x+z) \leq f(x) + \frac{f(x+\alpha z) - f(x)}{\alpha}$$

If f is differentiable the limit of the last term exists such that

$$f(x+z) \leq f(x) + \nabla f(x) \cdot z$$

Proof:

- $\Leftarrow$: Suppose that $f(x+z) - f(x) \leq \nabla f(x) \cdot z$ for any non-concave function. Since $f(.)$ is not concave

$$f(x+z) - f(x) > \frac{f(x+\alpha z) - f(x)}{\alpha}$$

for some x, z and $\alpha \in (0, 1]$.

- Taking the limit results in $f(x+z) - f(x) > \nabla f(x) \cdot z$, i.e. we arrive at a contradiction.

Quasiconcave Functions
Concave Functions (5)

- **Theorem - Hessian and Concave Functions**: If f is twice continuously differentiable and concave, then the Hessian matrix $D^2 f(x)$ is negative semidefinite; negative definite for strict concave functions (and vice versa). [Theorem M.C.2]

Quasiconcave Functions
Concave Functions (6)

Proof:

- $\Rightarrow$: A Taylor expansion of $f(x')$ around the point $\alpha = 0$ results in

$$f(x+\alpha z) = f(x) + \nabla f(x)\cdot(\alpha z) + \frac{\alpha^2}{2}(z^\top \cdot D^2(f(x+\beta(\alpha)z))z)$$

  By the former theorem we know that $f(x+\alpha z) - f(x) - \nabla f(x)\cdot(\alpha z) \leq 0$ for concave functions $\Rightarrow$ $z^\top D^2(f(x+\beta(\alpha)z))z \leq 0$. For arbitrary small α we get $z^\top D^2(f(x))z \leq 0$.

Quasiconcave Functions
Concave Functions (7)

Proof:

- $\Leftarrow$: If the right hand side of $f(x+\alpha z) - f(x) - \nabla f(x) \cdot (\alpha z) = 0.5\alpha^2 (z^\top D^2(f(x+\beta(\alpha)z))z)$ is ≤ 0 then the left hand side. By the former theorem f is concave.

Quasiconcave Functions
Quasiconcave Functions (1)

- **Definition - Quasiconcave Function**: A function $f : A \to \mathbb{R}$ is quasiconcave if

$$f(\nu x' + (1-\nu)x) \geq \min\{f(x'), f(x)\} \ , \ \nu \in [0,1].$$

If $>$ holds it is said to be strict quasiconcave; $\nu \in (0,1)$ and $x \neq x'$.

- Quasiconvex is defined by $f(\nu x' + (1-\nu)x) \leq \max\{f(x'), f(x)\}$. If f is quasiconcave than $-f$ is quasiconvex.

- If f is concave then f is quasiconcave but not vice versa. E.g. $f(x) = \sqrt{x}$ for $x > 0$ is concave and also quasiconcave. x^3 is quasiconcave but not concave.

Quasiconcave Functions (2)

- Transformation property: Positive monotone transformations of quasiconcave functions result in a quasiconcave function.

- **Definition - Superior Set**: $S(x) := \{x' \in A | f(x') \geq f(x)\}$ is called superior set of x (upper contour set of x).

- Note that if $f(x^\nu) \geq \min\{f(x'), f(x'')\}$, then if $f(x') \geq t$ and $f(x'') \geq t$ this implies that $f(x^\nu) \geq t$; where $t = f(x)$.

Quasiconcave Functions (3)

- **Theorem - Quasiconcave Function and Convex Sets**: The function f is quasiconcave if and only if $S(x)$ is convex for all $x \in A$.

Quasiconcave Functions (4)

Proof:

- Sufficient condition $\Rightarrow$: If f is quasiconcave then $S(x)$ is convex. Consider x^1 and x^2 in $S(x)$. We need to show that $f(x^\nu)$ in $S(x)$; $f(x) = t$.

- Since $f(x^1) \geq t$ and $f(x^2) \geq t$, the quasiconcave f implies $f(x^\nu) \geq \min\{f(x^1), f(x^2)\} \geq t$.

- Therefore $f(x^\nu) \in S(x)$; i.e. the set $S(x)$ is convex.

Quasiconcave Functions (5)

Proof:

- Necessary condition $\Leftarrow$: If $S(x)$ is convex then $f(x)$ has to be quasiconcave. W.l.g. assume that $f(x^1) \geq f(x^2)$, x^1 and x^2 in A.

- By assumption $S(x)$ is convex, such that $S(x^2)$ is convex. Since $f(x^1) \geq f(x^2)$, we get $x^1 \in S(x^2)$ and $x^\nu \in S(x^2)$.

- From the definition of $S(x^2)$ we conclude that $f(x^\nu) \geq f(x^2) = \min\{f(x^1), f(x^2)\}$.

- Therefore $f(x)$ has to be quasiconcave.

Quasiconcave Functions (6)

- **Theorem - Gradients and Quasiconcave Functions**: If f is continuously differentiable and quasiconcave, then $\nabla f(x) \cdot (x' - x) \geq 0$ whenever $f(x') \geq f(x)$ (and vice versa). [Theorem M.C.3]

- If $\nabla f(x) \cdot (x' - x) > 0$ whenever $f(x') \geq f(x)$ and $x \neq x'$ then $f(x)$ is strictly quasiconcave. If $f(x)$ is strictly quasiconcave and if $\nabla f(x) \neq 0$ for all $x \in A$, then $\nabla f(x) \cdot (x' - x) > 0$ whenever $f(x') \geq f(x)$ and $x \neq x'$.

Quasiconcave Functions
Quasiconcave Functions (7)

Proof:

- $\Rightarrow$: For $f(x') \geq f(x)$ and $\alpha \in (0,1]$ the definition of a quasiconcave function implies:

$$\frac{f(x+\alpha(x'-x)) - f(x)}{\alpha} \geq 0$$

If f is differentiable, then the limit exists such that

$$\nabla f(x) \cdot z \geq 0$$

Quasiconcave Functions
Quasiconcave Functions (8)

Proof:

- $\Leftarrow$: Suppose that $\nabla f(x) \cdot z \geq 0$ holds but f is not quasiconcave. Then $f(x + \alpha z) - f(x) < 0$ for some x, z and $\alpha \in (0, 1]$. Such that $(f(x + \alpha z) - f(x))/\alpha < 0$. Taking the limit results in a contradiction.

Quasiconcave Functions (9)

- **Theorem - Hessian Matrix and Quasiconcave Functions**: Suppose f is twice continuously differentiable. $f(x)$ is quasiconcave if and only if $D^2(f(x))$ is negative semidefinite in the subspace $\{z|\nabla f(x) \cdot z = 0\}$. I.e. $z^\top D^2(f(x))z \leq 0$ whenever $\nabla f(x) \cdot z = 0$. [Theorem M.C.4]

- If the Hessian $D^2(f(x))$ is negative definite in the subspace $\{z|\nabla f(x) \cdot z = 0\}$ for every $x \in A$ then $f(x)$ is strictly quasiconcave.

Quasiconcave Functions (10)

Proof:

- $\Rightarrow$: If f is quasiconcave then whenever $f(x^\nu) \geq f(x)$, so $\nabla f(x) \cdot (\alpha z) \geq 0$ has to hold.

- Thus $f(x^1) - f(x) \leq 0$ and the above theorem imply: $\nabla f(x) \cdot (z) \leq 0$, where $z = x^1 - x$.

- A first order Taylor series expansion of f in α (at $\alpha = 0$) results in

$$f(x + \alpha z) = f(x) + \nabla f(x) \alpha z + \frac{\alpha^2}{2} \cdot \left(z^\top D^2 f(x + \beta(\alpha) z) z \right) .$$

Quasiconcave Functions (11)

Proof:

- Apply this to x^1, x with $f(x^1) \leq f(x)$:

$$f(x + \alpha z) - f(x) - \nabla f(x)\alpha z = \frac{\alpha^2}{2} \cdot z^\top D^2 f(x + \beta(\alpha)z)z.$$

- If $z = x^1 - x$ fulfills $\nabla f(x)(x^1 - x) = 0$ the above inequality still has to hold.

- This implies $\alpha^2/2z^\top D^2 f(x + \beta(\alpha)z)z \leq 0$.

Quasiconcave Functions (12)

Proof:

- To fulfill this requirement on the subspace $\{z|\nabla f(x) \cdot z = 0\}$, where $\nabla f(x) \alpha z = 0$, this requires a negative definite Hessian of $f(x)$.

- $\Leftarrow$: In the above equation a negative semidefinite Hessian implies that . . . .

Envelope Theorem (1)

- Consider $f(x;q)$, x are variables in $\mathbb{R}^N$ and q are parameters in $\mathbb{R}^S$.

- We look at the constrained maximization problem

$$\max_x \; f(x;q) \; s.t. g_m(x;q) \leq b_m$$

  $m = 1, \ldots, M$.

- Assume that the solution of this optimization problem $x^* = x(q)$ is at least locally differentiable function (in a neighborhood of a $\bar{q}$ considered).

- $v(q) = f(x(q);q)$ is the maximum value function associated with this problem.

Envelope Theorem (2)

- With no constraints $(M = 0)$ and $S, N = 1$ the chain rule yields:

$$\frac{d}{dq}v(\bar{q}) = \frac{\partial f(x(\bar{q});\bar{q})}{\partial x}\frac{\partial x(\bar{q})}{\partial q} + \frac{\partial f(x(\bar{q});\bar{q})}{\partial q}.$$

- With an unconstrained maximization problem the first order condition $\frac{\partial f(x(\bar{q});\bar{q})}{\partial x} = 0$ results in

$$\frac{d}{dq}v(\bar{q}) = \frac{\partial f(x(\bar{q});\bar{q})}{\partial q}.$$

Envelope Theorem (3)

[T. M.L.1] Consider the value function $v(q)$ for the above constrained maximization problem. Assume that $v(q)$ is differentiable at $\bar{q} \in \mathbb{R}^S$ and $(\lambda_1, \ldots, \lambda_M)$ are the Lagrange multipliers associated with the maximizer solution $x(q)$ at $\bar{q}$. In addition the inequality constraints are remain unaltered in a neighborhood of $\bar{q}$. Then

$$\frac{\partial v(\bar{q})}{\partial q_s} = \frac{\partial f(x(\bar{q}); \bar{q})}{\partial q_s} - \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q}); \bar{q})}{\partial q_s}.$$

For $s = 1, \ldots, S$.

Envelope Theorem (4)

Proof:

- Let $x(.)$ stand for the maximizer of the function $f(.)$ and $v(q) = f(x(q), q)$ for all q. The chain rule yields:

$$\frac{\partial v(\bar{q})}{dq_s} = \sum_{n=1}^{N} \frac{\partial f(x(\bar{q}); \bar{q})}{\partial x_n} \frac{\partial x_n(\bar{q})}{\partial q_s} + \frac{\partial f(x(\bar{q}); \bar{q})}{\partial q_s}.$$

- The first order conditions tell us

$$\frac{\partial f(x(\bar{q}); \bar{q})}{\partial x_n} = \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q}); \bar{q})}{\partial x_n}.$$

Envelope Theorem (5)

Proof:

- In addition we observe

$$\sum_{n=1}^{N} \frac{\partial g_m(x(\bar{q}); \bar{q})}{\partial x_n} \frac{\partial x_n(\bar{q})}{\partial q_s} + \frac{\partial g_m(\bar{q})}{\partial q_s} = 0.$$

  if a constraint is binding; if not the multiplier λ_m is zero.

Envelope Theorem (6)

Proof:

- Plugging in and changing the order of summation results in :

$$\frac{\partial v(\bar{q})}{dq_s} = \sum_{m=1}^{M} \lambda_m \sum_{n=1}^{N} \frac{\partial g_m(x(\bar{q});\bar{q})}{\partial x_n} \frac{\partial x_n(\bar{q})}{\partial q_s} + \frac{\partial f(x(\bar{q});\bar{q})}{\partial q_s}.$$

- and

$$\frac{\partial v(\bar{q})}{dq_s} = -\sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q});\bar{q})}{\partial q_s} + \frac{\partial f(x(\bar{q});\bar{q})}{\partial q_s}.$$

- Remark: remember that the Lagrangian of the problem is $L(x,\lambda;q) = f(x;q) - \sum_m \lambda_m g_m(x;q)$. Hence we get $\frac{\partial v(\bar{q})}{dq_s}$ by means of the partial derivative of the Lagrangian with respect to q_l, evaluated at $\bar{q}$.

Quasiconcave Functions
Motivation (1)

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Proof:

- $\Rightarrow$: For $\alpha \in (0, 1]$ the definition of a concave function implies:

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Concave Functions (4)

Proof:

- $\Leftarrow$: Suppose that $f(x+z) - f(x) \leq \nabla f(x) \cdot z$ for any non-concave function. Since $f(.)$ is not concave

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Concave Functions (6)

Proof:

- $\Rightarrow$: A Taylor expansion of $f(x')$ around the point $\alpha = 0$ results in

$$f(x + \alpha z) = f(x) + \nabla f(x) \cdot (\alpha z) + \frac{\alpha^2}{2}(z^\top \cdot D^2(f(x + \beta(\alpha)z))z)$$

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Quasiconcave Functions
Concave Functions (7)

Proof:

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- If f is concave then f is quasiconcave but not vice versa. E.g. $f(x) = \sqrt{x}$ for $x > 0$ is concave and also quasiconcave. x^3 is quasiconcave but not concave.

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Quasiconcave Functions (2)

- Transformation property: Positive monotone transformations of quasiconcave functions result in a quasiconcave function.

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- Note that if $f(x^\nu) \geq \min\{f(x'), f(x'')\}$, then if $f(x') \geq t$ and $f(x'') \geq t$ this implies that $f(x^\nu) \geq t$; where $t = f(x)$.

Quasiconcave Functions (3)

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Proof:

- Sufficient condition $\Rightarrow$: If f is quasiconcave then $S(x)$ is convex. Consider x^1 and x^2 in $S(x)$. We need to show that $f(x^\nu)$ in $S(x)$; $f(x) = t$.

- Since $f(x^1) \geq t$ and $f(x^2) \geq t$, the quasiconcave f implies $f(x^\nu) \geq \min\{f(x^1), f(x^2)\} \geq t$.

- Therefore $f(x^\nu) \in S(x)$; i.e. the set $S(x)$ is convex.

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Proof:

- Necessary condition $\Leftarrow$: If $S(x)$ is convex then $f(x)$ has to be quasiconcave. W.l.g. assume that $f(x^1) \geq f(x^2)$, x^1 and x^2 in A.

- By assumption $S(x)$ is convex, such that $S(x^2)$ is convex. Since $f(x^1) \geq f(x^2)$, we get $x^1 \in S(x^2)$ and $x^\nu \in S(x^2)$.

- From the definition of $S(x^2)$ we conclude that $f(x^\nu) \geq f(x^2) = \min\{f(x^1), f(x^2)\}$.

- Therefore $f(x)$ has to be quasiconcave.

Quasiconcave Functions
Quasiconcave Functions (6)

- **Theorem - Gradients and Quasiconcave Functions**: If f is continuously differentiable and quasiconcave, then $\nabla f(x) \cdot (x' - x) \geq 0$ whenever $f(x') \geq f(x)$ (and vice versa). [Theorem M.C.3]

- If $\nabla f(x) \cdot (x' - x) > 0$ whenever $f(x') \geq f(x)$ and $x \neq x'$ then $f(x)$ is strictly quasiconcave. If $f(x)$ is strictly quasiconcave and if $\nabla f(x) \neq 0$ for all $x \in A$, then $\nabla f(x) \cdot (x' - x) > 0$ whenever $f(x') \geq f(x)$ and $x \neq x'$.

Quasiconcave Functions (7)

Proof:

- $\Rightarrow$: For $f(x') \geq f(x)$ and $\alpha \in (0,1]$ the definition of a quasiconcave function implies:

$$\frac{f(x+\alpha(x'-x))-f(x)}{\alpha} \geq 0$$

If f is differentiable, then the limit exists such that

$$\nabla f(x) \cdot z \geq 0$$

Quasiconcave Functions
Quasiconcave Functions (8)

Proof:

- $\Leftarrow$: Suppose that $\nabla f(x) \cdot z \geq 0$ holds but f is not quasiconcave. Then $f(x + \alpha z) - f(x) < 0$ for some x, z and $\alpha \in (0, 1]$. Such that $(f(x + \alpha z) - f(x))/\alpha < 0$. Taking the limit results in a contradiction.

Quasiconcave Functions
Quasiconcave Functions (9)

- **Theorem - Hessian Matrix and Quasiconcave Functions**: Suppose f is twice continuously differentiable. $f(x)$ is quasiconcave if and only if $D^2(f(x))$ is negative semidefinite in the subspace $\{z|\nabla f(x) \cdot z = 0\}$. I.e. $z^\top D^2(f(x))z \leq 0$ whenever $\nabla f(x) \cdot z = 0$. [Theorem M.C.4]

- If the Hessian $D^2(f(x))$ is negative definite in the subspace $\{z|\nabla f(x) \cdot z = 0\}$ for every $x \in A$ then $f(x)$ is strictly quasiconcave.

Quasiconcave Functions (10)

Proof:

- $\Rightarrow$: If f is quasiconcave then whenever $f(x^\nu) \geq f(x)$, so $\nabla f(x) \cdot (\alpha z) \geq 0$ has to hold.

- Thus $f(x^1) - f(x) \leq 0$ and the above theorem imply: $\nabla f(x) \cdot (z) \leq 0$, where $z = x^1 - x$.

- A first order Taylor series expansion of f in α (at $\alpha = 0$) results in

$$f(x + \alpha z) = f(x) + \nabla f(x)\alpha z + \frac{\alpha^2}{2} \cdot \left(z^\top D^2 f(x + \beta(\alpha)z)z\right).$$

Quasiconcave Functions (11)

Proof:

- Apply this to x^1, x with $f(x^1) \leq f(x)$:

$$f(x + \alpha z) - f(x) - \nabla f(x)\alpha z = \frac{\alpha^2}{2} \cdot z^\top D^2 f(x + \beta(\alpha)z)z.$$

- If $z = x^1 - x$ fulfills $\nabla f(x)(x^1 - x) = 0$ the above inequality still has to hold.
- This implies $\alpha^2/2z^\top D^2 f(x + \beta(\alpha)z)z \leq 0$.

Quasiconcave Functions (12)

Proof:

- To fulfill this requirement on the subspace $\{z|\nabla f(x) \cdot z = 0\}$, where $\nabla f(x)\alpha z = 0$, this requires a negative definite Hessian of $f(x)$.

- $\Leftarrow$: In the above equation a negative semidefinite Hessian implies that . . . .

Envelope Theorem (1)

- Consider $f(x;q)$, x are variables in $\mathbb{R}^N$ and q are parameters in $\mathbb{R}^S$.
- We look at the constrained maximization problem

$$\max_x \; f(x;q) \; s.t. g_m(x;q) \leq b_m$$

$m = 1, \ldots, M.$

- Assume that the solution of this optimization problem $x^* = x(q)$ is at least locally differentiable function (in a neighborhood of a $\bar{q}$ considered).
- $v(q) = f(x(q);q)$ is the maximum value function associated with this problem.

Envelope Theorem (2)

- With no constraints ($M = 0$) and $S, N = 1$ the chain rule yields:

$$\frac{d}{dq}v(\bar{q}) = \frac{\partial f(x(\bar{q});\bar{q})}{\partial x}\frac{\partial x(\bar{q})}{\partial q} + \frac{\partial f(x(\bar{q});\bar{q})}{\partial q}.$$

- With an unconstrained maximization problem the first order condition $\frac{\partial f(x(\bar{q});\bar{q})}{\partial x} = 0$ results in

$$\frac{d}{dq}v(\bar{q}) = \frac{\partial f(x(\bar{q});\bar{q})}{\partial q}.$$

Envelope Theorem (3)

[T. M.L.1] Consider the value function $v(q)$ for the above constrained maximization problem. Assume that $v(q)$ is differentiable at $\bar{q} \in \mathbb{R}^S$ and $(\lambda_1, \ldots, \lambda_M)$ are the Lagrange multipliers associated with the maximizer solution $x(q)$ at $\bar{q}$. In addition the inequality constraints are remain unaltered in a neighborhood of $\bar{q}$. Then

$$\frac{\partial v(\bar{q})}{\partial q_s} = \frac{\partial f(x(\bar{q}); \bar{q})}{\partial q_s} - \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q}); \bar{q})}{\partial q_s}.$$

For $s = 1, \ldots, S$.

Envelope Theorem (4)

Proof:

- Let $x(.)$ stand for the maximizer of the function $f(.)$ and $v(q) = f(x(q), q)$ for all q. The chain rule yields:

$$\frac{\partial v(\bar{q})}{dq_s} = \sum_{n=1}^{N} \frac{\partial f(x(\bar{q});\bar{q})}{\partial x_n}\frac{\partial x_n(\bar{q})}{\partial q_s} + \frac{\partial f(x(\bar{q});\bar{q})}{\partial q_s}.$$

- The first order conditions tell us

$$\frac{\partial f(x(\bar{q});\bar{q})}{\partial x_n} = \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q});\bar{q})}{\partial x_n}.$$

Envelope Theorem (5)

Proof:

- In addition we observe

$$\sum_{n=1}^{N} \frac{\partial g_m(x(\bar{q});\bar{q})}{\partial x_n} \frac{\partial x_n(\bar{q})}{\partial q_s} + \frac{\partial g_m(\bar{q})}{\partial q_s} = 0.$$

if a constraint is binding; if not the multiplier λ_m is zero.

Envelope Theorem (6)

Proof:

- Plugging in and changing the order of summation results in :

$$\frac{\partial v(\bar{q})}{dq_s} = \sum_{m=1}^{M} \lambda_m \sum_{n=1}^{N} \frac{\partial g_m(x(\bar{q});\bar{q})}{\partial x_n} \frac{\partial x_n(\bar{q})}{\partial q_s} + \frac{\partial f(x(\bar{q});\bar{q})}{\partial q_s}.$$

- and

$$\frac{\partial v(\bar{q})}{dq_s} = -\sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q});\bar{q})}{\partial q_s} + \frac{\partial f(x(\bar{q});\bar{q})}{\partial q_s}.$$

- Remark: remember that the Lagrangian of the problem is $L(x,\lambda;q) = f(x;q) - \sum_m \lambda_m g_m(x;q)$. Hence we get $\frac{\partial v(\bar{q})}{dq_s}$ by means of the partial derivative of the Lagrangian with respect to q_l, evaluated at $\bar{q}$.