Econometrics - Lecture 5

Autocorrelation, IV Estimator

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- Tests against Autocorrelation
- Inference under Autocorrelation
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- An Example
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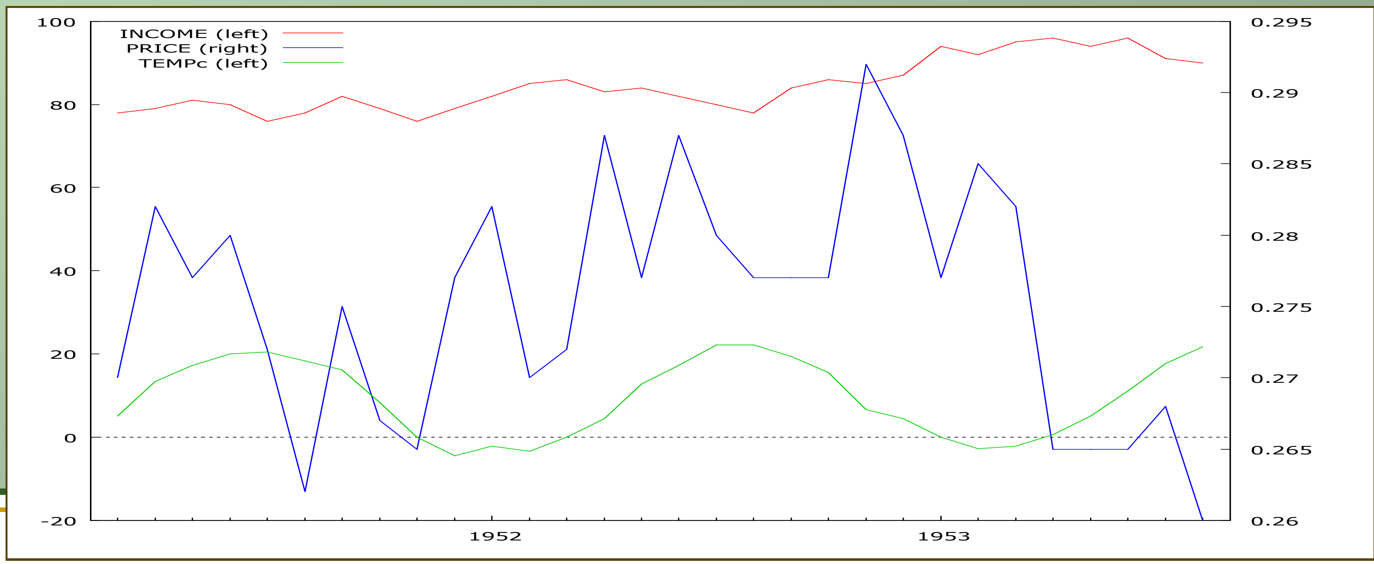
Example: Demand for Ice Cream

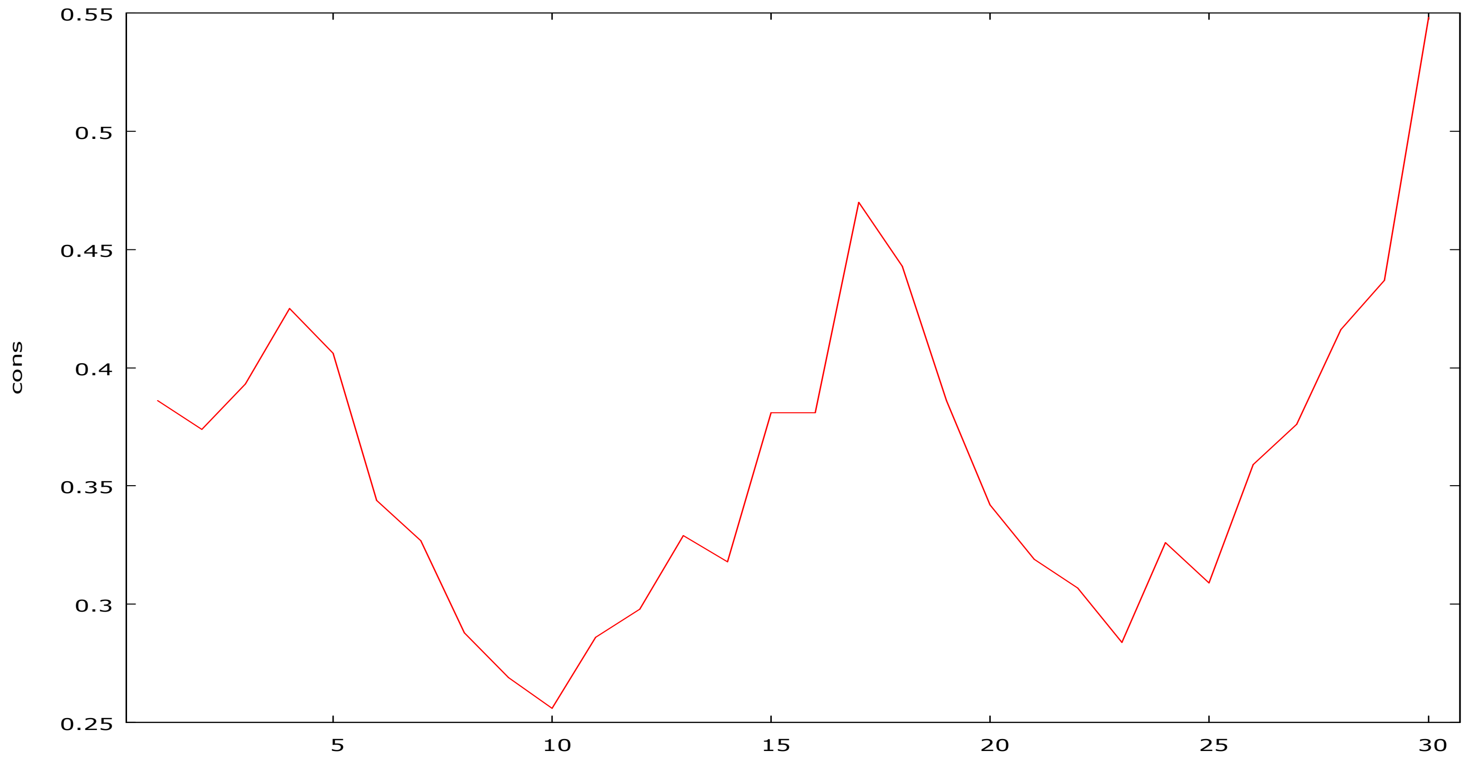
Verbeek's time series dataset "icecream"

- 30 four weekly observations (1951-1953)
- Variables
  - *cons*: consumption of ice cream per head (in pints)
  - *income*: average family income per week (in USD, red line)
  - *price*: price of ice cream (in USD per pint, blue line)
  - *temp*: average temperature (in Fahrenheit); tempc: (green, in °C)



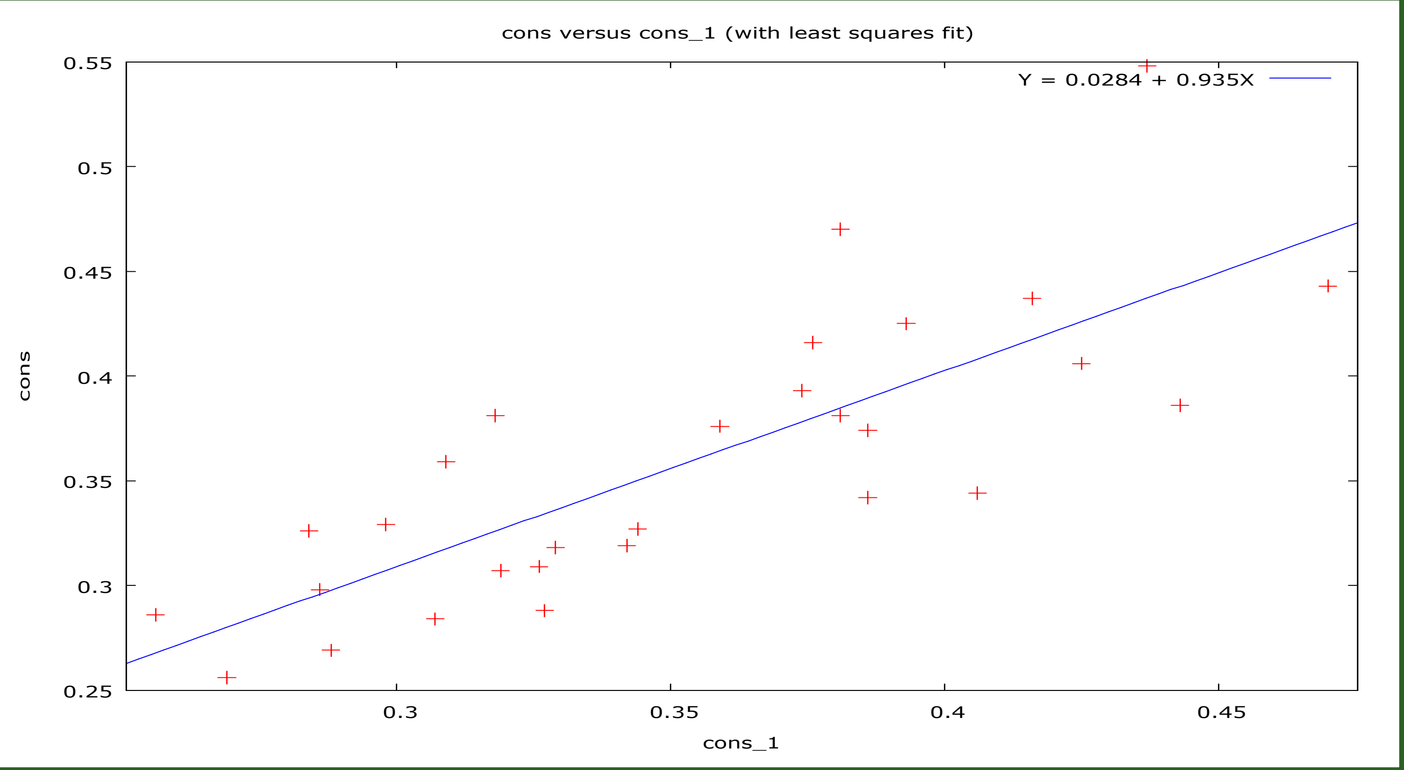
Demand for Ice Cream, cont'd

Time series plot of consumption of ice cream per head (in pints), *cons*, over observation periods



Demand for Ice Cream, cont'd

Consumption (*cons*) of ice cream per head (in pints): scatter diagramme of actual values *cons* over lagged values $cons_{-1}$



Autocorrelation

- Typical for time series data such as consumption, production, investments, etc.
- Autocorrelation of error terms is typically observed if
  - a relevant regressor with trend or seasonal pattern is not included in the model: miss-specified model
  - the functional form of a regressor is incorrectly specified
  - the dependent variable is correlated in a way that is not appropriately represented in the systematic part of the model
- Autocorrelation of the error terms indicates deficiencies of the model specification such as omitted regressors, incorrect functional form, incorrect dynamic
- Tests for autocorrelation are the most frequently used tool for diagnostic checking the model specification

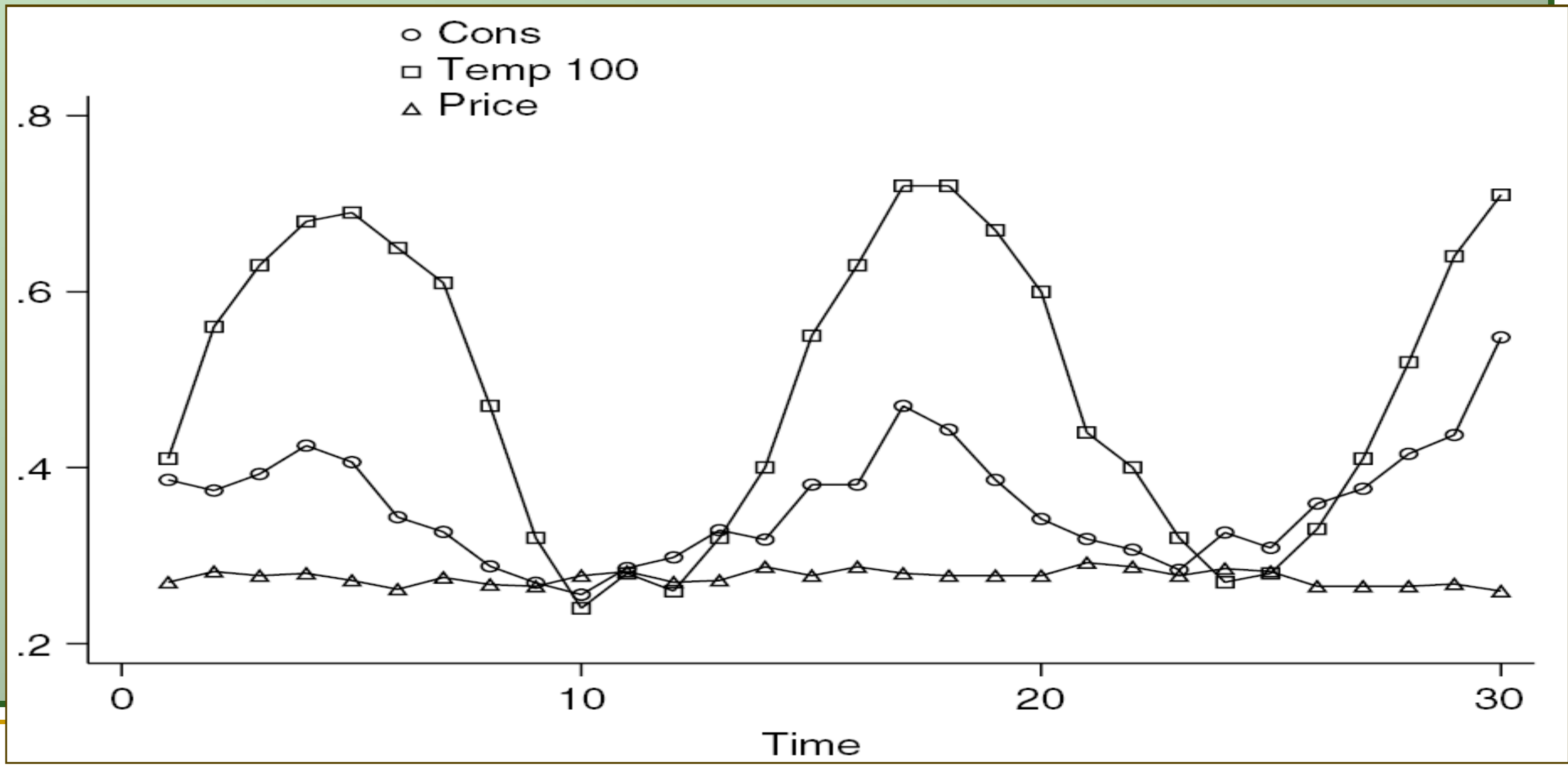
Demand for Ice Cream, cont'd

Time series plot of

Cons: consumption of ice cream per head (in pints); mean: 0.36

Temp/100: average temperature (in Fahrenheit)

Price (in USD per pint); mean: 0.275 USD



Demand for Ice Cream, cont'd

Demand for ice cream, measured by *cons*, explained by *price*, *income*, and *temp*

Table 4.9 OLS results

Dependent variable: *cons*

Variable	Estimate	Standard error	t-ratio
constant	0.197	0.270	0.730
price	−1.044	0.834	−1.252
income	0.00331	0.00117	2.824
temp	0.00345	0.00045	7.762

$s = 0.0368$ $R^2 = 0.7190$ $\bar{R}^2 = 0.6866$ $F = 22.175$
$dw = 1.0212$

Demand for Ice Cream, cont'd

Time series diagramme of demand for ice cream, actual values (o) and predictions (polygon), based on the model with income and price

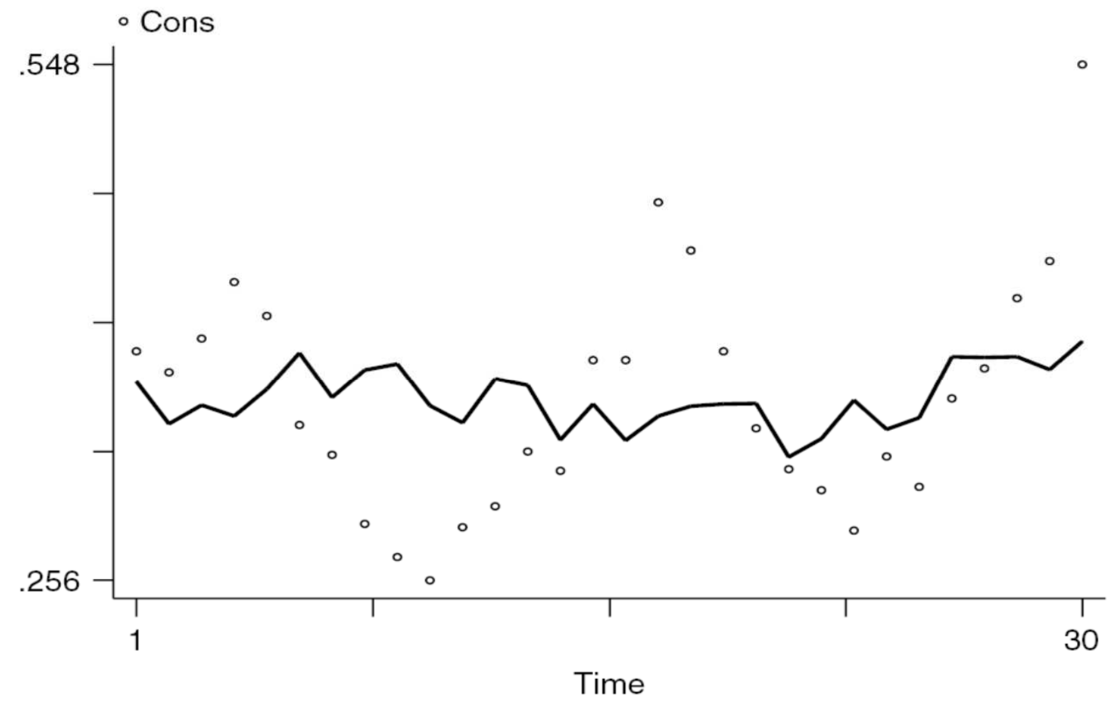
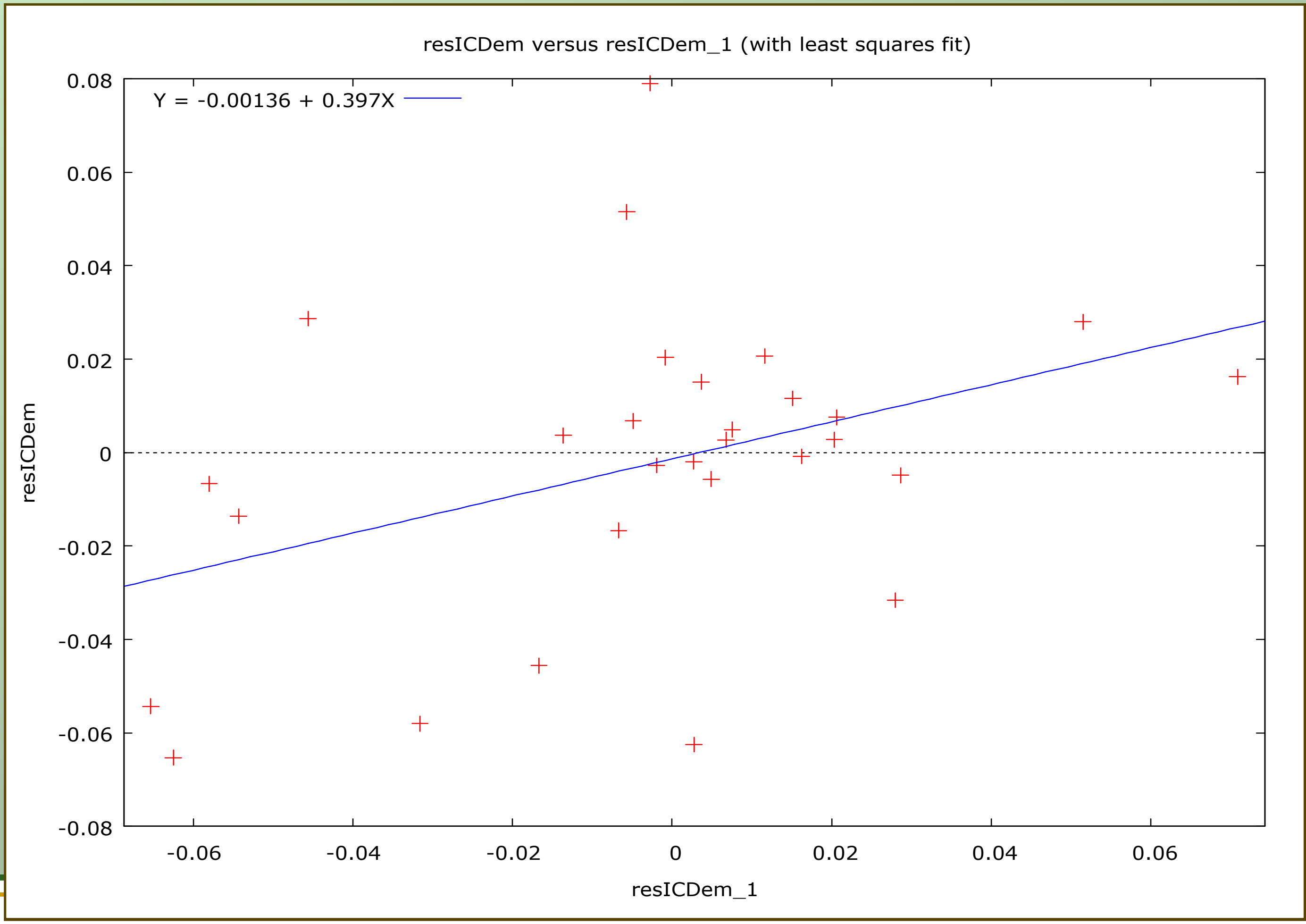


Figure 4.1 Actual and fitted consumption of ice cream, March 1951–July 1953

Demand for Ice Cream, cont'd

Ice cream model: Scatter-plot of residuals e_t vs e_{t-1} (r = 0.401)



A Model with AR(1) Errors

Linear regression

$$y_t = x_t'\beta + \varepsilon_t \; ^{1)}$$

with

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t \text{ with } -1 < \rho < 1 \text{ or } |\rho| < 1$$

where v_t are uncorrelated random variables with mean zero and constant variance σ_v^2

- For $\rho \neq 0$, the error terms ε_t are correlated; the Gauss-Markov assumption $V\{\varepsilon\} = \sigma_\varepsilon^2 I_N$ is violated
- The other Gauss-Markov assumptions are assumed to be fulfilled

The sequence ε_t, t = 0, 1, 2, … which follows $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$ is called an autoregressive process of order 1 or AR(1) process

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1) In the context of time series models, variables are indexed by „*t*"

Properties of AR(1) Processes

Repeated substitution of ε_{t-1}, ε_{t-2}, etc. results in

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t = v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \ldots$$

with v_t being uncorrelated and having mean zero and variance σ_v^2:

- $E\{\varepsilon_t\} = 0$
- $V\{\varepsilon_t\} = \sigma_\varepsilon^2 = \sigma_v^2(1-\rho^2)^{-1}$

This results from $V\{\varepsilon_t\} = \sigma_v^2 + \rho^2\sigma_v^2 + \rho^4\sigma_v^2 + \ldots = \sigma_v^2(1-\rho^2)^{-1}$ for $|\rho|<1$; the geometric series $1 + \rho^2 + \rho^4 + \ldots$ has the sum $(1- \rho^2)^{-1}$ given that $|\rho| < 1$

  - for $|\rho| > 1$, $V\{\varepsilon_t\}$ is undefined

- $Cov\{\varepsilon_t, \varepsilon_{t-s}\} = \rho^s \sigma_v^2 (1-\rho^2)^{-1}$ for $s > 0$

  all error terms are correlated; covariances – and correlations $Corr\{\varepsilon_t, \varepsilon_{t-s}\} = \rho^s (1-\rho^2)^{-1}$ – decrease with growing distance *s* in time

AR(1) Process, cont'd

The covariance matrix V{ε}:

$$V\{\varepsilon\} = \sigma_v^2 \Psi = \frac{\sigma_v^2}{1-\rho^2} \begin{pmatrix} 1 & \rho & \cdots & \rho^{N-1} \\ \rho & 1 & \cdots & \rho^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \cdots & 1 \end{pmatrix}$$

- V{ε} has a band structure
- Depends only of two parameters: ρ and σ_v^2

Consequences of $V\{\varepsilon\} \neq \sigma^2 I_T$

OLS estimators *b* for β

- are unbiased
- are consistent
- have the covariance-matrix

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

- are not efficient estimators, not BLUE
- follow – under general conditions – asymptotically the normal distribution

The estimator $s^2 = e'e/(T-K)$ for σ^2 is biased

For an AR(1)-process ε_t with $\rho > 0$, s.e. from $\sigma^2 (X'X)^{-1}$ underestimates the true s.e.

Inference in Case of Autocorrelation

Covariance matrix of *b*:

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

Use of $\sigma^2 (X'X)^{-1}$ (the standard output of econometric software) instead of $V\{b\}$ for inference on β may be misleading

Identification of autocorrelation:

- Statistical tests, e.g., Durbin-Watson test

Remedies

- Use of correct variances and standard errors
- Transformation of the model so that the error terms are uncorrelated

Estimation of ρ

Autocorrelation coefficient ρ: parameter of the AR(1) process

$\varepsilon_t = \rho\varepsilon_{t-1} + v_t$

Estimation of ρ

- by regressing the OLS residual e_t on the lagged residual e_{t-1}

$$r = \frac{\sum_{t=2}^{T} e_t e_{t-1}}{(T-K)s^2}$$

- estimator is
  - biased
  - but consistent under weak conditions

Autocorrelation Function

Autocorrelation of order *s*:

$$r_s = \frac{\sum_{t=s+1}^{T} e_t e_{t-s}}{(T-k)s^2}$$

- Autocorrelation function (ACF) assigns r_s to *s*
- Correlogram: graphical representation of the autocorrelation function

GRETL: Variable => Correlogram

Produces (a) the autocorrelation function (ACF) and (b) the graphical representation of the ACF (and the partial autocorrelation function)

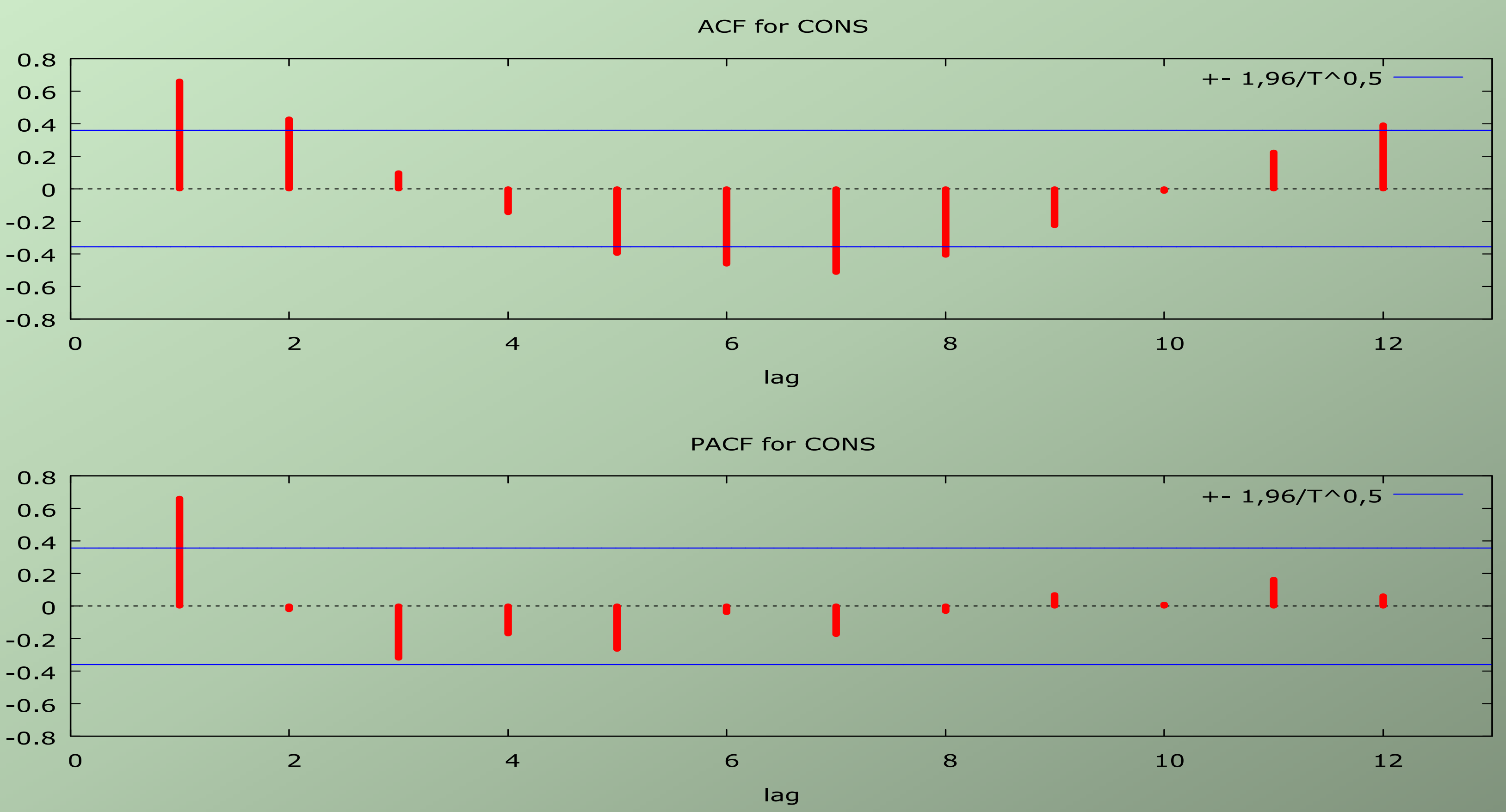
Example: Ice Cream Demand

Autocorrelation function (ACF) of *cons*

LAG	ACF		PACF		Q-stat. [p-value]
1	0,6627	***	0,6627	***	14,5389 [0,000]
2	0,4283	**	-0,0195		20,8275 [0,000]
3	0,0982		-0,3179	*	21,1706 [0,000]
4	-0,1470		-0,1701		21,9685 [0,000]
5	-0,3968	**	-0,2630		28,0152 [0,000]
6	-0,4623	**	-0,0398		36,5628 [0,000]
7	-0,5145	***	-0,1735		47,6132 [0,000]
8	-0,4068	**	-0,0299		54,8362 [0,000]
9	-0,2271		0,0711		57,1929 [0,000]
10	-0,0156		0,0117		57,2047 [0,000]
11	0,2237		0,1666		59,7335 [0,000]
12	0,3912	**	0,0645		67,8959 [0,000]

Example: Ice Cream Demand

Correlogram of *cons*



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Tests for Autocorrelation of Error Terms

Due to unbiasedness of *b*, residuals are expected to indicate autocorrelation

Graphical displays, e.g., the correlogram of residuals may give useful hints

Residual-based tests:

- Durbin-Watson test
- Box-Pierce test
- Breusch-Godfrey test

Durbin-Watson Test

Test of H_0: $\rho = 0$ against H_1: $\rho \neq 0$

Test statistic

$$dw = \frac{\sum_{t=2}^{T} (e_t - e_{t-1})^2}{\sum_{t=1}^{T} e_t^2} \approx 2(1-r)$$

- For $\rho > 0$, *dw* is expected to have a value in (0,2)
- For $\rho < 0$, *dw* is expected to have a value in (2,4)
- *dw* close to the value 2 indicates no autocorrelation of error terms
- Critical limits of *dw*
  - depend upon x_t's
  - exact critical value is unknown, but upper and lower bounds can be derived, which depend upon x_t's only via the number of regression coefficients
- Test can be inconclusive
- H_1: $\rho > 0$ may be more appropriate than H_1: $\rho \neq 0$

Durbin-Watson Test: Bounds for Critical Limits

Derived by Durbin and Watson

Upper (d_U) and lower (d_L) bounds for the critical limits and $\alpha = 0.05$

T	*K*=2		*K*=3		*K*=10	
	d_L	d_U	d_L	d_U	d_L	d_U
15	1.08	1.36	0.95	1.54	0.17	3.22
20	1.20	1.41	1.10	1.54	0.42	2.70
100	1.65	1.69	1.63	1.71	1.48	1.87

- $dw < d_L$: reject H_0
- $dw > d_U$: do not reject H_0
- $d_L < dw < d_U$: no decision (inconclusive region)

Durbin-Watson Test: Remarks

- Durbin-Watson test gives no indication of causes for the rejection of the null hypothesis and how the model to modify
- Various types of misspecification may cause the rejection of the null hypothesis
- Durbin-Watson test is a test against first-order autocorrelation; a test against autocorrelation of other orders may be more suitable, e.g., order four if the model is for quarterly data
- Use of tables unwieldy
  - Limited number of critical bounds (K, T, α) in tables
  - Inconclusive region
- **GRETL**: Standard output of the OLS estimation reports the Durbin-Watson statistic; to see the *p*-value:
  - OLS output => Tests => Durbin-Watson *p*-value

Asymptotic Tests

AR(1) process for error terms

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t$$

Auxiliary regression of e_t on (an intercept,) x_t and e_{t-1}: produces

- R_e^2

Test of H_0: $\rho = 0$ against H_1: $\rho > 0$ or H_1: $\rho \neq 0$

1. Breusch-Godfrey test (**GRETL**: OLS output => Tests => Autocorr.)
   - R_e^2 of the auxiliary regression: close to zero if $\rho = 0$
   - Under H_0: $\rho = 0$, $(T-1) R_e^2$ follows approximately the Chi-squared distribution with 1 d.f.
   - Lagrange multiplier *F* (LMF) statistic: *F*-test for explanatory power of e_{t-1}; follows approximately the $F(1, T-K-1)$ distribution if $\rho = 0$
   - General case of the Breusch-Godfrey test: Auxiliary regression based on higher order autoregressive process

Asymptotic Tests, cont'd

2. Similar the Ljung-Box test, based on

   $$Q^{LB} = T\,(T+2)\, \Sigma_s^m\, r_s^2/(T-s)$$

   with correlations r_s between e_t and e_{t-s}; Q^{LB} follows the Chi-squared distribution with m d.f. if $\rho = 0$
3. Box-Pierce test
   - The t-statistic based on the OLS estimate r of ρ from $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$,

     $$t = \sqrt{(T)}\, r$$

     follows approximately the t-distribution, $t^2 = T\, r^2$ the Chi-squared distribution with 1 d.f. if $\rho = 0$
   - Test based on $\sqrt{(T)}\, r$ is a special case of the Box-Pierce test which uses the test statistic $Q_m = T\, \Sigma_s^m\, r_s^2$

Asymptotic Tests, cont'd

GRETL:

- OLS output => Tests => Autocorrelation (shows the Breusch-Godfrey LMF statistic, the Box-Pierce statistic, and the Ljung-Box statistic as well as *p*-values)
- OLS output => Graphs => Residual correlogram (shows – besides the correlogram of the residuals – Ljung-Box statistic and *p*-value)

Remarks

- If the model of interest contains lagged values of *y* the auxiliary regression should also include all explanatory variables (just to make sure the distribution of the test is correct)
- If heteroskedasticity is suspected, White standard errors may be used in the auxiliary regression

Demand for Ice Cream, cont'd

OLS estimated demand function: Output from **GRETL**

Dependent variable : CONS

	coefficient	std. error	t-ratio	p-value
const	0.197315	0.270216	0.7302	0.4718
INCOME	0.00330776	0.00117142	2.824	0.0090 ***
PRICE	-1.04441	0.834357	-1.252	0.2218
TEMP	0.00345843	0.000445547	7.762	3.10e-08 ***

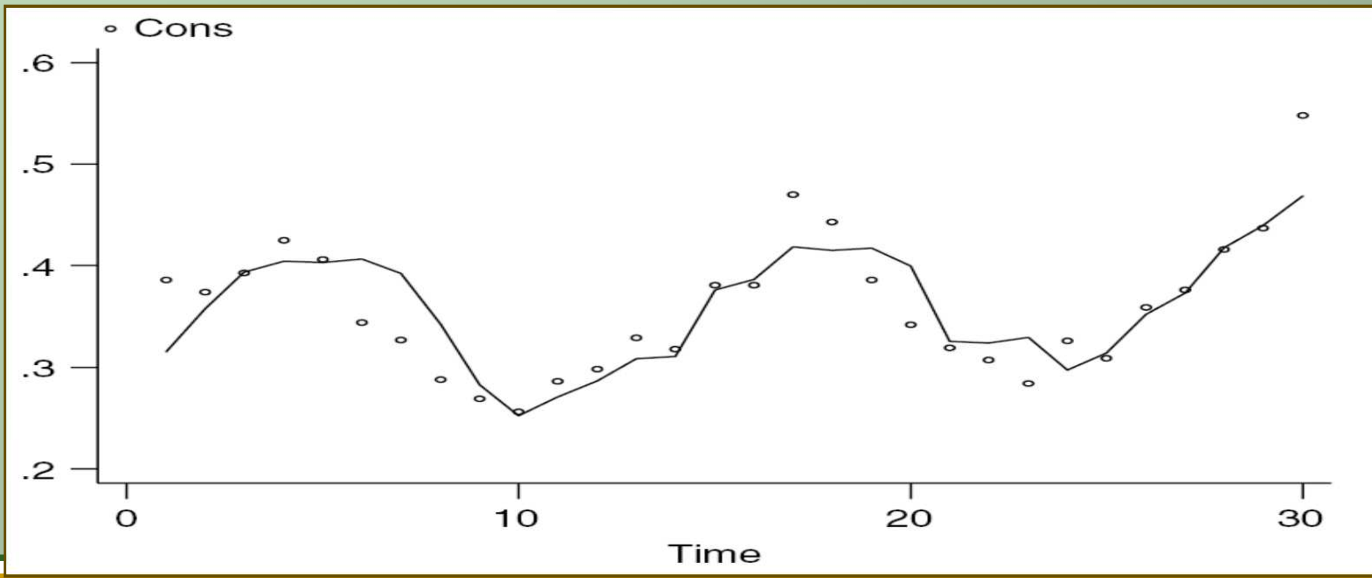
Mean dependent var	0.359433	S.D. dependent var	0,065791
Sum squared resid	0,035273	S.E. of regression	0,036833
R- squared	0,718994	Adjusted R-squared	0,686570
F(2, 129)	22,17489	P-value (F)	2,45e-07
Log-likelihood	58,61944	Akaike criterion	-109,2389
Schwarz criterion	-103,6341	Hannan-Quinn	-107,4459
rho	0,400633	Durbin-Watson	1,021170

Demand for Ice Cream, cont'd

Test for autocorrelation of error terms

- H_0: ρ = 0, H_1: ρ ≠ 0
- *dw* = 1.02 < 1.21 = d_L for *T* = 30, *K* = 4; *p* = 0.0003 (in GRETL: 0.0003025); reject H_0
- **GRETL** also shows the autocorrelation coefficient: *r* = 0.401

Plot of actual (o) and fitted (polygon) values



Demand for Ice Cream, cont'd

Auxiliary regression $\varepsilon_t = x_t'\beta + \rho\varepsilon_{t-1} + v_t$: OLS estimation gives

$r = 0.401$, $R^2 = 0.141$

Test of H_0: $\rho = 0$ against H_1: $\rho > 0$

1. Breusch-Godfrey test: LMF = 4.11, *p*-value: 0.053
2. Box-Pierce test: $t^2 = 4.237$, *p*-value: 0.040
3. Ljung-Box test: $Q^{LB} = 3.6$, *p*-value: 0.058

All three tests reject the null hypothesis

Contents

Inference under Autocorrelation

Covariance matrix of *b*:

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

Use of $\sigma^2 (X'X)^{-1}$ (the standard output of econometric software) instead of $V\{b\}$ for inference on β may be misleading

Remedies

- Use of correct variances and standard errors
- Transformation of the model so that the error terms are uncorrelated

HAC-estimator for V{*b*}

Substitution of Ψ in

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

by a suitable estimator

- Newey-West: substitution of $S_x = \sigma^2(X'\Psi X)/T = (\Sigma_t\Sigma_s\sigma_{ts}x_tx_s')/T$ by

$$\hat{S}_x = \frac{1}{T}\sum_t e_t^2 x_t x_t' + \frac{1}{T}\sum_{j=1}^{p}\sum_t (1-w_j)e_t e_{t-j}(x_t x_{t-j}' + x_{t-j}x_t')$$

  with $w_j = j/(p+1)$; p, the *truncation lag*, is to be chosen suitably
- The estimator

$$T (X'X)^{-1} \hat{S}_x (X'X)^{-1}$$

  for V{*b*} is called *heteroskedasticity and autocorrelation consistent* (HAC) estimator, the corresponding standard errors are the HAC s.e.

Demand for Ice Cream, cont'd

Demand for ice cream, measured by *cons*, explained by *price*, *income*, and *temp*, OLS and HAC standard errors

	coeff	s.e.	
		OLS	HAC
constant	0.197	0.270	0.288
price	-1.044	0.834	0.876
$income*10^{-3}$	3.308	1.171	1.184
$temp*10^{-3}$	3.458	0.446	0.411

Cochrane-Orcutt Estimator

GLS estimator

- With transformed variables $y_t^* = y_t - \rho y_{t-1}$ and $x_t^* = x_t - \rho x_{t-1}$, also called "quasi-differences", the model $y_t = x_t'\beta + \varepsilon_t$ with $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$ can be written as

$$y_t - \rho y_{t-1} = y_t^* = (x_t - \rho x_{t-1})'\beta + v_t = x_t^{*\prime}\beta + v_t \qquad \text{(A)}$$

- The model in quasi-differences has error terms which fulfill the Gauss-Markov assumptions
- Given observations for $t = 1, \ldots, T$, model (A) is defined for $t = 2, \ldots, T$
- Estimation of ρ using, e.g., the auxiliary regression $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$ gives the estimate r; substitution of r in (A) for ρ results in FGLS estimators for β
- The FGLS estimator is called Cochrane-Orcutt estimator

Cochrane-Orcutt Estimation

In following steps

1. OLS estimation of b for β from $y_t = x_t'\beta + \varepsilon_t$, $t = 1, \ldots, T$
2. Estimation of r for ρ from the auxiliary regression $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$
3. Calculation of quasi-differences $y_t^* = y_t - ry_{t-1}$ and $x_t^* = x_t - rx_{t-1}$
4. OLS estimation of β from

   $$y_t^* = x_t^{*\prime}\beta + v_t,\ t = 2, \ldots, T$$

   resulting in the Cochrane-Orcutt estimators

Steps 2. to 4. can be repeated in order to improve the estimate r: iterated Cochrane-Orcutt estimator

GRETL provides the iterated Cochrane-Orcutt estimator:

Model => Time series => Autoregressive estimation

Demand for Ice Cream, cont'd

Iterated Cochrane-Orcutt estimator

Table 4.10 EGLS (iterative Cochrane–Orcutt) results

Dependent variable: *cons*

Variable	Estimate	Standard error	*t*-ratio
constant	0.157	0.300	0.524
price	−0.892	0.830	−1.076
income	0.00320	0.00159	2.005
temp	0.00356	0.00061	5.800
$\hat{\rho}$	0.401	0.2079	1.927

$s = 0.0326^*$ $R^2 = 0.7961^*$ $\bar{R}^2 = 0.7621^*$ $F = 23.419$
$dw = 1.5486^*$

Durbin-Watson test: $dw = 1.55$; $d_L = 1.21 < dw < 1.65 = d_U$

Demand for Ice Cream, cont'd

Demand for ice cream, measured by *cons*, explained by *price*, *income*, and *temp,* OLS and HAC standard errors (se), and Cochrane-Orcutt estimates

	OLS-estimation			Cochrane-Orcutt	
	coeff	se	HAC	coeff	se
constant	0.197	0.270	0.288	0.157	0.300
price	-1.044	0.834	0.881	-0.892	0.830
income	3.308	1.171	1.151	3.203	1.546
temp	3.458	0.446	0.449	3.558	0.555

Demand for Ice Cream, cont'd

Model extended by $temp_{-1}$

Table 4.11 OLS results extended specification

Dependent variable: *cons*

Variable	Estimate	Standard error	t-ratio
constant	0.189	0.232	0.816
price	−0.838	0.688	−1.218
income	0.00287	0.00105	2.722
temp	0.00533	0.00067	7.953
$temp_{t-1}$	−0.00220	0.00073	−3.016

$s = 0.0299$ $R^2 = 0.8285$ $\bar{R}^2 = 0.7999$ $F = 28.979$
$dw = 1.5822$

Durbin-Watson test: $dw = 1.58$; $d_L=1.21 < dw < 1.65 = d_U$

Demand for Ice Cream, cont'd

Demand for ice cream, measured by *cons*, explained by *price*, *income*, and *temp,* OLS and HAC standard errors, Cochrane-Orcutt estimates, and OLS estimates for the extended model

	OLS		Cochrane-Orcutt		OLS	
	coeff	HAC	coeff	se	coeff	se
constant	0.197	0.288	0.157	0.300	0.189	0.232
price	-1.044	0.881	-0.892	0.830	-0.838	0.688
income	3.308	1.151	3.203	1.546	2.867	1.053
temp	3.458	0.449	3.558	0.555	5.332	0.670
$temp_{-1}$					-2.204	0.731

Adding $temp_{-1}$ improves the adj R^2 from 0.687 to 0.800

General Autocorrelation Structures

Generalization of model

$$y_t = x_t'\beta + \varepsilon_t$$

with $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$

Alternative dependence structures of error terms

- Autocorrelation of higher order than 1
- Moving average pattern

Higher Order Autocorrelation

For quarterly data, error terms may develop according to

$$\varepsilon_t = \gamma\varepsilon_{t-4} + v_t$$

or - more generally - to

$$\varepsilon_t = \gamma_1\varepsilon_{t-1} + \ldots + \gamma_4\varepsilon_{t-4} + v_t$$

- ε_t follows an AR(4) process, an autoregressive process of order 4
- More complex structures of correlations between variables with autocorrelation of order 4 are possible than with that of order 1

Moving Average Processes

Moving average process of order 1, MA(1) process

$$\varepsilon_t = v_t + \alpha v_{t-1}$$

- ε_t is correlated with ε_{t-1}, but not with ε_{t-2}, ε_{t-3}, …
- Generalizations to higher orders

Remedies against Autocorrelation

- Change functional form, e.g., use log(*y*) instead of *y*
- Extend the model by including additional explanatory variables, e.g., seasonal dummies, or additional lags
- Use HAC standard errors for the OLS estimators
- Reformulate the model in quasi-differences (FGLS) or in differences

Contents

OLS Estimator

Linear model for y_t

$$y_i = x_i'\beta + \varepsilon_i,\ i = 1, \ldots, N \quad (\text{or } y = X\beta + \varepsilon)$$

given observations x_{ik}, $k = 1, \ldots, K$, of the regressor variables, error term ε_i

OLS estimator

$$b = (\Sigma_i x_i x_i')^{-1}\Sigma_i x_i y_i = (X'X)^{-1}X'y$$

From

$$b = (\Sigma_i x_i x_i')^{-1}\Sigma_i x_i y_i = (\Sigma_i x_i x_i')^{-1}\Sigma_i x_i x_i' \beta + (\Sigma_i x_i x_i')^{-1}\Sigma_i x_i \varepsilon_i$$
$$= \beta + (\Sigma_i x_i x_i')^{-1}\Sigma_i x_i \varepsilon_i = \beta + (X'X)^{-1} X'\varepsilon$$

follows

$$E\{b\} = (\Sigma_i x_i x_i')^{-1}\Sigma_i x_i y_i = (\Sigma_i x_i x_i')^{-1}\Sigma_i x_i x_i' \beta + (\Sigma_i x_i x_i')^{-1}\Sigma_i x_i \varepsilon_i$$
$$= \beta + (\Sigma_i x_i x_i')^{-1} E\{\Sigma_i x_i \varepsilon_i\} = \beta + (X'X)^{-1} E\{X'\varepsilon\}$$

OLS Estimator: Properties

1. OLS estimator *b* is unbiased if
   - (A1) E{ε} = 0
   - E{$\Sigma_i x_i \varepsilon_i$ } = E{$X'\varepsilon$} = 0; is fulfilled if (A7) or a stronger assumption is true
     - (A2) {x_i, i =1, …,N} and {ε_i, i =1, …,N} are independent; is the strongest assumption
     - (A10) E{$\varepsilon|X$} = 0, i.e., X uninformative about E{ε_i} for all i (ε is conditional mean independent of X); is implied by (A2)
     - (A8) x_i and ε_i are independent for all i (no contemporaneous dependence); is less strong than (A2) and (A10)
     - (A7) E{$x_i \varepsilon_i$} = 0 for all i (no contemporaneous correlation); is even less strong than (A8)

OLS Estimator: Properties, cont'd

2. OLS estimator b is consistent for β if
   - (A8) x_i and ε_i are independent for all i
   - (A6) $(1/N)\Sigma_i x_i x_i'$ has as limit ($N \rightarrow \infty$) a non-singular matrix Σ_{xx}

   (A8) can be substituted by (A7) [$E\{x_i \varepsilon_i\} = 0$ for all i, no contemporaneous correlation]
3. OLS estimator b is asymptotically normally distributed if (A6), (A8) and
   - (A11) $\varepsilon_i \sim IID(0,\sigma^2)$

     are true;
   - for large N, b follows approximately the normal distribution

     $b \sim_a N\{\beta, \sigma^2(\Sigma_i x_i x_i')^{-1}\}$
   - Use White and Newey-West estimators for V{b} in case of heteroskedasticity and autocorrelation of error terms, respectively

Assumption (A7): $E\{x_i \varepsilon_i\} = 0$ for all *i*

Implication of (A7): for all *i*, each of the regressors is uncorrelated with the current error term, no contemporaneous correlation

- (A7) guaranties unbiasedness and consistency of the OLS estimator
- Stronger assumptions – (A2), (A10), (A8) – have same consequences

In reality, (A7) is not always true: alternative estimation procedures are required for ascertaining consistency and unbiasedness

Examples of situations with $E\{x_i \varepsilon_i\} \neq 0$ (see the following slides):

- Regressors with measurement errors
- Regression on the lagged dependent variable with autocorrelated error terms (dynamic regression)
- Unobserved heterogeneity
- Endogeneity of regressors, simultaneity

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Regressor with Measurement Error

$$y_i = \beta_1 + \beta_2 w_i + v_i$$

with white noise v_i, $V\{v_i\} = \sigma_v^2$, and $E\{v_i|w_i\} = 0$; conditional expectation of y_i given w_i : $E\{y_i|w_i\} = \beta_1 + \beta_2 w_i$

Example: y_i: household savings , w_i: household income

Measurement process: reported household income x_i may deviate from household income w_i

$$x_i = w_i + u_i$$

where u_i is (i) white noise with $V\{u_i\} = \sigma_u^2$, (ii) independent of v_i, and (iii) independent of w_i

The model to be analyzed is

$$y_i = \beta_1 + \beta_2 x_i + \varepsilon_i \quad \text{with } \varepsilon_i = v_i - \beta_2 u_i$$

- $E\{x_i \varepsilon_i\} = - \beta_2 \sigma_u^2 \neq 0$: requirement for consistency and unbiasedness of OLS estimates is violated
- x_i and ε_i are negatively (positively) correlated if $\beta_2 > 0$ ($\beta_2 < 0$)

Consequences of Measurement Errors

- Inconsistency of $b_2 = s_{xy}/s_x^2$

  plim $b_2 = \beta_2$ + (plim $s_{x\varepsilon}$)/(plim s_x^2) = β_2 + E{$x_i\ \varepsilon_i$} / V{x_i}

  $$= \beta_2 \left(1 - \frac{\sigma_u^2}{\sigma_w^2 + \sigma_u^2} \right)$$

  β_2 is underestimated
- Inconsistency of $b_1 = \bar{y} - b_2 \bar{x}$

  plim (b_1 - β_1) = - plim (b_2 - β_2) E{x_i}

  given E{x_i} > 0 for the reported income: β_1 is overestimated; inconsistency of b_2 "carries over"
- The model does not correspond to the conditional expectation of y_i given x_i:

  E{$y_i|x_i$} = $\beta_1 + \beta_2 x_i$ - β_2 E{$u_i|x_i$} ≠ $\beta_1 + \beta_2 x_i$

  as E{$u_i|x_i$} ≠ 0

Dynamic Regression

Allows modelling dynamic effects of changes of *x* on *y*:

$$y_t = \beta_1 + \beta_2 x_t + \beta_3 y_{t-1} + \varepsilon_t$$

with ε_t following the AR(1) model

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t$$

v_t white noise with σ_v^2

From $y_t = \beta_1 + \beta_2 x_t + \beta_3 y_{t-1} + \rho\varepsilon_{t-1} + v_t$ follows

$$E\{y_{t-1}\varepsilon_t\} = \beta_3\, E\{y_{t-2}\varepsilon_t\} + \rho^2\sigma_v^2(1 - \rho^2)^{-1}$$

i.e., y_{t-1} is correlated with ε_t

Remember: $E\{\varepsilon_t, \varepsilon_{t-s}\} = \rho^s \sigma_v^2 (1-\rho^2)^{-1}$ for $s > 0$

OLS estimators not consistent if $\rho \neq 0$

The model does not correspond to the conditional expectation of y_t given the regressors x_t and y_{t-1}:

$$E\{y_t|x_t, y_{t-1}\} = \beta_1 + \beta_2 x_t + \beta_3 y_{t-1} + E\{\varepsilon_t\, |x_t, y_{t-1}\}$$

Omission of Relevant Regressors

Two models:

$y_i = x_i'\beta + z_i'\gamma + \varepsilon_i$ (A)

$y_i = x_i'\beta + v_i$ (B)

- True model (A), fitted model (B)
- OLS estimates b_B of β from (B)

$$b_B = \beta + \left(\sum_i x_i x_i'\right)^{-1} \sum_i x_i z_i' \gamma + \left(\sum_i x_i x_i'\right)^{-1} \sum_i x_i \varepsilon_i$$

- Omitted variable bias: $E\{(\Sigma_i x_i x_i')^{-1} \Sigma_i x_i z_i'\}\gamma = E\{(X'X)^{-1} X'Z\}\gamma$
- No bias if (a) γ = 0, i.e., model (A) is correct, or if (b) variables in x_i and z_i are uncorrelated (orthogonal)

OLS estimators are biased, if relevant regressors are omitted that are correlated with regressors in x_i

Unobserved Heterogeneity

Example: Wage equation with y_i: log wage, x_{1i}: personal characteristics, x_{2i}: years of schooling, u_i: abilities (unobservable)

$$y_i = x_{1i}'\beta_1 + x_{2i}\beta_2 + u_i\gamma + v_i$$

- Model for analysis (unobserved u_i covered in error term)

  $$y_i = x_i'\beta + \varepsilon_i$$

  with $x_i = (x_{1i}', x_{2i})'$, $\beta = (\beta_1', \beta_2)'$, $\varepsilon_i = u_i\gamma + v_i$
- Given $E\{x_i\, v_i\} = 0$

  $$\text{plim } b = \beta + \Sigma_{xx}^{-1}\, E\{x_i\, u_i\}\, \gamma$$
- OLS estimators b are not consistent if x_i and u_i are correlated ($\gamma \neq 0$), e.g., if higher abilities induce more years at school: estimator for β_2 might be overestimated, hence effects of years at school etc. are overestimated: “ability bias”

Unobserved heterogeneity: observational units differ in other aspects than ones that are observable

Endogenous Regressors

Regressors in X which are correlated with error term, $E\{X'\varepsilon\} \neq 0$, are called endogenous

- OLS estimators $b = \beta + (X'X)^{-1}X'\varepsilon$
  - $E\{b\} \neq \beta$, b is biased; bias $E\{(X'X)^{-1}X'\varepsilon\}$ difficult to assess
  - plim $b = \beta + \Sigma_{xx}^{-1}q$ with $q = \text{plim}(N^{-1}X'\varepsilon)$
    - For $q = 0$ (regressors and error term asymptotically uncorrelated), OLS estimators b are consistent also in case of endogenous regressors
    - For $q \neq 0$ (error term and at least one regressor asymptotically correlated): plim $b \neq \beta$, the OLS estimators b are not consistent
- Endogeneity bias
- Relevant for many economic applications

Exogenous regressors: with error term uncorrelated, all regressors that are not endogenous

Consumption Function

AWM data base, 1970:1-2003:4

- C: private consumption (PCR), growth rate p.y.
- Y: disposable income of households (PYR), growth rate p.y.

  $C_t = \beta_1 + \beta_2 Y_t + \varepsilon_t$ (A)

  β_2: marginal propensity to consume, $0 < \beta_2 < 1$
- OLS estimates:

  $\hat{C}_t = 0.011 + 0.718\ Y_t$

  with t = 15.55, R^2 = 0.65, DW = 0.50
- I_t: per capita investment (exogenous, $E\{I_t\ \varepsilon_t\} = 0$)

  $Y_t = C_t + I_t$ (B)
- Both Y_t and C_t are endogenous: $E\{C_t\ \varepsilon_i\} = E\{Y_t\ \varepsilon_i\} = \sigma_\varepsilon^2(1 - \beta_2)^{-1}$
- The regressor Y_t has an impact on C_t; at the same time C_t has an impact on Y_t

Simultaneous Equation Models

Illustrated by the preceding consumption function:

$C_t = \beta_1 + \beta_2 Y_t + \varepsilon_t$ (A)

$Y_t = C_t + I_t$ (B)

Variables Y_t and C_t are simultaneously determined by equations (A) and (B)

- Equations (A) and (B) are the structural equations or the structural form of the simultaneous equation model that describes both Y_t and C_t
- The coefficients β_1 and β_2 are behavioural parameters
- Reduced form of the model: one equation for each of the endogenous variables C_t and Y_t, with only the exogenous variable I_t as regressor

The OLS estimators are biased and not consistent

Consumption Function, cont'd

- Reduced form of the model:

$$C_t = \frac{\beta_1}{1-\beta_2} + \frac{\beta_2}{1-\beta_2} I_t + \frac{1}{1-\beta_2}\varepsilon_t$$

$$Y_t = \frac{\beta_1}{1-\beta_2} + \frac{1}{1-\beta_2} I_t + \frac{1}{1-\beta_2}\varepsilon_t$$

- OLS estimator b_2 from (A) is inconsistent; E$\{Y_t\ \varepsilon_t\} \neq 0$

  plim $b_2 = \beta_2 + \mathrm{Cov}\{Y_t\ \varepsilon_t\} / \mathrm{V}\{Y_t\} = \beta_2 + (1 - \beta_2)\ \sigma_\varepsilon^2(\mathrm{V}\{I_t\} + \sigma_\varepsilon^2)^{-1}$

  for $0 < \beta_2 < 1$, b_2 overestimates β_2
- The OLS estimator b_1 is also inconsistent

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An Alternative Estimator

Model

$$y_i = \beta_1 + \beta_2 x_i + \varepsilon_i$$

with $E\{\varepsilon_i x_i\} \neq 0$, i.e., endogenous regressor x_i : OLS estimators are biased and inconsistent

Instrumental variable z_i satisfying

1. Exogeneity: $E\{\varepsilon_i z_i\} = 0$: is uncorrelated with error term
2. Relevance: $Cov\{x_i, z_i\} \neq 0$: is correlated with endogenous regressor

Transformation of model equation

$$Cov\{y_i, z_i\} = \beta_2 Cov\{x_i, z_i\} + Cov\{\varepsilon_i, z_i\}$$

gives

$$\beta_2 = \frac{Cov\{y_i, z_i\}}{Cov\{x_i, z_i\}}$$

IV Estimator for β_2

Substitution of sample moments for covariances gives the instrumental variables (IV) estimator

$$\hat{\beta}_{2,IV} = \frac{\sum_i (z_i - \bar{z})(y_i - \bar{y})}{\sum_i (z_i - \bar{z})(x_i - \bar{x})}$$

- Consistent estimator for β_2 given that the instrumental variable z_i is valid , i.e., it is
  - Exogenous, i.e. $E\{\varepsilon_i z_i\} = 0$
  - Relevant, i.e. $Cov\{x_i, z_i\} \neq 0$
- Typically, nothing can be said about the bias of an IV estimator; small sample properties are unknown
- Coincides with OLS estimator for $z_i = x_i$

Consumption Function, cont'd

Alternative model: $C_t = \beta_1 + \beta_2 Y_{t-1} + \varepsilon_t$

- Y_{t-1} and ε_t are certainly uncorrelated; avoids risk of inconsistency due to correlated Y_t and ε_t
- Y_{t-1} is certainly highly correlated with Y_t, is almost as good as regressor as Y_t

Fitted model:

$$\hat{C} = 0.012 + 0.660\ Y_{-1}$$

with t = 12.86, R^2 = 0.56, DW = 0.79 (instead of

$$\hat{C} = 0.011 + 0.718\ Y$$

with t = 15.55, R^2 = 0.65, DW = 0.50)

Deterioration of t-statistic and R^2 are price for improvement of the estimator

IV Estimator: The Concept

Alternative to OLS estimator

- Avoids inconsistency in case of endogenous regressors

Idea of the IV estimator:

Replace regressors which are correlated with error terms by regressors which are

- uncorrelated with the error terms
- (highly) correlated with the regressors that are to be replaced

and use OLS estimation

The hope is that the IV estimator is consistent (and less biased than the OLS estimator)

Price: IV estimator is less efficient; deteriorated model fit as measured by, e.g., *t*-statistic, R^2

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IV Estimator: General Case

The model is

$y_i = x_i'\beta + \varepsilon_i$

with $V\{\varepsilon_i\} = \sigma_\varepsilon^2$ and

$E\{\varepsilon_i x_i\} \neq 0$

- at least one component of x_i is correlated with the error term

The vector of instruments z_i (with the same dimension as x_i) fulfils

$E\{\varepsilon_i z_i\} = 0$

$Cov\{x_i , z_i\} \neq 0$

IV estimator based on the instruments z_i

$$\hat{\beta}_{IV} = \left(\sum\nolimits_i z_i x_i'\right)^{-1} \left(\sum\nolimits_i z_i y_i\right)$$

IV Estimator: Distribution

The (asymptotic) covariance matrix of the IV estimator is given by

$$V\left\{\hat{\beta}_{IV}\right\} = \sigma^2 \left[\left(\sum_i x_i z_i'\right)\left(\sum_i z_i z_i'\right)^{-1}\left(\sum_i z_i x_i'\right)\right]^{-1}$$

In the estimated covariance matrix $\hat{V}\{\hat{\beta}_{IV}\}$, σ^2 is substituted by

$$\hat{\sigma}^2 = \tfrac{1}{N}\sum_i \left(y_i - x_i'\hat{\beta}_{IV}\right)^2$$

which is based on the IV residuals $y_i - x_i'\hat{\beta}_{\mathrm{IV}}$

The asymptotic distribution of IV estimators, given IID(0, σ_ε^2) error terms, leads to the approximate distribution

$$N\left(\beta, \hat{V}\{\hat{\beta}_{IV}\}\right)$$

with the estimated covariance matrix $\hat{V}\{\hat{\beta}_{IV}\}$

Derivation of the IV Estimator

The model is

$$y_i = x_i'\beta + \varepsilon_t = x_{0i}'\beta_0 + \beta_K x_{Ki} + \varepsilon_i$$

with $x_{0i} = (x_{1i}, \ldots, x_{K-1,i})'$ containing the first K-1 components of x_i, and $E\{\varepsilon_i \, x_{0i}\} = 0$

K-th component is endogenous: $E\{\varepsilon_i \, x_{Ki}\} \neq 0$

The instrumental variable z_{Ki} fulfils

$$E\{\varepsilon_i \, z_{Ki}\} = 0$$

Moment conditions: K conditions to be satisfied by the coefficients, the K-th condition with z_{Ki} instead of x_{Ki}:

$$E\{\varepsilon_i \, x_{0i}\} = E\{(y_i - x_{0i}'\beta_0 - \beta_K x_{Ki}) \, x_{0i}\} = 0 \quad (K\text{-1 conditions})$$

$$E\{\varepsilon_i \, z_i\} = E\{(y_i - x_{0i}'\beta_0 - \beta_K x_{Ki}) \, z_{Ki}\} = 0$$

Number of conditions – and of corresponding linear equations – equals the number of coefficients to be estimated

Derivation of the IV Estimator, cont'd

The system of linear equations for the K coefficients β to be estimated can be uniquely solved for the coefficients β: the coefficients β are said "to be identified"

To derive the IV estimators from the moment conditions, the expectations are replaced by sample averages

$$\tfrac{1}{N}\sum_i (y_i - x_i'\hat{\beta}_{IV})\, x_{ki} = 0,\ k = 1,\ldots,K-1$$

$$\tfrac{1}{N}\sum_i (y_i - x_i'\hat{\beta}_{IV})\, z_{Ki} = 0$$

The solution of the linear equation system – with $z_i' = (x_{0i}', z_{Ki})$ – is

$$\hat{\beta}_{IV} = \left(\sum_i z_i x_i'\right)^{-1} \sum_i z_i y_i$$

Identification requires that the KxK matrix $\Sigma_i\, z_i\, x_i'$ is finite and invertible; instrument z_{Ki} is relevant when this is fulfilled

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Calculation of IV Estimators

The model in matrix notation

$y = X\beta + \varepsilon$

The IV estimator

$$\hat{\beta}_{IV} = \left(\sum_i z_i x_i'\right)^{-1} \sum_i z_i y_i = (Z'X)^{-1} Z'y$$

with z_i obtained from x_i by substituting instrumental variable(s) for all endogenous regressors

Calculation in two steps:

1. Reduced form: Regression of the explanatory variables $x_1, \ldots, x_K$ – including the endogenous ones – on the columns of Z: fitted values

$$\hat{X} = Z(Z'Z)^{-1} Z'X$$

2. Regression of y on the fitted explanatory variables:

$$\hat{\beta}_{IV} = (\hat{X}'\hat{X})^{-1} \hat{X}'y$$

Calculation of IV Estimators: Remarks

- The KxK matrix $Z'X = \Sigma_i z_i x_i'$ is required to be finite and invertible
- From
$$(\hat{X}'\hat{X})^{-1}\hat{X}'y = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y$$
$$= (Z'X)^{-1}Z'Z(X'Z)^{-1}X'Z(Z'Z)^{-1}Z'y = (Z'X)^{-1}Z'y = \hat{\beta}_{IV}$$
it is obvious that the estimator obtained in the second step is the IV estimator
- However, the estimator obtained in the second step is more general; see below
- In **GRETL**: The sequence „Model > Instrumental variables > Two-Stage Least Squares…" leads to the specification window with boxes (i) for the regressors and (ii) for the instruments

Choice of Instrumental Variables

Instrumental variable are required to be

- exogenous, i.e., uncorrelated with the error terms
- relevant, i.e., correlated with the endogenous regressors

Instruments

- must be based on subject matter arguments, e.g., arguments from economic theory
- should be explained and motivated
- must show a significant effect in explaining an endogenous regressor
- Choice of instruments often not easy

Regression of endogenous variables on instruments

- Best linear approximation of endogenous variables
- Economic interpretation not of importance and interest

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Returns to Schooling: Causality?

Human capital earnings function:

$$w_i = \beta_1 + \beta_2 S_i + \beta_3 E_i + \beta_4 E_i^2 + \varepsilon_i$$

with w_i: log of individual earnings, S_i: years of schooling, E_i: years of experience ($E_i = age_i - S_i - 6$)

Empirically, more education implies higher income

Question: Is this effect causal?

- If yes, one year more at school increases wage by β_2 (Theory A)
- Alternatively, personal abilities of an individual causes higher income and also more years at school; more years at school do not necessarily increase wage (Theory B)

Issue of substantial attention in literature

Returns to Schooling: Endogenous Regressors

Wage equation: besides S_i and E_i, additional explanatory variables like gender, regional, racial dummies, family background

Model for analysis:

$$w_i = \beta_1 + z_i'\gamma + \beta_2 S_i + \beta_3 E_i + \beta_4 E_i^2 + \varepsilon_i$$

z_i: observable variables besides E_i, S_i

- z_i is assumed to be exogenous, i.e., $E\{z_i \varepsilon_i\} = 0$
- S_i may be endogenous, i.e., $E\{S_i \varepsilon_i\} \neq 0$
  - Ability bias: unobservable factors like intelligence, family background, etc. enable to more schooling and higher earnings
  - Measurement error in measuring schooling
  - Etc.
- With S_i, also $E_i = age_i - S_i - 6$ and E_i^2 are endogenous
- OLS estimators may be inconsistent

Returns to Schooling: Data

- Verbeek's data set "schooling"
- National Longitudinal Survey of Young Men (Card, 1995)
- Data from 3010 males, survey 1976
- Individual characteristics, incl. experience, race, region, family background, etc.
- Human capital earnings or wage function

$$\log(wage_i) = \beta_1 + \beta_2\, ed_i + \beta_3\, exp_i + \beta_3\, exp_i^2 + \varepsilon_i$$

  with ed_i: years of schooling (S_i), exp_i: years of experience (E_i)
- Variables: *wage76* (wage in 1976, raw, cents p.h.), *ed76* (years at school in 1976), *exp76* (experience in 1976), *exp762* (*exp76* squared)
- Further explanatory variables: *black*: dummy for afro-american, *smsa*: dummy for living in metropolitan area, *south*: dummy for living in the south

OLS Estimation

OLS estimated wage function

Model 2: OLS, using observations 1-3010
Dependent variable: l_WAGE76

	coefficient	std. error	*t*-ratio	*p*-value	
const	4.73366	0.0676026	70.02	0.0000	***
ED76	0.0740090	0.00350544	21.11	2.28e-092	***
EXP76	0.0835958	0.00664779	12.57	2.22e-035	***
EXP762	-0.00224088	0.000317840	-7.050	2.21e-012	***
BLACK	-0.189632	0.0176266	-10.76	1.64e-026	***
SMSA76	0.161423	0.0155733	10.37	9.27e-025	***
SOUTH76	-0.124862	0.0151182	-8.259	2.18e-016	***

Mean dependent var	6.261832	S.D. dependent var	0.443798
Sum squared resid	420.4760	S.E. of regression	0.374191
R-squared	0.290505	Adjusted R-squared	0.289088
F(6, 3003)	204.9318	P-value(F)	1.5e-219
Log-likelihood	-1308.702	Akaike criterion	2631.403
Schwarz criterion	2673.471	Hannan-Quinn	2646.532

Instruments for S_i, E_i, E_i^2

Potential instrumental variables

- Factors which affect schooling but are uncorrelated with error terms, in particular with unobserved abilities that are determining wage
- For years of schooling (S_i)
  - Costs of schooling, e.g., distance to school (*lived near college*), number of siblings
  - Parents' education
- For years of experience (E_i, E_i^2): *age* is natural candidate

Step 1 of IV Estimation

Reduced form for *schooling* (*ed76*), gives predicted values *ed76_h*,

Model 3: OLS, using observations 1-3010
Dependent variable: ED76

	coefficient	std. error	t-ratio	p-value	
const	-1.81870	4.28974	-0.4240	0.6716	
AGE76	1.05881	0.300843	3.519	0.0004	***
sq_AGE76	-0.0187266	0.00522162	-3.586	0.0003	***
BLACK	-1.46842	0.115245	-12.74	2.96e-036	***
SMSA76	0.841142	0.105841	7.947	2.67e-015	***
SOUTH76	-0.429925	0.102575	-4.191	2.85e-05	***
NEARC4A	0.441082	0.0966588	4.563	5.24e-06	***

Mean dependent var	13.26346	S.D. dependent var	2.676913
Sum squared resid	18941.85	S.E. of regression	2.511502
R-squared	0.121520	Adjusted R-squared	0.119765
F(6, 3003)	69.23419	P-value(F)	5.49e-81
Log-likelihood	-7039.353	Akaike criterion	14092.71
Schwarz criterion	14134.77	Hannan-Quinn	14107.83

Step 2 of IV Estimation

Wage equation, estimated by IV with instruments *age*, *age*^{2}, and *nearc4a*

Model 4: OLS, using observations 1-3010
Dependent variable: l_WAGE76

	coefficient	std. error	t-ratio	p-value
const	3.69771	0.435332	8.494	3.09e-017 ***
ED76_h	0.164248	0.036887	4.453	8.79e-06 ***
EXP76_h	0.044588	0.022502	1.981	0.0476 **
EXP762_h	-0.000195	0.001152	-0.169	0.8655
BLACK	-0.057333	0.056772	-1.010	0.3126
SMSA76	0.079372	0. 037116	2.138	0.0326 **
SOUTH76	-0.083698	0.022985	-3.641	0.0003 ***

Mean dependent var	6.261832	S.D. dependent var	0.443798
Sum squared resid	446.8056	S.E. of regression	0.385728
R-squared	0.246078	Adjusted R-squared	0.244572
F(6, 3003)	163.3618	P-value(F)	4.4e-180
Log-likelihood	-1516.471	Akaike criterion	3046.943
Schwarz criterion	3089.011	Hannan-Quinn	3062.072

Returns to Schooling: Summary of Estimates

Estimated regression coefficients and *t*-statistics

	OLS	IV[1)]	TSLS[1)]	IV (M.V.)
ed76	0.0740	0.1642	0.1642	0.1329
	21.11	4.45	3.92	2.59
exp76	0.0836	0.0445	0.0446	0.0560
	12.75	1.98	1.74	2.15
exp762	-0.0022	-0.0002	-0.0002	-0.0008
	-7.05	-0.17	-0.15	-0.59
black	-0.1896	-0. 0573	-0.0573	-0.1031
	-10.76	-1.01	-0.89	-1.33
R^2	0.291	0.246		
F-test	204.9	163.4		

[1)] The model differs from that used by Verbeek

Some Comments

Instrumental variables (*age*, age^2, *nearc4a*)

- are relevant, i.e., have explanatory power for *ed76*, *exp76*, $exp76^2$
- Whether they are exogenous, i.e., uncorrelated with the error terms, is not answered
- Test for exogeneity of regressors: Wu-Hausman test

Estimates of *ed76*-coefficient:

- IV estimate: 0.16 (0.13), i.e., 16% higher wage for one additional year of schooling; more than the double of the OLS estimate (0.07); not in line with "ability bias" argument!
- s.e. of IV estimate (0.04) much higher than s.e. of OLS estimate (0.004)
- Loss of efficiency especially in case of weak instruments: R^2 of model for *ed76*: 0.12; Corr{*ed76*, *ed76_h*} = 0.35

GRETL's TSLS Estimation

Wage equation, estimated by GRETL's TSLS

Model 8: TSLS, using observations 1-3010
Dependent variable: l_WAGE76
Instrumented: ED76 EXP76 EXP762
Instruments: const AGE76 sq_AGE76 BLACK SMSA76 SOUTH76 NEARC4A

	coefficient	std. error	t-ratio	p-value	
const	3.69771	0.495136	7.468	8.14e-014	***
ED76	0.164248	0.0419547	3.915	9.04e-05	***
EXP76	0.0445878	0.0255932	1.742	0.0815	*
EXP762	-0.00019526	0.0013110	-0.1489	0.8816	
BLACK	-0.0573333	0.0645713	-0.8879	0.3746	
SMSA76	0.0793715	0.0422150	1.880	0.0601	*
SOUTH76	-0.0836975	0.0261426	-3.202	0.0014	***

Mean dependent var	6.261832	S.D. dependent var	0.443798
Sum squared resid	577.9991	S.E. of regression	0.438718
R-squared	0.195884	Adjusted R-squared	0.194277
F(6, 3003)	126.2821	P-value(F)	8.9e-143

Returns to Schooling: Summary of Estimates

Estimated regression coefficients and *t*-statistics

	OLS	IV[1)]	TSLS[1)]	IV (M.V.)
ed76	0.0740	0.1642	0.1642	0.1329
	21.11	4.45	3.92	2.59
exp76	0.0836	0.0445	0.0446	0.0560
	12.75	1.98	1.74	2.15
exp762	-0.0022	-0.0002	-0.0002	-0.0008
	-7.05	-0.17	-0.15	-0.59
black	-0.1896	-0. 0573	-0.0573	-0.1031
	-10.76	-1.01	-0.89	-1.33
R^2	0.291	0.246	0.196	
F-test	204.9	163.4	126.3	

[1)] The model differs from that used by Verbeek

Some Comments

Verbeek‘s IV estimates

- Deviate from GRETL results
- No report of R^2; definition of R^2 does not apply to IV estimated models

IV estimates of coefficients

- are smaller than the OLS estimates; exception is *ed76*
- have higher s.e. than OLS estimates, smaller *t*-statistics

Questions

- Robustness of IV estimates to changes in the specification
- Exogeneity of instruments
- Weak instruments

Contents

Some Tests

Questions of interest

1. Is it necessary to use IV estimation, must violation of exogeneity be expected? To be tested: the null hypothesis of exogeneity of suspected variables
2. If IV estimation is used: Are the chosen instruments valid (relevant)?

For testing

- exogeneity of regressors: Wu-Hausman test, also called Durbin-Wu-Hausman test, in GRETL: Hausman test
- relevance of potential instrumental variables: Sargan test or over-identifying restrictions test
- Weak instruments, i.e., only weak correlation between endogenous regressor and instrument: Cragg-Donald test

Wu-Hausman Test

For testing whether one or more regressors x_i are endogenous (correlated with the error term); H_0: E{$\varepsilon_i\, x_i$} = 0

- If the null hypothesis
  - is true, OLS estimates are more efficient than IV estimates
  - is not true, OLS estimates are inefficient, the less efficient but consistent IV estimates to be used

Based on the assumption that the instrumental variables are valid, i.e., given that E{$\varepsilon_i\, z_i$} = 0, the null hypothesis E{$\varepsilon_i\, x_i$} = 0 can be tested against the alternative E{$\varepsilon_i\, x_i$} ≠ 0

The idea of the test:

- Under the null hypothesis, both the OLS and IV estimator are consistent; they should differ by sampling errors only
- Rejection of the null hypothesis indicates inconsistency of the OLS estimator

Wu-Hausman Test, cont'd

Based on the differences between OLS- and IV-estimators; various versions of the Wu-Hausman test

Added variable interpretation of the Wu-Hausman test: checks whether the residuals v_i from the reduced form equation of potentially endogenous regressors contribute to explaining

$$y_i = x_{1i}'\beta_1 + x_{2i}'\beta_2 + v_i'\gamma + \varepsilon_i$$

- x_2: potentially endogenous regressors
- v_i: residuals from reduced form equation for x_2 (predicted values for x_2: $x_2 + v$)
- H_0: $\gamma = 0$; corresponds to: x_2 is exogenous

For testing H_0: use of

- t-test, if γ has one component, x_2 is just one regressor
- F-test, if more than 1 regressors are tested for exogeneity

Hausman Test Statistic

Based on the quadratic form of differences between OLS- estimators b_{LS} and IV-estimators b_{IV}

- H_0: both b_{LS} and b_{IV} are consistent, b_{LS} is efficient relative to b_{IV}
- H_1: b_{IV} is consistent, b_{LS} is inconsistent

Hausman test statistic

$$H = (b_{IV} - b_{LS})' \mathrm{V} (b_{IV} - b_{LS})$$

with estimated covariance matrix V of $b_{IV} - b_{LS}$ follows the approximate Chi-square distribution with J d.f.

Wu-Hausman Test: Remarks

Remarks

- Test requires valid instruments
- Test has little power if instruments are weak or invalid
- Various versions of the test, all based on differences between OLS- and IV-estimators

In GRETL: Whenever the TSLS estimation is used, GRETL produces automatically the Hausman test statistic

Sargan Test

For testing whether the instruments are valid

The validity of the instruments z_i requires that all moment conditions are fulfilled; for the *R*-vector z_i, the *R* sums

$$\frac{1}{N}\sum_i e_i z_i = 0$$

must be close to zero

Test statistic

$$\xi = NQ_N(\hat{\beta}_{IV}) = \left(\sum_i e_i z_i\right)'\left(\hat{\sigma}^2\sum_i z_i z_i'\right)^{-1}\left(\sum_i e_i z_i\right)$$

has, under the null hypothesis, an asymptotic Chi-squared distribution with *R-K* df

Calculation of ξ: $\xi = NR_e^2$ using R_e^2 from the auxiliary regression of IV residuals $e_i = y_i - x_i'\hat{\beta}_{\text{IV}}$ on the instruments z_i

Sargan Test: Remarks

Remarks

- In case of an identified model ($R = K$), all R moment conditions are fulfilled, $\xi = 0$
- Over-identified model: $R > K$; the Sargan test is also called *over-identifying restrictions test*
- Rejection implies: the joint validity of all moment conditions and hence of all instruments is not acceptable
- The Sargan test gives no indication of invalid instruments

In GRETL: Whenever the TSLS estimation is used and $R > K$, GRETL produces automatically the Sargan test statistic

Cragg-Donald Test

Weak (only marginally valid) instruments, i.e., only weak correlation between endogenous regressor and instrument :

- Biased IV estimates
- Inconsistent IV estimates
- Inappropriate large-sample approximations to the finite-sample distributions even for large N

Definition of weak instruments: estimates are biased to an extent that is unacceptably large

Null hypothesis: instruments are weak, i.e., can lead to an asymptotic relative bias greater than some value b

Cragg-Donald Test, cont'd

Test procedure

- Regression of the endogenous regressor on all instruments, both external, i.e., ones not included among the regressors, and internal
- *F*-test of the null hypothesis that the coefficients of all external instruments are zero
- If *F*-statistic is less a not too large value, e.g., 10: consider the instruments as weak

Contents

From OLS to IV Estimation

Linear model $y_i = x_i'\beta + \varepsilon_i$

- OLS estimator: solution of the *K* normal equations

  $$1/N \, \Sigma_i (y_i - x_i' b) \, x_i = 0$$

- Corresponding moment conditions

  $$E\{\varepsilon_i \, x_i\} = E\{(y_i - x_i'\beta) \, x_i\} = 0$$

- IV estimator given *R* instrumental variables z_i which may overlap with x_i: based on the *R* moment conditions

  $$E\{\varepsilon_i \, z_i\} = E\{(y_i - x_i'\beta) \, z_i\} = 0$$

- IV estimator: solution of corresponding sample moment conditions

Number of Instruments

Moment conditions

$$E\{\varepsilon_i z_i\} = E\{(y_i - x_i'\beta) z_i\} = 0$$

one equation for each component of z_i

- z_i possibly overlapping with x_i

General case: *R* moment conditions

Substitution of expectations by sample averages gives *R* equations

$$\frac{1}{N}\sum_i (y_i - x_i'\hat{\beta}_{IV}) z_i = 0$$

1. *R* = *K*: one unique solution, the IV estimator; identified model

$$\hat{\beta}_{IV} = \left(\sum_i z_i x_i'\right)^{-1} \sum_t z_i y_i = (Z'X)^{-1} Z'y$$

2. *R* < *K:* infinite number of solutions, not enough instruments for a unique solution; under-identified or not identified model

The GIV Estimator

3. *R* > *K*: more instruments than necessary for identification; over-identified model

For *R* > *K*, in general, no unique solution of all *R* sample moment conditions can be obtained; instead:

- the weighted quadratic form in the sample moments

$$Q_N(\beta) = \left[\tfrac{1}{N} \sum_i (y_i - x_i'\beta) z_i \right]' W_N \left[\tfrac{1}{N} \sum_i (y_i - x_i'\beta) z_i \right]$$

  with a *RxR* positive definite weighting matrix W_N is minimized
- gives the generalized instrumental variable (GIV) estimator

$$\hat{\beta}_{IV} = (X'ZW_N Z'X)^{-1} X'ZW_N Z'y$$

The weighting matrix W_N

W_N: positive definite, order RxR

- Different weighting matrices result in different consistent GIV estimators with different covariance matrices
- Optimal choice for W_N?
- For $R = K$, the matrix $Z'X$ is square and invertible; the IV estimator is $(Z'X)^{-1}Z'y$ for any W_N

GIV and TSLS Estimator

Optimal weighting matrix: $W_N^{opt} = [1/N(Z'Z)]^{-1}$; corresponds to the most efficient IV estimator

$$\hat{\beta}_{IV} = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y$$

- If the error terms are heteroskedastic or autocorrelated, the optimal weighting matrix has to be adapted
- Regression of each regressor, i.e., each column of X, on Z, i.e., on the R column of Z, results in $\hat{X} = Z(Z'Z)^{-1}Z'X$ and

$$\hat{\beta}_{IV} = (\hat{X}'\hat{X})^{-1}\hat{X}'y$$

- This explains why the GIV estimator is also called "two stage least squares" (TSLS) estimator:
  1. First step: regress each column of X on Z
  2. Second step: regress y on predictions of X

GIV Estimator and Properties

- GIV estimator is consistent
- The asymptotic distribution of the GIV estimator, given IID(0, σ_ε^2) error terms, leads to

  $$N\left(\beta, \hat{V}\{\hat{\beta}_{IV}\}\right)$$

  which is used as approximate distribution in case of finite *N*
- The (asymptotic) covariance matrix of the GIV estimator is given by

  $$V\left\{\hat{\beta}_{IV}\right\} = \sigma^2\left[\left(\sum_i x_i z_i'\right)\left(\sum_i z_i z_i'\right)^{-1}\left(\sum_i z_i x_i'\right)\right]^{-1}$$
- In the estimated covariance matrix, σ^2 is substituted by

  $$\hat{\sigma}^2 = \tfrac{1}{N}\sum_i\left(y_i - x_i'\hat{\beta}_{IV}\right)^2$$

  the estimate based on the IV residuals $y_i - x_i'\hat{\beta}_{\mathrm{IV}}$

Your Homework

1. Use the data set “icecream” of Verbeek for the following analyses:
   a) Estimate the model where *cons* is explained by *price* and *temp*; show a diagramme of the residuals which may indicate autocorrelation of the error terms.
   b) Use the Durbin-Watson and the Breusch-Godfrey test against autocorrelation; state suitably H_0 and H_1.
   c) Compare (i) the OLS and (ii) the HAC standard errors of the estimated coefficients.
   d) Repeat a), using (i) the iterative Cochrane-Orcutt estimation and (ii) OLS estimation of the model in differences; compare and interpret the results.
2. For the Durbin-Watson test: (a) show that $dw \approx 2 - 2r$; (b) can you agree with the statement “The Durbin-Watson test is a misspecification test”.

Your Homework, cont'd

3. Use the data set "schooling" of Verbeek for the following analyses based on the wage equation

$$\log(\mathit{wage76}) = \beta_1 + \beta_2\, \mathit{ed76} + \beta_3\, \mathit{exp76} + \beta_4\, \mathit{exp762} + \beta_5\, \mathit{black} + \beta_6\, \mathit{momed} + \beta_7\, \mathit{smsa76} + \varepsilon$$

   a) Assuming that *ed76* is endogenous, (i) estimate the reduced form for *ed76*, including external instruments *smsa66, sinmom14*, *south66*, and *mar76*; (ii) assess the validity of the potential instruments; what indicate the correlation coefficients?
   b) Estimate, by means of the GRETL Instrumental variables (Two-Stage Least Squares …) procedure, the wage equation, using the external instruments *black*, *momed*, *sinmom14*, *smsa66*, *south76*, *mar76,* and *age76*. Interpret the results including the Hausman and the Sargan test.
   c) Compare the estimates for β_2 (i) from the model in b), (ii) from the model with instruments *black*, *momed*, *smsa66*, *south76*, *mar76,* and *age76*, and (iii) with the OLS estimates.

Your Homework, cont'd

4. The model for consumption and income consists of two equations:

   $C_t = \beta_1 + \beta_2 Y_t + \varepsilon_t$

   $Y_t = C_t + I_t$

   a. Show that both C_t and Y_t are endogenous:

   $$E\{C_i\, \varepsilon_i\} = E\{Y_i\, \varepsilon_i\} = \sigma_\varepsilon^2 (1 - \beta_2)^{-1}$$

   b. Derive the reduced form of the model