Econometrics - Lecture 4

Heteroskedasticity

Contents

- Violations of $V\{\varepsilon\} = \sigma^2 I_N$: Illustrations and Consequences
- Heteroskedasticity
- Tests against Heteroskedasticity
- GLS Estimation

Gauss-Markov Assumptions

Observation y_i is a linear function

$$y_i = x_i'\beta + \varepsilon_i$$

of observations x_{ik}, k =1, …, K, of the regressor variables and the error term ε_i

for i = 1, …, N; $x_i' = (x_{i1}, \ldots, x_{iK})$; $X = (x_{ik})$

A1	$E\{\varepsilon_i\} = 0$ for all i
A2	all ε_i are independent of all x_i (exogeneous x_i)
A3	$V\{\varepsilon_i\} = \sigma^2$ for all i (homoskedasticity)
A4	$Cov\{\varepsilon_i, \varepsilon_j\} = 0$ for all i and j with $i \neq j$ (no autocorrelation)

In matrix notation: $E\{\varepsilon\} = 0$, $V\{\varepsilon\} = \sigma^2 I_N$

OLS Estimator: Properties

Under assumptions (A1) and (A2):

1. The OLS estimator b is unbiased: $E\{b\} = \beta$

Under assumptions (A1), (A2), (A3) and (A4):

2. The variance of the OLS estimator is given by

$$V\{b\} = \sigma^2(\Sigma_i x_i x_i')^{-1} = \sigma^2(X' X)^{-1}$$

3. The sampling variance s^2 of the error terms ε_i,

$$s^2 = (N - K)^{-1} \Sigma_i e_i^2$$

is unbiased for σ^2

4. The OLS estimator b is BLUE (best linear unbiased estimator)

Violations of $V\{\varepsilon\} = \sigma^2 I_N$

Implications of the Gauss-Markov assumptions for ε:

$$V\{\varepsilon\} = \sigma^2 I_N$$

Violations:

- Heteroskedasticity

  $$V\{\varepsilon\} = \text{diag}(\sigma_1^2, \ldots, \sigma_N^2)$$

  with $\sigma_i^2 \neq \sigma_j^2$ for at least one pair $i \neq j$, or using $\sigma_i^2 = \sigma^2 h_i^2$,

  $$V\{\varepsilon\} = \sigma^2 \Psi = \sigma^2 \text{diag}(h_1^2, \ldots, h_N^2)$$

- Autocorrelation: $V\{\varepsilon_i, \varepsilon_j\} \neq 0$ for at least one pair $i \neq j$ or

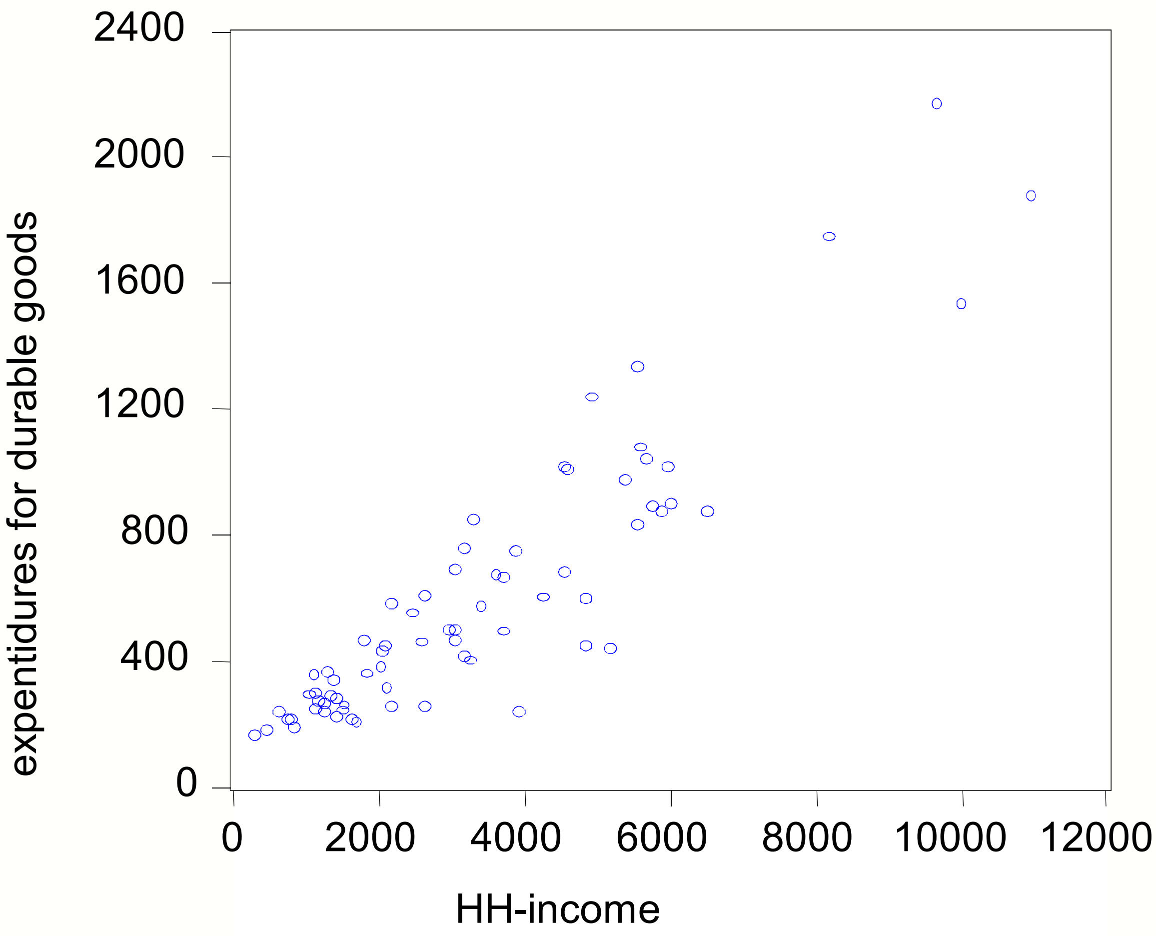
  $$V\{\varepsilon\} = \sigma^2 \Psi$$

  with non-diagonal elements different from zero

Example: Household Income and Expenditures

70 households (HHs):

monthly HH-income and expenditures for durable goods



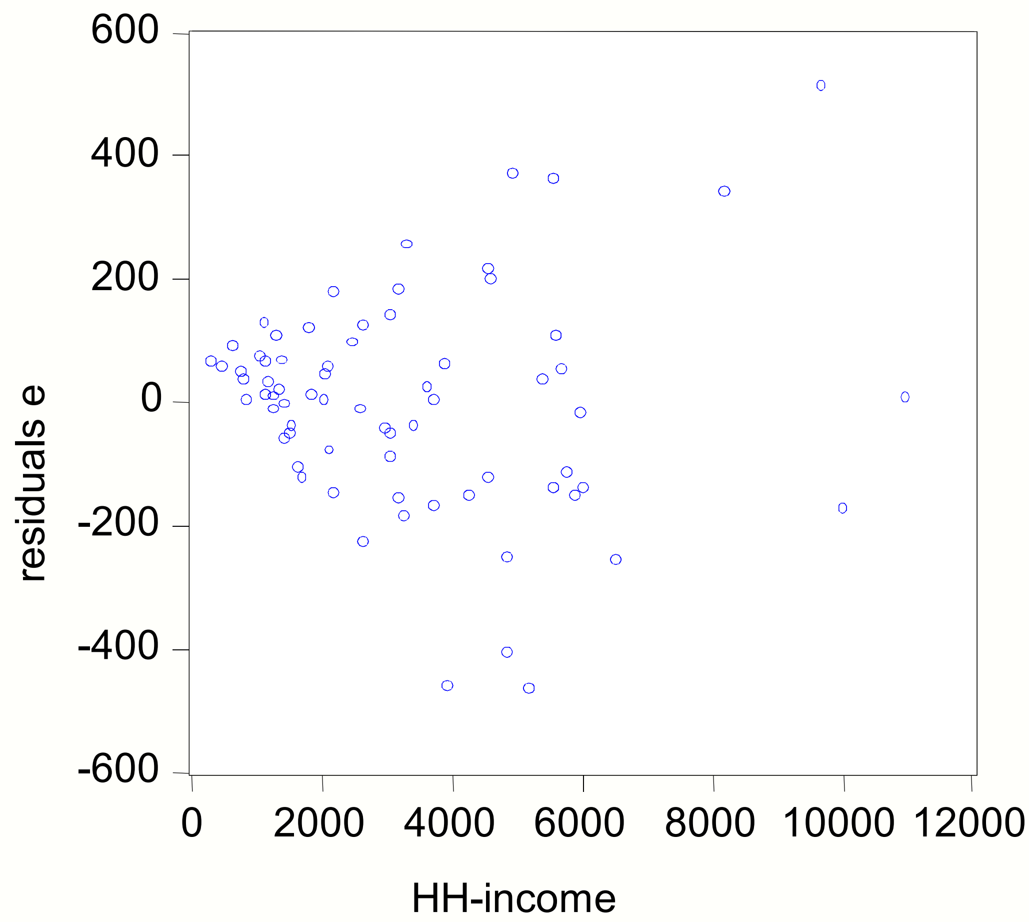
Household Income and Expenditures, cont`d

Residuals e = y- ŷ from

$$\hat{Y} = 44.18 + 0.17\,X$$

X: monthly HH-income
Y: monthly expenditures for durable goods

the larger the income, the more scattered are the residuals



Typical Situations for Heteroskedasticity

Heteroskedasticity is typically observed

- in data from cross-sectional surveys, e.g., surveys in households, firms, or regions
- in data with variance that depends of one or several explanatory variables, e.g., variance of the firms' turnover depends on firm size (in number of staff)
- in data from financial markets, e.g., exchange rates, stock returns

Example: Household Expenditures for Food

Variation of expenditures for food, increasing with growing income; from Verbeek, Fig. 4.1

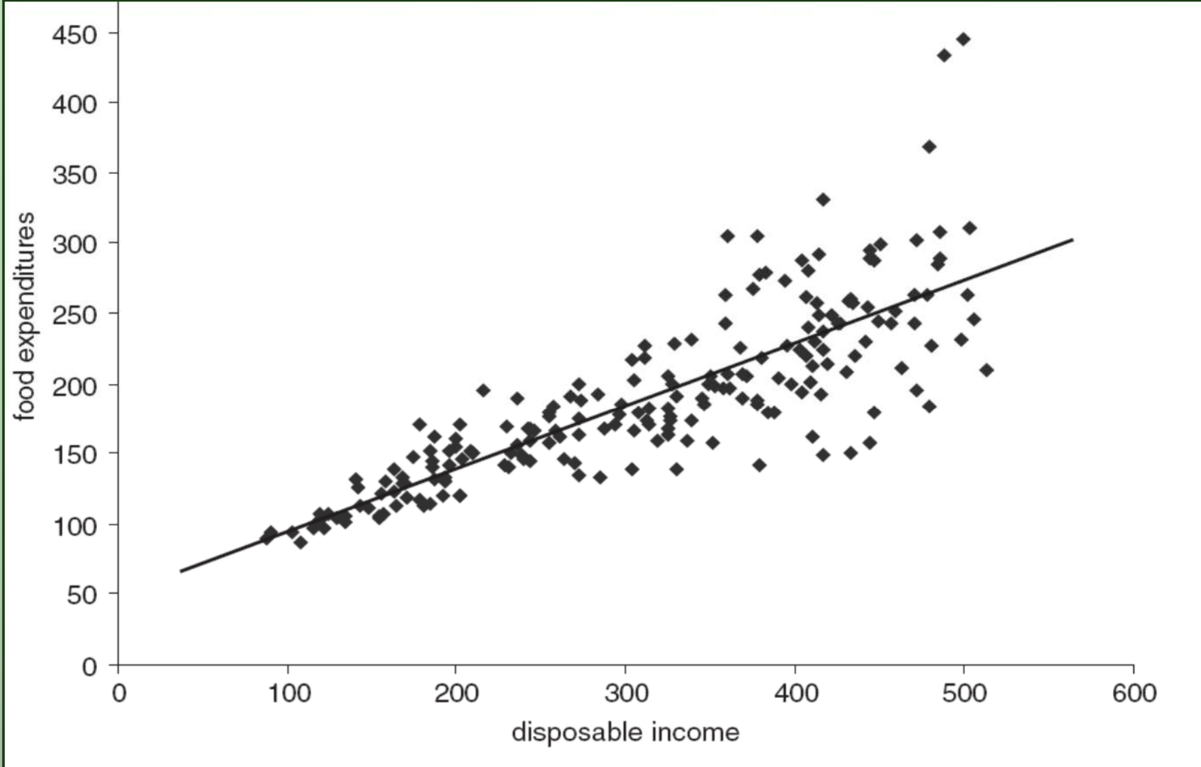


Figure 4.1 An Engel curve with heteroskedasticity

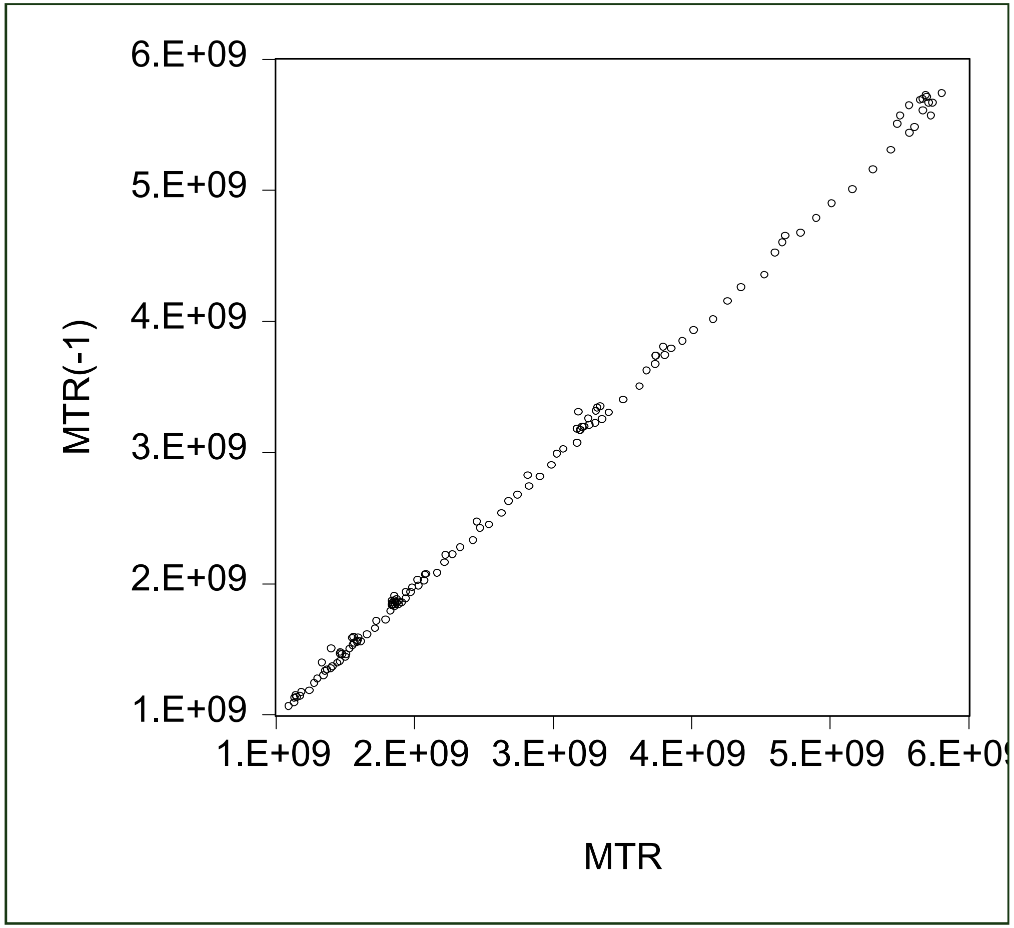
Autocorrelation of Economic Time-series

- Consumption in actual period is similar to that of the preceding period; the actual consumption „depends“ on the consumption of the preceding period
- Consumption, production, investments, etc.: to be expected that successive observations of economic variables correlate positively
- Seasonal adjustment: application of smoothing and filtering algorithms induces correlation of the smoothed data

Example: Imports

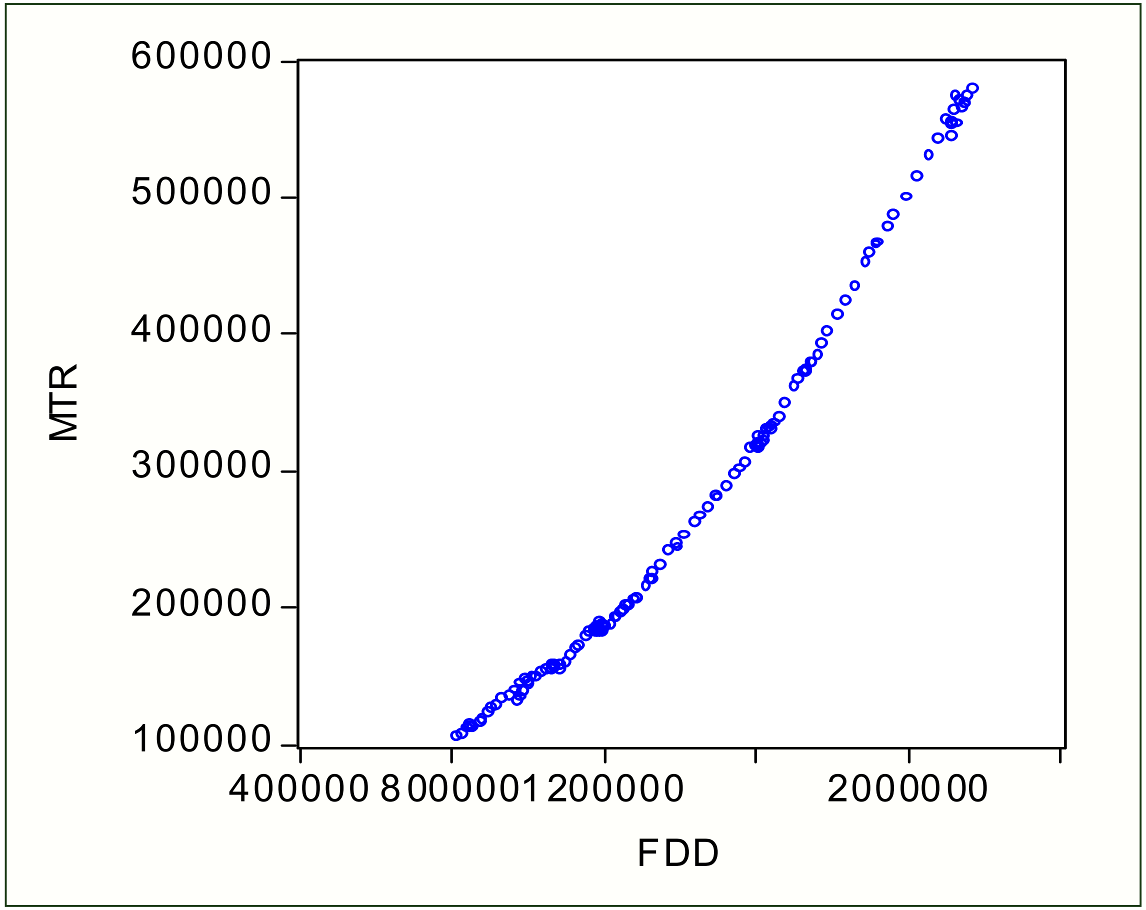
Scatter-diagram of by one period lagged imports [MTR(-1)] against actual imports [MTR]
(MRT from the AWM-database)

Correlation coefficient between MTR und MTR(-1): 0.9994



Example: Import Function

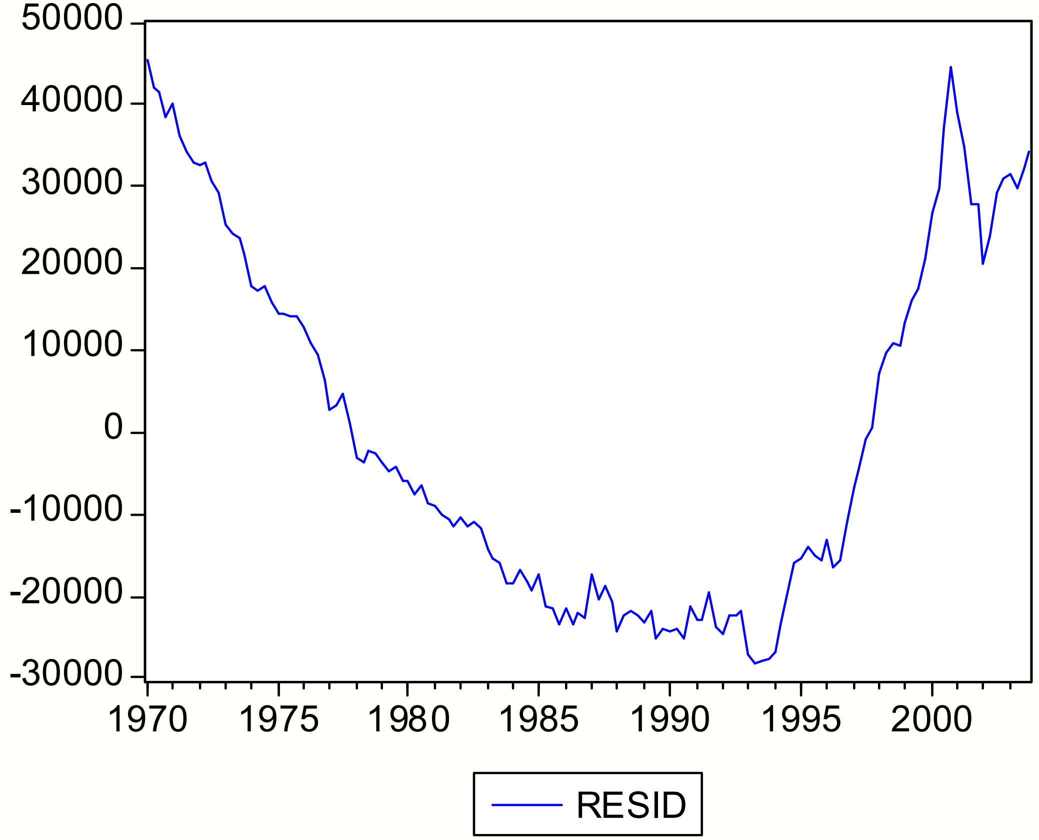
MTR: Imports
FDD: Total Demand
(from AWM-database)



Import function: MTR = -227320 + 0.36 FDD
$R^2 = 0.977$, $t_{FFD} = 74.8$

Import Function: Residuals

MTR: Imports
FDD: Total Demand
(from AWM-database)

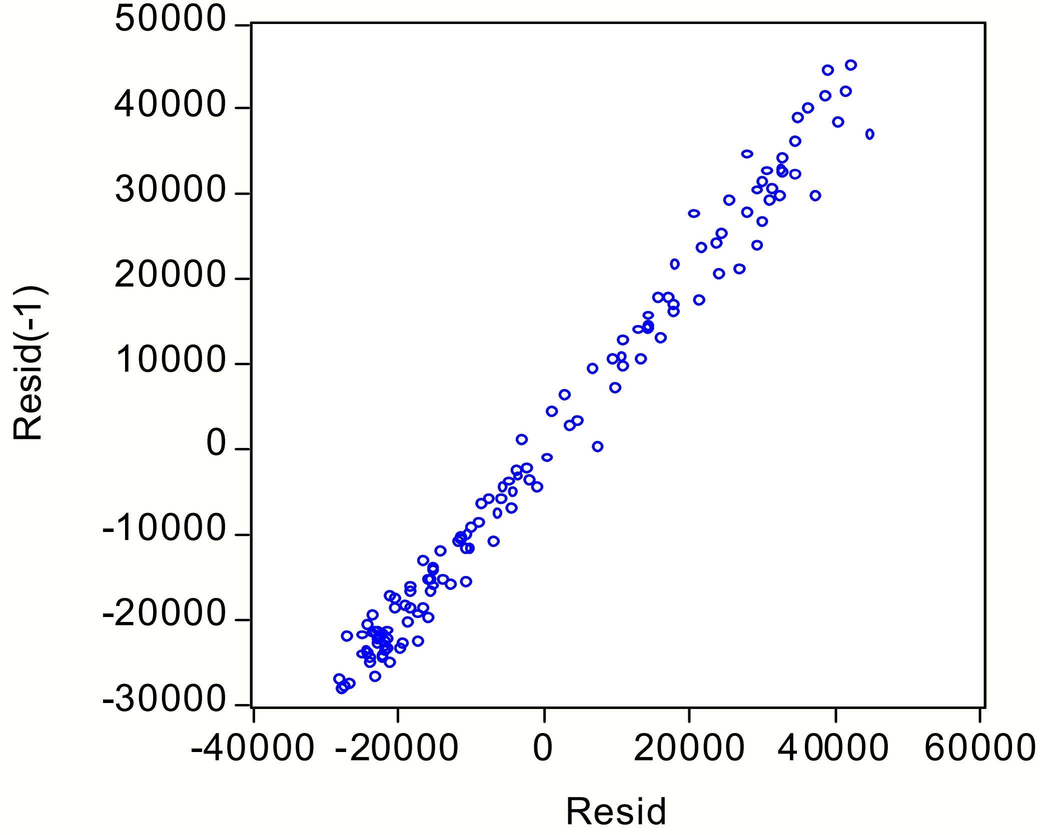


RESID: e_t = MTR - (-227320 + 0.36 FDD)

Import Function: Residuals, cont'd

Scatter-diagram of by one period lagged residuals [Resid(-1)] against actual residuals [Resid]

Serial correlation!



Typical Situations for Autocorrelation

Autocorrelation is typically observed if

- a relevant regressor with trend or seasonal pattern is not included in the model: miss-specified model
- the functional form of a regressor is incorrectly specified
- the dependent variable is correlated in a way that is not appropriately represented in the systematic part of the model

Warning! Omission of a relevant regressor with trend implies autocorrelation of the error terms; in econometric analyses, autocorrelation of the error terms is always to be suspected!

- Autocorrelation of the error terms indicates deficiencies of the model specification
- Tests for autocorrelation are the most frequently used tool for diagnostic checking the model specification

Some Import Functions

- Regression of imports (MTR) on total demand (FDD)

  $MTR = -2.27 \times 10^9 + 0.357\ FDD$, $t_{FDD} = 74.9$, $R^2 = 0.977$

  Autocorrelation (of order 1) of residuals:

  $Corr(e_t, e_{t-1}) = 0.993$
- Import function with trend (T)

  $MTR = -4.45 \times 10^9 + 0.653\ FDD - 0.030 \times 10^9\ T$

  $t_{FDD} = 45.8$, $t_T = -21.0$, $R^2 = 0.995$

  Multicollinearity? Corr(FDD, T) = 0.987!
- Import function with lagged imports as regressor

  $MTR = -0.124 \times 10^9 + 0.020\ FDD + 0.956\ MTR_{-1}$

  $t_{FDD} = 2.89$, $t_{MTR(-1)} = 50.1$, $R^2 = 0.999$

Consequences of $V\{\varepsilon\} \neq \sigma^2 I_N$ for OLS estimators

OLS estimators *b* for β

- are unbiased
- are consistent
- have the covariance-matrix

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

- are not efficient estimators, not BLUE
- follow – under general conditions – asymptotically the normal distribution

The estimator $s^2 = e'e/(N-K)$ for σ^2 is biased

Consequences of $V\{\varepsilon\} \neq \sigma^2 I_N$ for Applications

- OLS estimators *b* for β are still unbiased
- Routinely computed standard errors are biased; the bias can be positive or negative
- *t*- and *F*-tests may be misleading

Remedies

- Alternative estimators
- Corrected standard errors
- Modification of the model

Tests for identification of heteroskedasticity and for autocorrelation are important tools

Contents

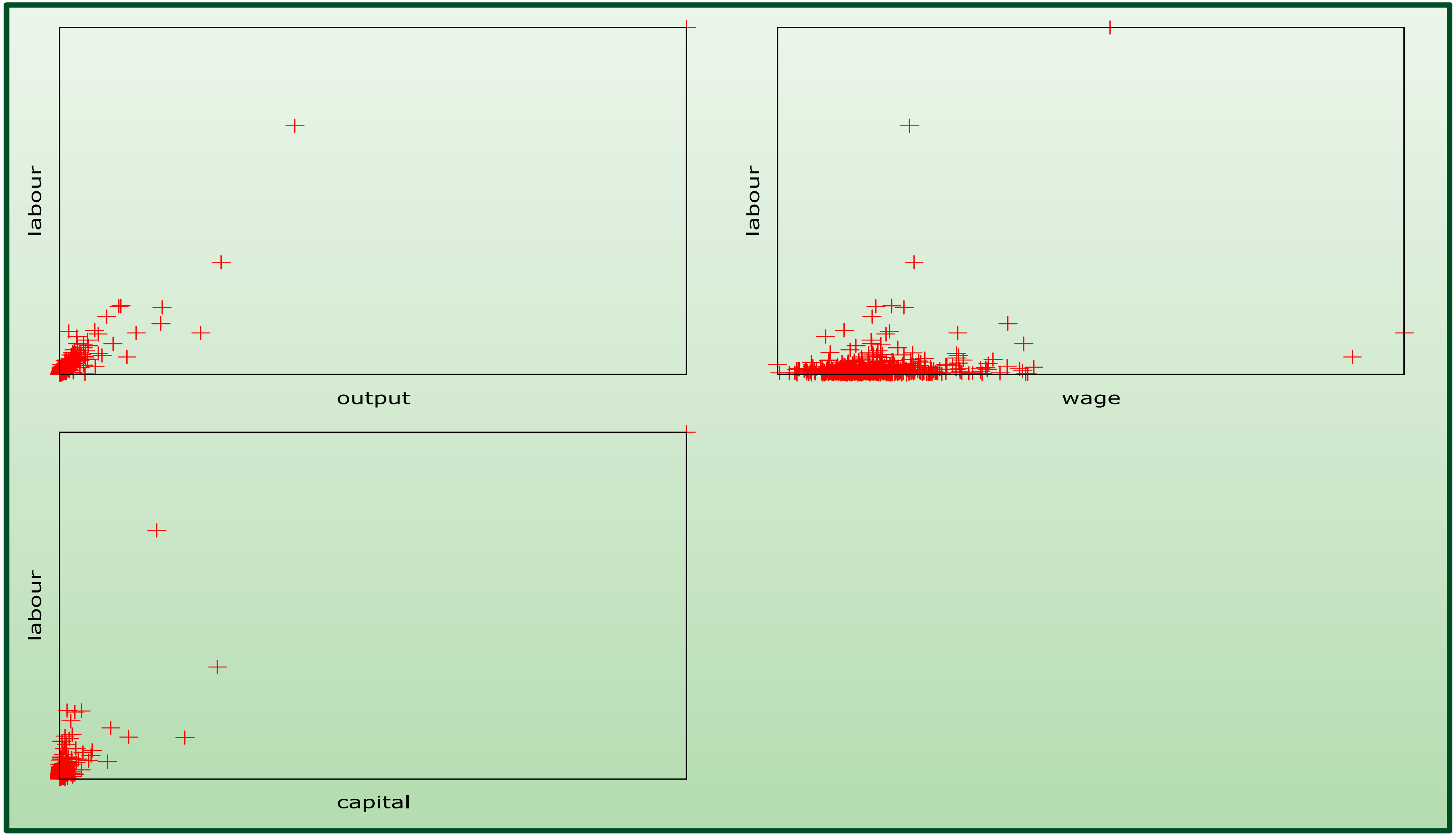
Example: Labor Demand

Verbeek's data set "labour2": Sample of 569 Belgian companies (data from 1996)

- Variables
  - *labour*: total employment (number of employees)
  - *capital*: total fixed assets
  - *wage*: total wage costs per employee (in 1000 EUR)
  - *output*: value added (in million EUR)
- Labour demand function

$$labour = \beta_1 + \beta_2 * wage + \beta_3 * output + \beta_4 * capital$$

Labor Demand and Potential Regressors



Inference under Heteroskedasticity

Covariance matrix of b:

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

with $\Psi = \text{diag}(h_1^2, \ldots, h_N^2)$

Use of $\sigma^2 (X'X)^{-1}$ (the standard output of econometric software) instead of $V\{b\}$ for inference on β may be misleading

Remedies

- Use of correct variances and standard errors
- Transformation of the model so that the error terms are homoskedastic

The Correct Variances

- $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$, $i = 1,\ldots,N$: each observation has its own unknown parameter h_i
- N observation for estimating N unknown parameters?

To estimate σ^2_i – and $V\{b\}$

- Known form of the heteroskedasticity, specific correction
  - E.g., $h_i^2 = z_i'\alpha$ for some variables z_i
  - Requires estimation of α
- White's heteroskedasticity-consistent covariance matrix estimator (HCCME)

$$\tilde{V}\{b\} = \sigma^2 (X'X)^{-1} (\Sigma_i \hat{h}_i^2 x_i x_i')\ (X'X)^{-1}$$

  with $\hat{h}_i^2 = e_i^2$
  - Denoted as HC_0
  - Inference based on HC_0: "heteroskedasticity-robust inference"

White’s Standard Errors

White’s standard errors for *b*

- Square roots of diagonal elements of HCCME
- Underestimate the true standard errors
- Various refinements, e.g., $HC_1 = HC_0[N/(N-K)]$

In **GRETL**: HC_0 is the default HCCME, HC_1 and other modifications are available as options

Labor Demand Function

For Belgian companies, 1996; Verbeek's data set "labour2"

Table 4.1 OLS results linear model

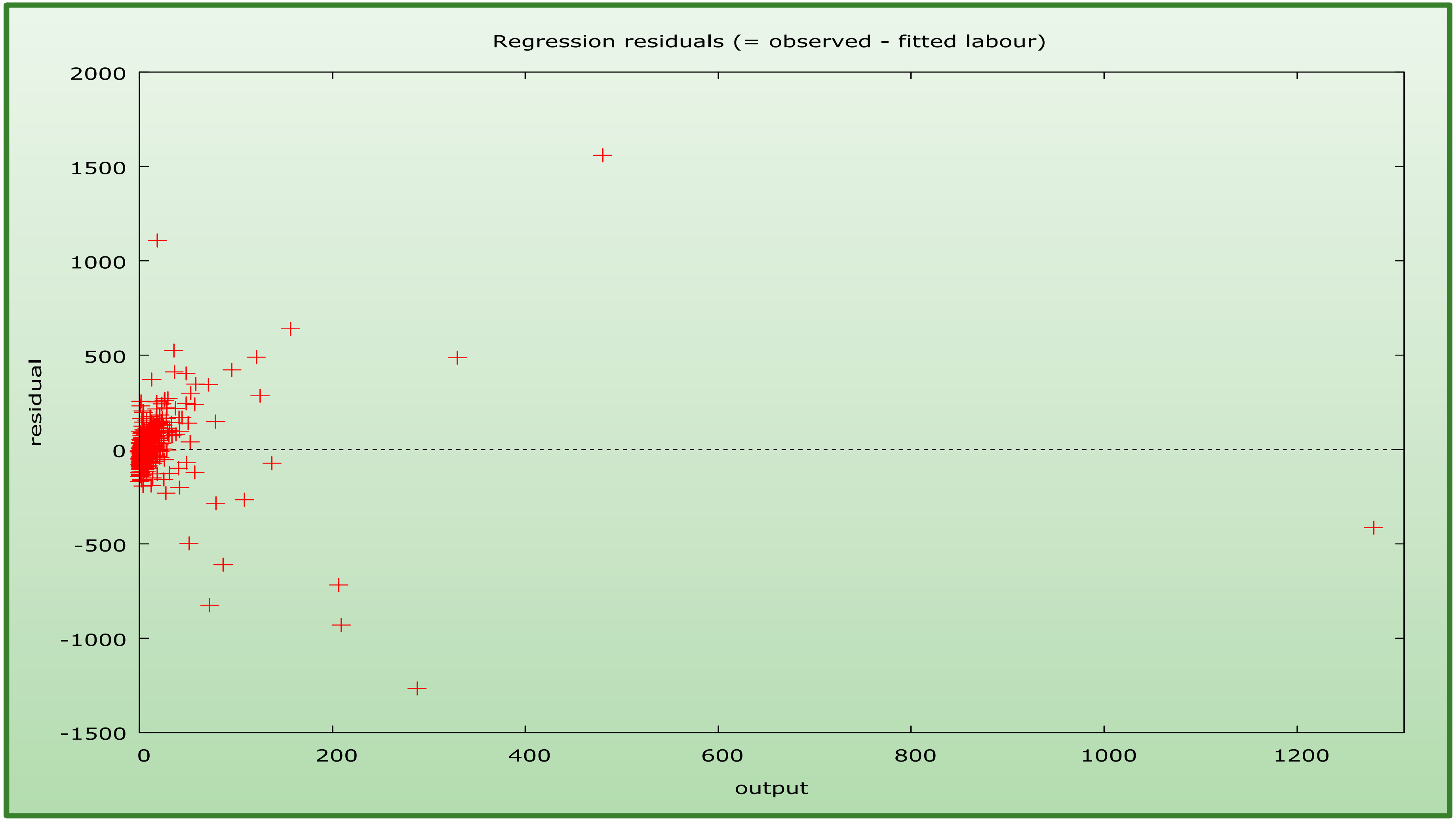
Dependent variable: *labour*

Variable	Estimate	Standard error	t-ratio
constant	287.72	19.64	14.648
wage	−6.742	0.501	−13.446
output	15.40	0.356	43.304
capital	−4.590	0.269	−17.067

$s = 156.26$ $R^2 = 0.9352$ $\bar{R}^2 = 0.9348$ $F = 2716.02$

$labour = \beta_1 + \beta_2 * wage + \beta_3 * output + \beta_4 * capital$

Labor Demand Function: Residuals vs *output*



Labor Demand Function, cont'd

Can the error terms be assumed to be homoskedastic?

- They may vary depending on the company size, measured by, e.g., size of output or capital
- Regression of squared residuals on appropriate regressors will indicate heteroskedasticity

Labor Demand Function, cont'd

Auxiliary regression of squared residuals, Verbeek

Table 4.2 Auxiliary regression Breusch–Pagan test

Dependent variable: e_i^2

Variable	Estimate	Standard error	t-ratio
constant	−22719.51	11838.88	−1.919
wage	228.86	302.22	0.757
output	5362.21	214.35	25.015
capital	−3543.51	162.12	−21.858

$s = 94182 \quad R^2 = 0.5818 \quad \bar{R}^2 = 0.5796 \quad F = 262.05$

Indicates dependence of error terms on *output*, *capital*, not on *wage*

Labor Demand Function, cont'd

With White standard errors: Output from **GRETL**

Dependent variable : LABOR
Heteroskedastic-robust standard errors, variant HC0,

	coefficient	std. error	t-ratio	p-value	
const	287,719	64,8770	4,435	1,11e-05	***
WAGE	-6,7419	1,8516	-3,641	0,0003	***
CAPITAL	-4,5905	1,7133	-2,679	0,0076	***
OUTPUT	15,4005	2,4820	6,205	1,06e-09	***

Mean dependent var	201,024911	S.D. dependent var	611,9959
Sum squared resid	13795027	S.E. of regression	156,2561
R- squared	0,935155	Adjusted R-squared	0,934811
F(2, 129)	225,5597	P-value (F)	3,49e-96
Log-likelihood	455,9302	Akaike criterion	7367,341
Schwarz criterion	-3679,670	Hannan-Quinn	7374,121

Labor Demand Function, cont'd

Estimated function

$$labour = \beta_1 + \beta_2 * wage + \beta_3 * output + \beta_4 * capital$$

OLS estimates and standard errors: without (s.e.) and with White correction (White s.e.) and GLS estimates with $w_i = 1/(e_i^2)$

	β_1	β_2	β_3	β_4
Coeff OLS	287.19	-6.742	15.400	-4.590
s.e.	19.642	0.501	0.356	0.269
White s.e.	64.877	1.852	2.482	1.713
Coeff GLS	321.17	-7.404	15.585	-4.740
s.e.	20.328	0.506	0.349	0.255

The White standard errors are inflated by factors 3.7 (*wage*), 6.4 (*capital*), 7.0 (*output*) with respect to the OLS s.e.

An Alternative Estimator for *b*

Idea of the estimator

1. Transform the model so that it satisfies the Gauss-Markov assumptions
2. Apply OLS to the transformed model

Results in an (at least approximately) BLUE

Transformation often depends upon unknown parameters that characterizing heteroskedasticity: two-step procedure

1. Estimate the parameters that characterize heteroskedasticity and transform the model
2. Estimate the transformed model

The procedure results in an approximately BLUE

An Example

Model:

$y_i = x_i'\beta + \varepsilon_i$ with $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$

Division by h_i results in

$y_i/h_i = (x_i/h_i)'\beta + \varepsilon_i/h_i$

with a homoskedastic error term

$V\{\varepsilon_i/h_i\} = \sigma_i^2/h_i^2 = \sigma^2$

OLS applied to the transformed model gives

$$\hat{\beta} = \left(\sum_i h_i^{-2} x_i x_i'\right)^{-1} \sum_i h_i^{-2} x_i y_i$$

This estimator is an example of the "generalized least squares" (GLS) or "weighted least squares" (WLS) estimator

Weighted Least Squares Estimator

- A GLS or WLS estimator is a least squares estimator where each observation is weighted by a non-negative factor $w_i > 0$:

$$\hat{\beta}_w = \left(\sum_i w_i x_i' x_i \right)^{-1} \sum_i w_i x_i' y_i$$

- Weights w_i proportional to the inverse of the error term variance $\sigma^2 h_i^2$: Observations with a higher error term variance have a lower weight; they provide less accurate information on β
- Needs knowledge of the h_i
  - Is seldom available
  - Estimates of h_i can be based on assumptions on the form of h_i
  - E.g., $h_i^2 = z_i'\alpha$ or $h_i^2 = \exp(z_i'\alpha)$ for some variables z_i
- Analogous with general weights w_i
- White's HCCME uses $w_i = e_i^{-2}$

Labor Demand Function, cont'd

Regression of "l_usq1", i.e., log(e_i^2), on *capital* and *output*

Dependent variable : l_usq1

	coefficient	std. error	t-ratio	p-value
const	7,24526	0,0987518	73,37	2,68e-291 ***
CAPITAL	−0,0210417	0,00375036	−5,611	3,16e-08 ***
OUTPUT	0,0359122	0,00481392	7,460	3,27e-013 ***

Mean dependent var	7,531559	S.D. dependent var	2,368701
Sum squared resid	2797,660	S.E. of regression	2,223255
R- squared	0,122138	Adjusted R-squared	0,119036
F(2, 129)	39,37427	P-value (F)	9,76e-17
Log-likelihood	−1260,487	Akaike criterion	2526,975
Schwarz criterion	2540,006	Hannan-Quinn	2532,060

Labor Demand Function, cont'd

Estimated function

labour = β_1 + β_2***wage* + β_3***output* + β_4***capital*

OLS estimates and standard errors: without (s.e.) and with White correction (White s.e.); and GLS estimates with $w_i = e_i^{-2}$, with fitted values for e_i from the regression of $\log(e_i^2)$ on *capital* and *output*

	β_1	*wage*	*output*	*capital*
OLS coeff	**287.19**	**-6.742**	**15.400**	**-4.590**
s.e.	19.642	0.501	0.356	0.269
White s.e.	64.877	1.852	2.482	1.713
FGLS coeff	**321.17**	**-7.404**	**15.585**	**-4.740**
s.e.	20.328	0.506	0.349	0.255

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Tests against Heteroskedasticity

Due to unbiasedness of *b*, residuals are expected to indicate heteroskedasticity

Graphical displays of residuals may give useful hints

Residual-based tests:

- Breusch-Pagan test
- Koenker test
- Goldfeld-Quandt test
- White test

Breusch-Pagan Test

For testing whether the error term variance is a function of $Z_2, \ldots, Z_p$

Model for heteroskedasticity

$$\sigma_i^2/\sigma^2 = h(z_i'\alpha)$$

with function h with $h(0)=1$, p-vectors z_i und α, z_i containing an intercept and p-1 variables $Z_2, \ldots, Z_p$

Null hypothesis

$$H_0: \alpha = 0$$

implies $\sigma_i^2 = \sigma^2$ for all i, i.e., homoskedasticity

Breusch-Pagan Test, cont'd

Typical functions h for $h(z_i'\alpha)$

- Linear regression: $h(z_i'\alpha) = z_i'\alpha$
- Exponential function $h(z_i'\alpha) = \exp\{z_i'\alpha\}$
  - Auxiliary regression of the log (e_i^2) upon z_i
  - "Multiplicative heteroskedasticity"
  - Variances are non-negative

For $h(z_i'\alpha) = z_i'\alpha$

- Auxiliary regression of the "scaled" squared residuals $u_i^2 = e_i^2/s^2$ with $s^2 = e'e/N$ on z_i (and squares of z_i);
- Test statistic BP = N*R^2 with R^2 from the auxiliary regression
- Test statistic BP follows approximately the Chi-squared distribution with p -1 d.f.

Koenker Test

Koenker test: variant of the BP test which is robust against non-normality of the error terms

- For testing whether the error term variance is a function of $Z_2, \ldots, Z_p$
- Auxiliary regression of the squared OLS residuals e_i^2 on z_i

$$e_i^2 = z_i'\alpha + v_i$$

Test statistic: $N^*R_v^2$ with R_v^2 of the auxiliary regression; follows approximately the Chi-squared distribution with p -1 d.f.

- **GRETL**: The output window of OLS estimation allows the execution of the Breusch-Pagan test with $h(z_i'\alpha) = z_i'\alpha$
  - OLS output => Tests => Heteroskedasticity => Breusch-Pagan
  - Koenker test: OLS output => Tests => Heteroskedasticity => Koenker

Labor Demand Function, cont'd

Auxiliary regression of squared residuals, Verbeek

Tests of the null hypothesis of homoskedasticity

Table 4.2 Auxiliary regression Breusch–Pagan test

Dependent variable: e_i^2

Variable	Estimate	Standard error	t-ratio
constant	−22719.51	11838.88	−1.919
wage	228.86	302.22	0.757
output	5362.21	214.35	25.015
capital	−3543.51	162.12	−21.858

$s = 94182$ $R^2 = 0.5818$ $\bar{R}^2 = 0.5796$ $F = 262.05$

Breusch-Pagan: BP = 5931.8, *p*-value = 0

Koenker: NR^2 = 569*0.5818 = 331.04, *p*-value = 2.17E-70

Goldfeld-Quandt Test

For testing whether the error term variance has values σ_A^2 and σ_B^2 for observations from regime A and B, respectively, $\sigma_A^2 \neq \sigma_B^2$

Regimes can be urban vs rural area, economic prosperity vs stagnation, etc.

Example (in matrix notation):

$y_A = X_A\beta_A + \varepsilon_A, \;\; V\{\varepsilon_A\} = \sigma_A^2 I_{NA}$ (regime A)

$y_B = X_B\beta_B + \varepsilon_B, \;\; V\{\varepsilon_B\} = \sigma_B^2 I_{NB}$ (regime B)

Null hypothesis: $\sigma_A^2 = \sigma_B^2$

Test statistic:

$$F = \frac{S_A}{S_B} \frac{N_B - K}{N_A - K}$$

with S_i: sum of squared residuals for *i*-th regime; follows under H_0 exactly or approximately the *F*-distribution with N_A-K and N_B-K d.f.

Goldfeld-Quandt Test, cont'd

Test procedure in three steps:

1. Sort the observations with respect to the regimes A and B
2. Separate fittings of the model to the N_A and N_B observations; sum of squared residuals S_A and S_B
3. Calculate the test statistic F

White Test

For testing whether the error term variance is a function of the model regressors, their squares and their cross-products; generalizes the Breusch-Pagan test

Auxiliary regression of the squared OLS residuals upon x_i's, squares of x_i's, and cross-products

Test statistic: NR^2 with R^2 of the auxiliary regression; follows the Chi-squared distribution with the number of coefficients in the auxiliary regression as d.f.

The number of coefficients in the auxiliary regression may become large, maybe conflicting with the size of N, resulting in low power of the White test

Labor Demand Function, cont'd

White's test for heteroskedasticity
OLS, using observations 1-569
Dependent variable: uhat^2

	coefficient	std. error	t-ratio	p-value	
const	-260,910	18478,5	-0,01412	0,9887	
WAGE	554,352	833,028	0,6655	0,5060	
CAPITAL	2810,43	663,073	4,238	2,63e-05	***
OUTPUT	-2573,29	512,179	-5,024	6,81e-07	***
sq_WAGE	-10,0719	9,29022	-1,084	0,2788	
X2_X3	-48,2457	14,0199	-3,441	0,0006	***
X2_X4	58,5385	8,11748	7,211	1,81e-012	***
sq_CAPITAL	14,4176	2,01005	7,173	2,34e-012	***
X3_X4	-40,0294	3,74634	-10,68	2,24e-024	***
sq_OUTPUT	27,5945	1,83633	15,03	4,09e-043	***

Unadjusted R-squared = 0,818136

Test statistic: TR^2 = 465,519295,
with p-value = P(Chi-square(9) > 465,519295) = 0

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Transformed Model Satisfying Gauss-Markov Assumptions

Model:

$$y_i = x_i'\beta + \varepsilon_i \quad \text{with } V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$$

Division by h_i results in

$$y_i / h_i = (x_i / h_i)'\beta + \varepsilon_i / h_i$$

with a homoskedastic error term

$$V\{\varepsilon_i / h_i\} = \sigma_i^2 / h_i^2 = \sigma^2$$

OLS applied to the transformed model gives

$$\hat{\beta} = \left(\sum_i h_i^{-2} x_i x_i' \right)^{-1} \sum_i h_i^{-2} x_i y_i$$

This estimator is an example of the "generalized least squares" (GLS) or "weighted least squares" (WLS) estimator

Properties of GLS Estimators

The GLS estimator

$$\hat{\beta} = \left(\sum_i h_i^{-2} x_i x_i' \right)^{-1} \sum_i h_i^{-2} x_i y_i$$

is a least squares estimator; standard properties of OLS estimator apply

- The covariance matrix of the GLS estimator is

$$V\{\hat{\beta}\} = \sigma^2 \left(\sum_i h_i^{-2} x_i x_i' \right)^{-1}$$

- Unbiased estimator of the error term variance

$$\hat{\sigma}^2 = \frac{1}{N-K} \sum_i h_i^{-2} \left(y_i - x_i' \hat{\beta} \right)^2$$

- Under the assumption of normality of errors, *t*- and *F*-tests can be used; for large *N*, these properties hold approximately without normality assumption

Generalized Least Squares Estimator

- A GLS or WLS estimator is a least squares estimator where each observation is weighted by a non-negative factor
- Example:

  $y_i = x_i'\beta + \varepsilon_i$ with $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$

  - Division by h_i results in a model with homoskedastic error terms

    $V\{\varepsilon_i / h_i\} = \sigma_i^2 / h_i^2 = \sigma^2$

  - OLS applied to the transformed model results in the weighted least squares (GLS) estimator with $w_i = h_i^{-2}$:

    $$\hat{\beta} = \left(\sum_i h_i^{-2} x_i x_i'\right)^{-1} \sum_i h_i^{-2} x_i y_i$$

  - Transformation corresponds to the multiplication of each observation with the non-negative factor h_i^{-1}
- The GLS estimator is a LS estimator $b_W = (\Sigma_i w_i x_i x_i')^{-1}(\Sigma_i w_i x_i x_i)$ that weights the *i*-th observation with $w_i = h_i^{-2}$, so that the Gauss-Markov assumptions are satisfied

Feasible GLS Estimator

Is a GLS estimator with estimated weights $w_i = h_i^{-2}$

- Substitution of the weights $w_i = h_i^{-2}$ by estimates $\hat{h}_i^{-2}$

$$\hat{\beta}^* = \left(\sum_i \hat{h}_i^{-2} x_i x_i' \right)^{-1} \sum_i \hat{h}_i^{-2} x_i y_i$$

- Feasible (or estimated) GLS or FGLS or EGLS estimator
- For consistent estimates $\hat{h}_i$, the FGLS and GLS estimators are asymptotically equivalent
- For small values of N, FGLS estimators are in general not BLUE
- For consistent estimates $\hat{h}_i$, the FGLS estimator is consistent and asymptotically efficient with covariance matrix (estimate for σ^2: based on FGLS residuals)

$$V\left\{ \hat{\beta}^* \right\} = \hat{\sigma}^2 \left(\sum_i \hat{h}_i^{-2} x_i x_i' \right)^{-1}$$

- Warning: The transformed model is uncentered

Multiplicative Heteroskedasticity

Assume $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2 = \sigma^2 \exp\{z_i'\alpha\}$

- The auxiliary regression

  $$\log e_i^2 = \log \sigma^2 + z_i'\alpha + v_i$$

  provides a consistent estimator a for α
- Transform the model $y_i = x_i'\beta + \varepsilon_i$ with $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$ by dividing through $\hat{h}_i$ from $\hat{h}_i^2 = \exp\{z_i'a\}$
- Error term in this model is (approximately) homoskedastic
- Applying OLS to the transformed model gives the FGLS estimator for β

FGLS Estimation

In the following steps ($y_i = x_i'\beta + \varepsilon_i$):

1. Calculate the OLS estimates *b* for β
2. Compute the OLS residuals $e_i = y_i - x_i' b$
3. Regress $\log(e_i^2)$ on z_i and a constant, obtaining estimates *a* for α

   $$\log e_i^2 = \log \sigma^2 + z_i'\alpha + v_i$$

4. Compute $\hat{h}_i^2 = \exp\{z_i' a\}$, transform all variables and estimate the transformed model to obtain the FGLS estimators:

   $$y_i / \hat{h}_i = (x_i / \hat{h}_i)'\beta + \varepsilon_i / \hat{h}_i$$

5. The consistent estimate s^2 for σ^2, based on the FGLS-residuals, and the consistently estimated covariance matrix

   $$\hat{V}\left\{\hat{\beta}^*\right\} = s^2 \left(\sum_i \hat{h}_i^{-2} x_i x_i' \right)^{-1}$$

   are part of the standard output when regressing the transformed model

FGLS Estimation in GRETL

Preparatory steps:

1. Calculate the OLS estimates b for β of $y_i = x_i'\beta + \varepsilon_i$
2. Under the assumption $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$, conduct an auxiliary regression for e_i^2 or $\log(e_i^2)$ that provides estimates $\hat{h}_i^2$
3. Define *wtvar* as weight variable with $wtvar_i = (\hat{h}_i^2)^{-1}$

FGLS estimation:

4. Model => Other linear models => Weighted least squares
5. Use of variable *wtvar* as "Weight variable": both the dependent and all independent variables are multiplied with the square roots $(wtvar)^{1/2}$

Labor Demand Function

For Belgian companies, 1996; Verbeek

Table 4.5 OLS results loglinear model with White standard errors

Dependent variable: log(*labour*)

Variable	Estimate	Heteroskedasticity-consistent Standard error	*t*-ratio
constant	6.177	0.294	21.019
log(*wage*)	−0.928	0.087	−10.706
log(*output*)	0.990	0.047	21.159
log(*capital*)	−0.004	0.038	−0.098

$s = 0.465 \quad R^2 = 0.8430 \quad \bar{R}^2 = 0.8421 \quad F = 544.73$

Log-transformation is expected to reduce heteroskedasticity

Labor Demand Function, cont'd

Estimated function

$\log(labour) = \beta_1 + \beta_2 * \log(wage) + \beta_3 * \log(output) + \beta_4 * \log(capital)$

The table shows: OLS estimates and standard errors: without (s.e.) and with White correction (White s.e.); FGLS estimates and standard errors

	β_1	*wage*	*output*	*capital*
OLS coeff	**6.177**	**-0.928**	**0.990**	**-0.0037**
s.e.	0.246	0.071	0.026	0.0188
White s.e.	0.293	0.086	0.047	0.0377
FGLS coeff	**5.895**	**-0.856**	**1.035**	**-0.0569**
s.e.	0.248	0.072	0.027	0.0216

Labor Demand Function, cont'd

For Belgian companies, 1996; Verbeek

Table 4.6 Auxiliary regression multiplicative heteroskedasticity

Dependent variable: $\log e_i^2$

Variable	Estimate	Standard error	t-ratio
constant	−3.254	1.185	−2.745
log(*wage*)	−0.061	0.344	−0.178
log(*output*)	0.267	0.127	2.099
log(*capital*)	−0.331	0.090	−3.659

$s = 2.241 \quad R^2 = 0.0245 \quad \bar{R}^2 = 0.0193 \quad F = 4.73$

Breusch-Pagan test: BP = 66.23, *p*-value: 1,42E-13

Labor Demand Function, cont'd

For Belgian companies, 1996; Verbeek

Weights estimated assuming multiplicative heteroskedasticity

Table 4.7 EGLS results loglinear model

Dependent variable: log(*labour*)

Variable	Estimate	Standard error	*t*-ratio
constant	5.895	0.248	23.806
log(*wage*)	−0.856	0.072	−11.903
log(*output*)	1.035	0.027	37.890
log(*capital*)	−0.057	0.022	−2.636

$s = 2.509 \quad R^2 = 0.9903 \quad \bar{R}^2 = 0.9902 \quad F = 14401.3$

Labor Demand Function, cont'd

Estimated function

$\log(labour) = \beta_1 + \beta_2 * \log(wage) + \beta_3 * \log(output) + \beta_4 * \log(capital)$

The table shows: OLS estimates and standard errors: without (s.e.) and with White correction (White s.e.); FGLS (or EGLS) estimates and standard errors

	β_1	*wage*	*output*	*capital*
OLS coeff	**6.177**	**-0.928**	**0.990**	**-0.0037**
s.e.	0.246	0.071	0.026	0.0188
White s.e.	0.293	0.086	0.047	0.0377
FGLS coeff	**5.895**	**-0.856**	**1.035**	**-0.0569**
s.e.	0.248	0.072	0.027	0.0216

Labor Demand Function, cont'd

Some comments:

- Reduction of standard errors in FGLS estimation as compared to heteroskedasticity-robust estimation, efficiency gains
- Comparison with OLS estimation not appropriate
- FGLS estimates differ slightly from OLS estimates; effect of capital is indicated to be relevant (*p*-value: 0.0086)
- R^2 of FGLS estimation is misleading
  - Model has no intercept, is uncentered
  - Comparison with that of OLS estimation not appropriate, explained variables are different

Your Homework

1. Use the data set “labour2” of Verbeek for the following analyses:
   a) (i) Estimate (OLS) the model for log(*labor)* with regressors log(*output)* and log(*wage)*; (ii) generate a graphical display of the residuals which may indicate heteroskedasticity of the error term.
   b) Perform (i) the Breusch-Pagan test with $h(z_i'\alpha) = \exp\{z_i'\alpha\}$, z_i = (log(*capital*$_i$), log(*output*$_i$), log(*wage*$_i$))’, and the White test (ii) without and (iii) with interactions; explain the tests and compare the results.
   c) For the model of a): Compare (i) the OLS and (ii) the White standard errors with HC_0 of the estimated coefficients.
   d) Estimate (i) the model of a), using FGLS and weights obtained in the auxiliary regression of the Breusch-Pagan test in b); (ii) comment on the estimates of the coefficients, the standard errors, and the R^2 of this model and those of c)(i) and (ii).

Your Homework, cont'd

2. Transform the variables of the model $y_i = x_i'\beta + \varepsilon_i$ with $E\{\varepsilon_i\} = 0$ and $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$ for $i = 1, \ldots, N$, by dividing each variable through h_i: $y_i \rightarrow y_i / h_i$ and $(x_i)' \rightarrow (x_i / h_i)'$. Show that for the model in transformed variables,

   $$y_i / h_i = (x_i / h_i)'\beta + \varepsilon_i / h_i$$

   the Gauss-Markov assumptions A3 and A4 are satisfied.