MUNI
ECON

SUPTECH WORKSHOP III

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Background Session I – Introduction to modern portfolio theory

A primer on microeconomics

Utility and choice

Preference relation

$$a \succ b \qquad a \sim b \qquad a \prec b$$

Rationality assumptions:

- Every investor possesses a **complete** preference relation.
- The preference relation satisfies the property of **transitivity**.
- The preference relation is **continuous**.

Utility

Previous are sufficient to guarantee the existence of a continuous function $u : \mathbb{R}^N \to \mathbb{R}$ such that, for any consumption bundles a and b,

$$a \succeq b \Leftrightarrow u(a) \geq u(b)$$

This real-valued function u is called a **utility function**.

Risk aversion

Consider an investor with wealth Y and a fair-game lottery $L = (h, -h, 0.5)$ with $h > 0$.

We say an investor is **risk averse** iff

$$Y \succ Y + L$$

This implies the utility function to be strictly concave:

$$E[U(Y)] > E[U(Y + L)]$$

$$U(Y) > \frac{1}{2}U(Y + h) + \frac{1}{2}U(Y - h)$$

Thus, $U''(Y) < 0$ and we have decreasing marginal utility of wealth.

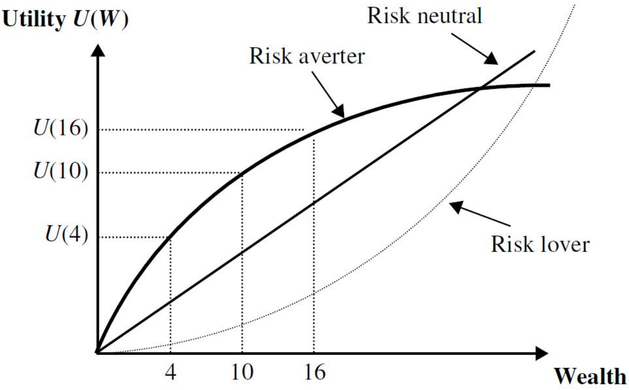


Figure 2 Utility functions

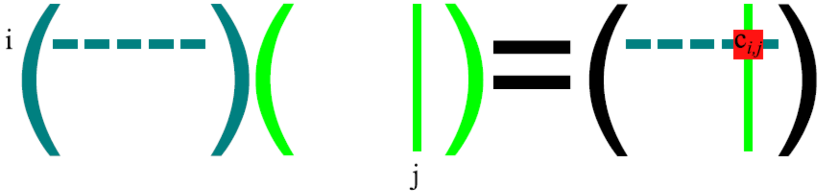
Linear algebra basics

Linear algebra basics

A vector $\boldsymbol{x} \in \mathbb{R}^n$ and a matrix $\boldsymbol{A} \in \mathbb{R}^n \times \mathbb{R}^m$ for $n, m \in \mathbb{N}$.

$$\mathbf{1}_n = (1, 1, ..., 1) \qquad \boldsymbol{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 4 & 5 & 6 \end{pmatrix}, \quad \mathbf{A}^T = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \\ 3 & 0 & 6 \end{pmatrix}$$



$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix} = ?$$

Some examples

$$\mathbf{r} = (r_1, r_2, ..., r_n)$$
$$E(\mathbf{r}) = (E(r_1), E(r_2), ..., E(r_n))$$
$$\mathbf{w}^T\mathbf{r} = (w_1, w_2, ..., w_n)\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = w_1 r_1 + w_2 r_2 + ... + w_n r_n = \sum_{i=1}^{n} w_i r_i$$
$$\mathbf{1}_n^T\mathbf{w} = (1, 1, ..., 1)\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \sum_{i=1}^{n} w_i$$

More examples

Let $\boldsymbol{w} \in \mathbb{R}^n$ and $\boldsymbol{\Sigma} \in \mathbb{R}^N \times \mathbb{R}^n$.

$$\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} = (w_1, w_2, ..., w_n) \begin{pmatrix} \text{cov}(r_1, r_1) & \text{cov}(r_1, r_2) & \cdots & \text{cov}(r_1, r_n) \\ \text{cov}(r_2, r_1) & \text{cov}(r_2, r_2) & \cdots & \text{cov}(r_2, r_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(r_n, r_1) & \text{cov}(r_n, r_2) & \cdots & \text{cov}(r_n, r_n) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{cov}(r_i, r_j)$$

Expected value

$$E(X) = \sum p_i x_i$$

Properties:

- **E1** $E(a) = a$
- **E2** $E(aX) = aE(X)$
- **E3** $E(X + Y) = E(X) + E(Y)$

Covariance

$$cov(X,Y) = E[(X - E(X))(Y - E(Y))]$$

Properties:

- **C1** $cov(a, X) = 0$
- **C2** $cov(X, Y) = cov(Y, X)$
- **C3** $cov(a + bX, Y) = bcov(X, Y)$
- **C4** $cov(X + Y, Z) = cov(X, Z) + cov(Y, Z)$

Variance

$$var(X) = E[(X - E(X))^2] = cov(X, X)$$

Properties:

- **D1** $var(a) = 0$
- **D2** $var(a + bX) = b^2 var(X)$
- **D3** $var(X + Y) = var(X) + var(Y) + 2cov(X, Y)$

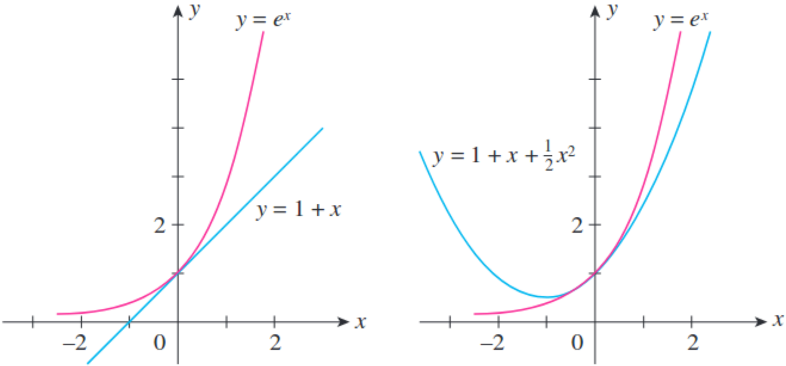
Correlation

$$\rho(X, Y) = \frac{cov(X, Y)}{\sqrt{var(X) var(Y)}}$$

Mean-variance portfolio theory

Mean-variance preferences

The general n-variate problem is difficult, often simplified to M-V. Taylor expansion of $U(\tilde{Y}_1)$ around $E(\tilde{Y}_1)$:



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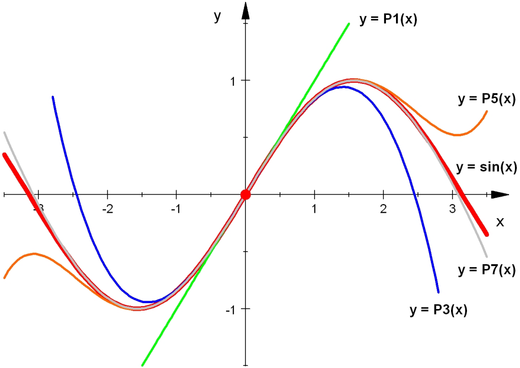


Mean-variance preferences

The general n-variate problem is difficult, often simplified to M-V. Taylor expansion of $U(\tilde{Y}_1)$ around $E(\tilde{Y}_1)$:

$$\begin{aligned} U(\tilde{Y}_1) &= U(E[\tilde{Y}_1]) + \\ & U'(E[\tilde{Y}_1]) \cdot (\tilde{Y}_1 - E[\tilde{Y}_1]) + \\ & \frac{1}{2} U''(E[\tilde{Y}_1]) \cdot (\tilde{Y}_1 - E[\tilde{Y}_1])^2 + \varepsilon \end{aligned}$$

thus for the **expected utility**

$$E[U(\tilde{Y}_1)] = U(E[\tilde{Y}_1]) + \frac{1}{2} U''(E[\tilde{Y}_1]) \cdot Var[\tilde{Y}_1] + E[\varepsilon]$$

The case with two assets

$$r_1, r_2 \qquad \sigma_1^2, \sigma_2^2 \qquad \rho_{1,2} = \mathrm{cov}(r_1, r_2) \quad w_1, w_2$$

$$r_p = w_1 r_1 + w_2 r_2$$

$$E(r_p) = ??$$

$$\sigma_p = ??$$

The case with two assets

$$r_1, r_2 \qquad \sigma_1^2, \sigma_2^2 \qquad \rho_{1,2} = \text{cov}(r_1, r_2) \quad w_1, w_2$$

$$r_p = w_1 r_1 + w_2 r_2$$

$$E(r_p) = w_1 E(r_1) + w_2 E(r_2)$$

$$\sigma_p^2 = \text{var}(r_p) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \text{cov}(r_1, r_2)$$

Portfolio – initial setup, n assets

Let $\boldsymbol{w} \in \mathbb{R}^n$ and $\boldsymbol{\Sigma} \in \mathbb{R}^N \times \mathbb{R}^n$.

$$\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} = (w_1, w_2, ..., w_n) \begin{pmatrix} \text{cov}(r_1, r_1) & \text{cov}(r_1, r_2) & \cdots & \text{cov}(r_1, r_n) \\ \text{cov}(r_2, r_1) & \text{cov}(r_2, r_2) & \cdots & \text{cov}(r_2, r_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(r_n, r_1) & \text{cov}(r_n, r_2) & \cdots & \text{cov}(r_n, r_n) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \text{cov}(r_i, r_j)$$

The diversification/insurance principle

$$\begin{aligned}\sigma_p^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{cov}(r_i, r_j) \\ &= \sum_{i=1}^{n} w_i^2 \mathrm{var}(r_i) + 2\sum_{i=1}^{n}\sum_{j>i} w_i w_j \mathrm{cov}(r_i, r_j)\end{aligned}$$

For mutually uncorrelated r_i with equal variance $\sigma_i = \sigma$, and an equal weights strategy $w_i = 1/n$ we have

$$\sigma_p^2 = \sum_{i=1}^{n} \frac{1}{n^2}\sigma^2 = \frac{n}{n^2}\sigma^2 = \frac{\sigma^2}{n}$$

And for the limiting case $\lim_{n\to\infty} \sigma_p^2 = \lim_{n\to\infty} \sigma^2/n = 0$

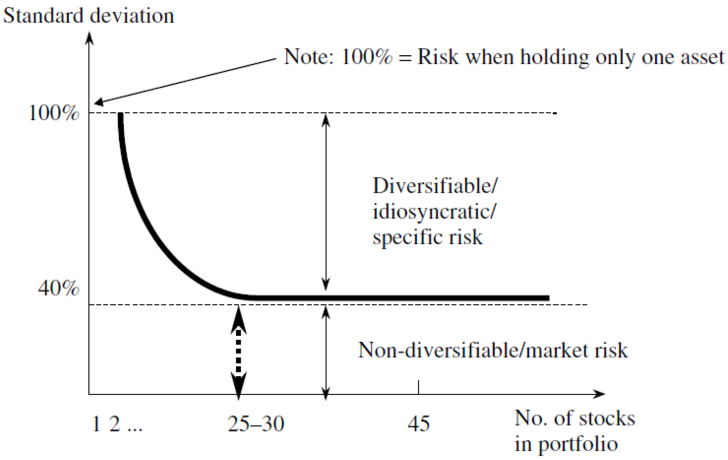


Figure 1 Random selection of stocks

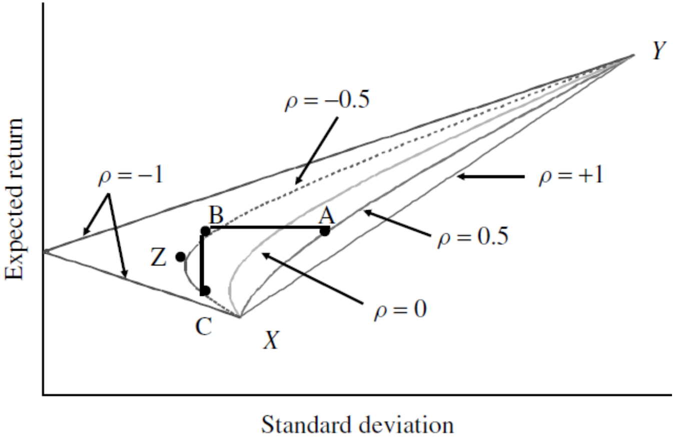


Figure 3 Efficient frontier and correlation

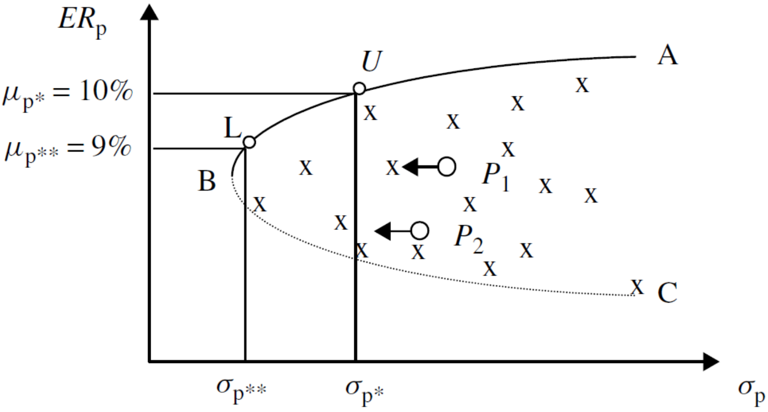


Figure 4 Efficient and inefficient portfolios

Risk-free asset: transformation line

Assume a risk-free asset with $E(r) = r$ and $\sigma_r = 0$.
Now we mix a risky asset $(r_m, E(r_m), \sigma_m)$ with the risk free asset. For the portfolio with w_1 invested into the risky asset, we get

$$E(r_p) = w_m E(r_m) + (1 - w_m) r$$

$$\sigma_p = w_m \sigma_m$$

$$\begin{aligned} E(r_p) &= \frac{\sigma_p}{\sigma_m} E(r_m) + \left(1 - \frac{\sigma_p}{\sigma_m}\right) r \\ &= r + \frac{E(r_m) - r}{\sigma_m} \sigma_p \\ &= \delta_0 + \delta_m \sigma_p \end{aligned}$$

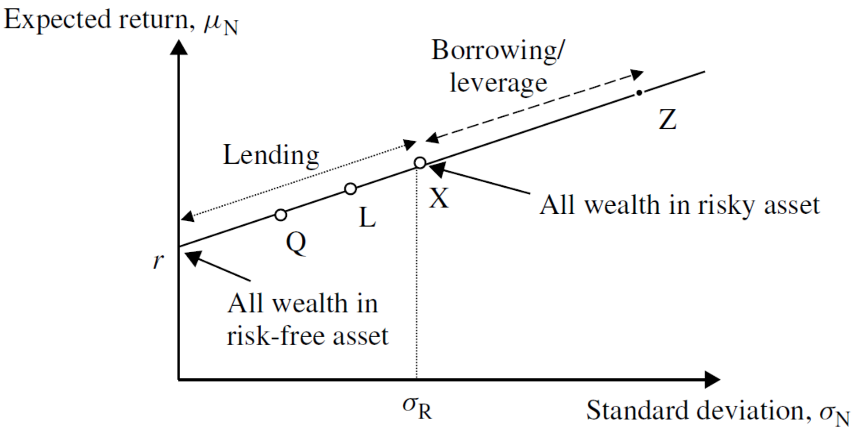
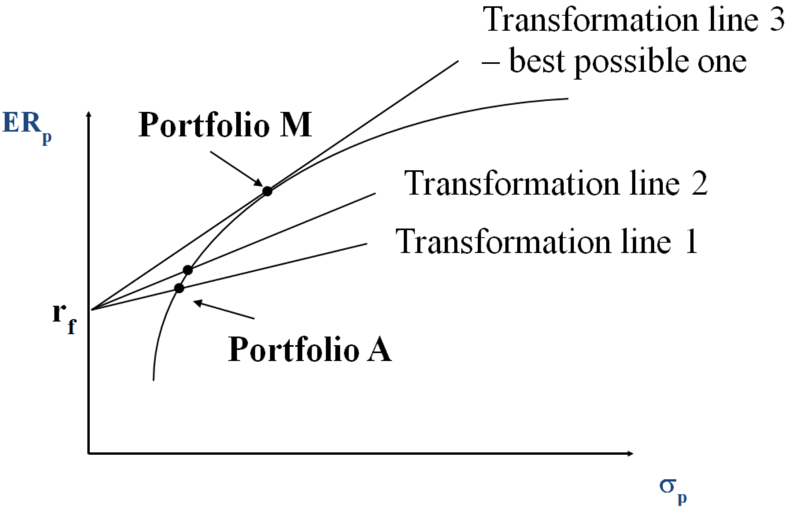


Figure 5 Transformation line


Transformation line 3
– best possible one
ER_p
Portfolio M
Transformation line 2
Transformation line 1
r_f
Portfolio A
σ_p

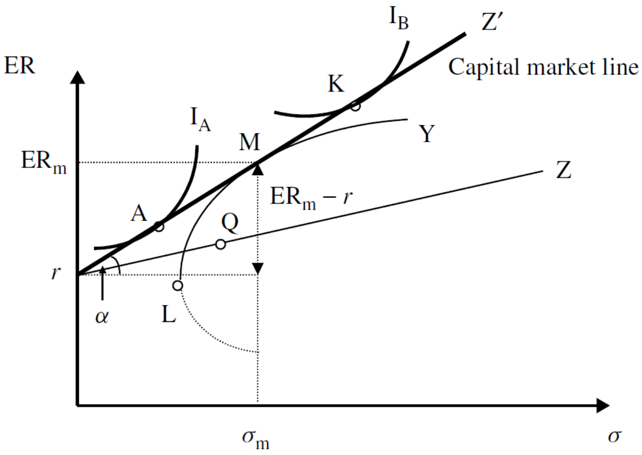


Figure 6 Portfolio choice

Separation principle

The investor makes two separate decisions:

- without any recourse to the individual's preferences, the investor determines the point of tangency, the **market portfolio**.
- the investor then determines how he will **combine** the market portfolio of risky assets with the riskless asset

The slope of the CML is called **market price of risk**.

$$\frac{dE(r_p)}{d\sigma_p} = \frac{E(r_m) - r}{\sigma_m}$$

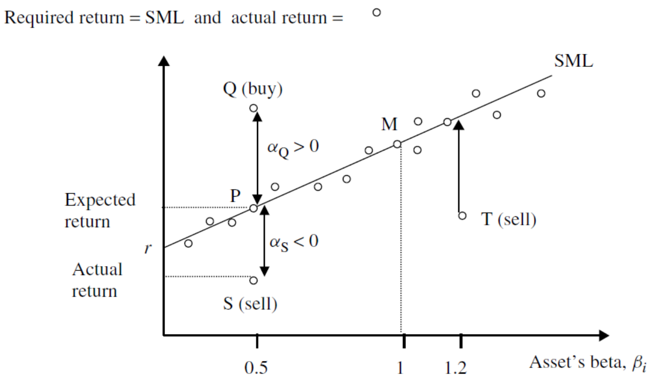
Capital Asset Pricing Model – CAPM

$$E(r_i) = r + \beta[E(r_m) - r]$$

$$\beta = \frac{\operatorname{cov}(r_i, r_m)}{\sigma_m^2}$$

CAPM assumptions

- Assumption 1 :
  - Investors agree in their forecasts of expected returns, standard deviation and correlations
  - Therefore all investors optimally hold risky assets in the same relative proportions
- Assumption 2 :
  - Investors generally behave optimally. In equilibrium prices of securities adjust so that when investors are holding their optimal portfolio, aggregate demand equals its supply.



Securities that lie above (below) the SML have a positive (negative) 'alpha', indicating a positive (negative) 'abnormal return', after correcting for 'beta risk'.

Figure 8 Security market line, SML

Systematic risk

$$\sigma_p = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{cov}(r_i, r_j)$$
$$= \sum_{i=1}^{n} w_i^2 \mathrm{var}(r_i) + 2\sum_{i=1}^{n}\sum_{j>i} w_i w_j \mathrm{cov}(r_i, r_j)$$

For constant covariance $cov(r_i, r_j) = cov, i \neq j$ with equal variance $\sigma_i^2 = \sigma^2$, and an equal weights strategy $w_i = 1/n$ we have

$$\sigma_p^2 = \frac{n}{n^2}\sigma^2 + \frac{n(n-1)}{n^2} cov = \frac{1}{n}\sigma^2 + \left(1 - \frac{1}{n}\right) cov$$

And for the limiting case $\lim_{n\to\infty} \sigma_p^2 = cov$

Properties of Betas

Betas represent an asset's systematic (market or non-diversifiable) risk.

Beta of the market portfolio : $\beta_m = 1$

Beta of the risk-free asset: $\beta_r = 0$

Portfolio beta: $\beta_p = \sum w_i \beta_i$

Applications of betas: market timing (bull/bear markets), portfolio construction, performance measures, risk management.

The Efficient Frontier: Markowitz Model

$$\min \frac{1}{2}\sigma_p^2 = \frac{1}{2}\mathbf{w}^T\boldsymbol{\Omega}\mathbf{w}$$

$$\mathbf{w}^T E(\mathbf{r}) = E(r_p)$$

$$\mathbf{w}^T\mathbf{1} = 1$$

The Two-Fund Theorem: if $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ represent efficient portfolios, then $\alpha\boldsymbol{w}_1 + (1-\alpha)\boldsymbol{w}_2$ is also an efficient portfolio for any $\alpha \in \mathbb{R}$.

Borrowing and Lending: Market Portfolio

$$\max \frac{E(r_p) - r}{\sigma_p}$$

$$\mathbf{w}^T E(\mathbf{r}) = E(r_p)$$

$$\mathbf{w}^T \mathbf{1} = 1$$

$$\sigma_p = \mathbf{w}^T \Omega \mathbf{w}$$

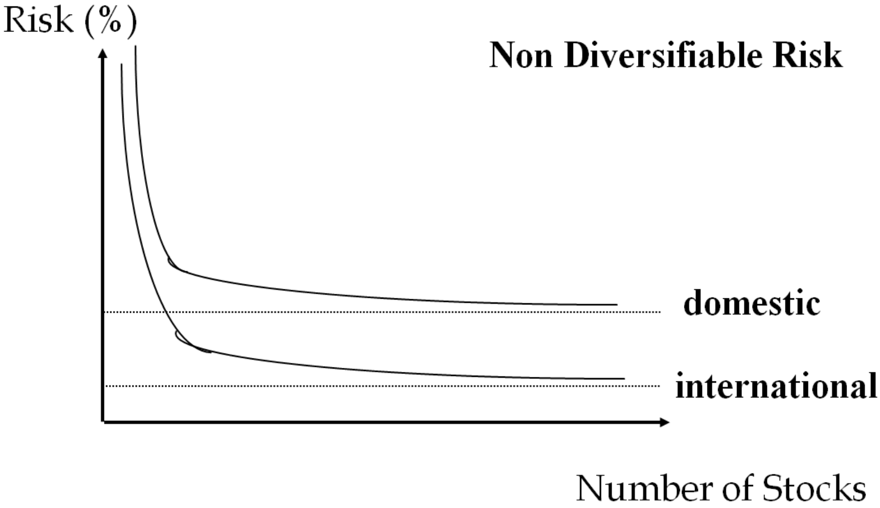
International Diversification

International investments:

- Can you enhance your risk return profile ?
  - US investors seem to overweight US stocks
  - Other investors prefer their home country (**Home country bias**)

International diversification is easy (and 'cheap')

- Improvements in technology (the internet)
- 'Customer friendly' products : Mutual funds, investment trusts, index funds


Risk (%)
Non Diversifiable Risk
domestic
international
Number of Stocks

Benefits and Costs of Intl. Investments

Benefits:

- Interdependence of domestic and international stock markets
- Interdependence between the foreign stock returns and exchange rate

Costs:

- Equity risk: could be more (or less than domestic market)
- Exchange rate risk
- Political risk
- Information risk

Performance Measures / Risk Adjusted Rate of Return

- **Sharpe ratio:**

$$SR_i = (E(r_i) - r)/\sigma_i$$

Risk is measured by the standard deviation (total risk of security). Aim: maximize. (CML)

- **Treynor ratio:**

$$TR_i = (E(r_i) - r)/\beta_i$$

Risk is measured by beta (market risk only). Aim: maximize. (SML)

Estimating the Betas

Time series regression:

$$r_{i,t} - r_t = \alpha_i + \beta_i (r_{m,t} - r_t) + \varepsilon_{i,t}$$

Arbitrage pricing theory

$$R_{i,t} = a_i + \sum_{j=1}^{k} b_{i,j} F_{j,t} + \varepsilon_{i,t}$$

Note that the factors are common to all stocks, ie, they are *pervasive* risk factors. The parameters $\beta_{i,j}$, called *factor loadings*, measure the sensitivity of security i to factor j. The random variable $\varepsilon_{i,t}$ is the residual, the part of return not explained by the common factors ($var[\varepsilon_{i,j}]$ is idiosyncratic risk).

Exact factor pricing, single factor

Assume a single factor is responsible for return dynamics:

$$r_j = a_j + \beta_j F$$

Construct a portfolio consisting of a risk-free asset and the factor with weights $(w_f, w_F)^T = (1 - \beta_j, \beta_j)^T$. The portfolio return is then

$$r_p = w_f r_f + w_F F = (1 - \beta_j) r_f + \beta_j F$$

The slopes are the same, and by *no arbitrage condition*, so must be the intercepts, $a_j = (1 - \beta_j) r_f$. Thus, $r_j = r_f + \beta_j (F - r_f)$ and for the expectation

$$E(r_j) = r_f + \beta_j (E(F) - r_f)$$

When the risk factor is the market, we replace F with R_m and get CAPM.

Fama and French (1993) 3-factor model

$$E(r_j) - r_f = \beta_j[E(r_m) - r_f] + \beta_{js}E[SMB] + \beta_{jh}E[HML]$$

Small Value	Big Value
Small Neutral	Big Neutral
Small Growth	Big Growth

Thresholds: size (median), book to market equity (30th & 70th percentile)

Size premium: $SMB = (SmallValue + SmallNeutral + SmallGrowth)/3 - (BigValue + BigNeutral + BigGrowth)/3$,

Value premium: $(SmallValue + BigValue)/2 - (SmallGrowth + BigGrowth)/2$.

<Ken French data library>

Empirical testing of CAPM

Time-series tests:

$$E(R_{i,t}) - r_t = \alpha_i + \beta_i[E(R_{m,t}) - r_t]$$

Cross-section: **Fama-MacBeth** (1973) rolling regression For any single t, run

$$\mathbf{R}_t = \alpha_t^T \mathbf{e} + \gamma_t \boldsymbol{\beta} + \theta_t \mathbf{Z}_t + \boldsymbol{\varepsilon}$$

If CAPM holds, then for any t, $\alpha_t = \theta_t = 0$ and $\gamma_t > 0$. Thus, it is easy to test for $E(\alpha_t) = 0$, $E(\theta_t) = 0$ and $E(\gamma_t) > 0$ from a sample from $t = 1, 2, ..., T$.