Seminář 3 **Příklad 1: Určete symetrickou matici A, která je určena kvadratickou formou.**

a) $Q := (x, y) \to x^{\wedge}2 + x\,y + y^{\wedge}2$

$$(x, y) \to x^2 + x\,y + y^2$$

$with(VectorCalculus) : A := \dfrac{Hessian(\,Q(x, y), [x, y]\,)}{2};$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

b) $Q := (x, y) \to a\,x^{\wedge}2 + b\,x\,y + c\,y^{\wedge}2$

$$(x, y) \to a\,x^2 + b\,x\,y + c\,y^2$$

$A := \dfrac{Hessian(\,Q(x, y)\,, [x, y]\,)}{2};$

$$\begin{bmatrix} a & \frac{1}{2}\,b \\ \frac{1}{2}\,b & c \end{bmatrix}$$

c) $Q := (x1, x2, x3) \to \ 3\,x1^{\wedge}2 - 2\cdot x1\,x2 + 3\,x1\,x3 + x2^{\wedge}2 + x3^{\wedge}2$

$$(x1, x2, x3) \to 3\,x1^2 + VectorCalculus{:-}\grave{}-\grave{}(2\,x1\,x2) + 3\,x1\,x3 + x2^2 + x3^2$$

$A := \dfrac{1}{2}(Hessian(\,3\,x1^{\wedge}2 - 2\cdot x1\,x2 + 3\,x1\,x3 + x2^{\wedge}2 + x3^{\wedge}2,$
$[x1, x2, x3]\,));$

$$\begin{bmatrix} 3 & -1 & \frac{3}{2} \\ -1 & 1 & 0 \\ \frac{3}{2} & 0 & 1 \end{bmatrix}$$

Příklad 2: Pomocí Sylvestrova kritéria rozhodněte o definitnosti kvadratickéformy (s využitím sw ověřte také dle znamének vlastních čísel):

a) $Q := (x, y) \rightarrow 2\ x\text{^}2\ -2\cdot x\, y\ +\ y\text{^}2$

$$(x, y) \rightarrow 2\,x^2 + VectorCalculus\text{:-`-`}(2\,x\,y) + y^2$$

$A := \dfrac{Hessian(\ Q(x, y), [x, y]\ )}{2};$

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$with(LinearAlgebra) : Eigenvalues\,(A)$

$$\begin{bmatrix} \frac{3}{2} + \frac{1}{2}\sqrt{5} \\ \frac{3}{2} - \frac{1}{2}\sqrt{5} \end{bmatrix}$$

b) $Q := (x, y) \rightarrow\ -3\ x\text{^}2\ +\ 8\cdot x\, y\ -6\, y\text{^}2$

$$(x, y) \rightarrow VectorCalculus\text{:-`-`}\left(3\,x^2\right) + 8\,x\,y + VectorCalculus\text{:-`-`}\left(6\,y^2\right)$$

$A := \dfrac{Hessian(\ Q(x, y), [x, y]\ )}{2};$

$$\begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$$

$Eigenvalues\,(\%)$

$$\begin{bmatrix} -\frac{9}{2} + \frac{1}{2}\sqrt{73} \\ -\frac{9}{2} - \frac{1}{2}\sqrt{73} \end{bmatrix}$$

c) $Q := (x1, x2, x3) \rightarrow\ \ -x1\text{^}2\ +\ 4\cdot x1\ x2\ +\ 10\, x2\ x3\ +\ 4\, x2\text{^}2\ +\ 6\, x3\text{^}2$

$$(x1, x2, x3) \rightarrow VectorCalculus\text{:-`-`}\left(x1^2\right) + 4\,x1\ x2 + 10\,x2\ x3 + 4\,x2^2 + 6\,x3^2$$

$A := \dfrac{1}{2}(Hessian(\ -x1\text{^}2\ +\ 4\cdot x1\ x2\ +\ 10\, x2\ x3\ +\ 4\, x2\text{^}2 + 6\, x3\text{^}2, [x1, x2, x3]\ ));$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{bmatrix}$$

$A[1,1], Determinant(A[1..2,1..2]), Determinant(A)$

$$-1, -8, -23$$

$Eigenvalues(A)$

$$\begin{bmatrix} 1 \\ 4+\sqrt{39} \\ 4-\sqrt{39} \end{bmatrix}$$

Příklad 3: Vypočítejte gradienty všech následujících funkcí v zadaných bodech.a) f(x,y) v bodě [2,1] $f := (x,y) \to x^2 + x\,y$

$$(x,y) \to x^2 + x\,y$$

$with(Student[MultivariateCalculus]) : Gradient(f(x,y),$
$[x, y])$

$$\begin{bmatrix} 2x+y \\ x \end{bmatrix}$$

$Gradient(f(x,y), [x, y] = [2, 1])$

$$\left[\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right]$$

b) g(x,y,z) v bodě [0,0,1]

$g := (x,y,z) \to x\exp(x\,y) - z^2$

$$(x,y,z) \to x\,e^{x\,y} + VectorCalculus{:-}`-`(z^2)$$

$Gradient(x\exp(x\,y) - z^2, [x, y, z])$

$$\begin{bmatrix} e^{x\,y} + x\,y\,e^{x\,y} \\ x^2\,e^{x\,y} \\ -2\,z \end{bmatrix}$$

$Gradient(x\exp(x\,y) - z^2, [x, y, z] = [0, 0, 1])$

$$\left[\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right]$$

c) h(x,y,z) v bodě [0,0,0]

$h := (x, y, z) \rightarrow \exp(x) + \exp(2y) + \exp(3z)$

$$(x, y, z) \rightarrow e^{x} + e^{2y} + e^{3z}$$

Gradient$(\exp(x) + \exp(2y) + \exp(3z), [x, y, z])$

$$\begin{bmatrix} e^{x} \\ 2\,e^{2y} \\ 3\,e^{3z} \end{bmatrix}$$

Gradient$(\exp(x) + \exp(2y) + \exp(3z), [x, y, z] = [0, 0, 0])$

$$\begin{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\end{bmatrix}$$

c) k(x,y,z) v bodě [0,0,0]

$k := (x, y, z) \rightarrow \exp(x + 2y + 3z)$

$$(x, y, z) \rightarrow e^{x+2y+3z}$$

Gradient$(\exp(x + 2y + 3z), [x, y, z])$

$$\begin{bmatrix} e^{x+2y+3z} \\ 2\,e^{x+2y+3z} \\ 3\,e^{x+2y+3z} \end{bmatrix}$$

Gradient$(\exp(x + 2y + 3z), [x, y, z] = [0, 0, 0])$

$$\begin{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\end{bmatrix}$$

Příklad 4: Vypočítejte směrovou derivaci následujících funkcí v daném bodě av daném směru.

a) f(x,y) v bodě [2,1] a směru (1,1): $f := (x, y) \rightarrow 2x + y - 1$

$$(x, y) \rightarrow 2x + y + \mathit{VectorCalculus}\text{:-`-`}(1)$$

$\mathit{Diff}(f(x, y), x) = \mathit{diff}(f(x, y), x);$

$$\frac{\partial}{\partial x}(2x + y - 1) = 2$$

$\mathit{Diff}(f(x, y), y) = \mathit{diff}(f(x, y), y);$

$$\frac{\partial}{\partial y}(2x + y - 1) = 1$$

$DirectionalDerivative\left(f(x,y),\ [x,y]=[2,1],\ [1,1]\right)$

$$\frac{3}{2}\sqrt{2}$$

b) g(x,y,z) v bodě [0,1,1] a směru (1,1,1): $g := (x,y,z) \rightarrow x\ \exp(x\,y) - x\,y - z^2$

$$(x,y,z) \rightarrow x\, e^{x\,y} + VectorCalculus:\text{-}`\text{-}`(x\,y) + VectorCalculus:\text{-}`\text{-}`(z^2)$$

$Gradient\left(x\ \exp(x\,y) - x\,y - z^2,\ [x,y,z]\right)$

$$\begin{bmatrix} e^{x\,y} + x\,y\,e^{x\,y} - y \\ x^2\,e^{x\,y} - x \\ -2\,z \end{bmatrix}$$

$DirectionalDerivative\left(x\ \exp(x\,y) - x\,y - z^2,\ [x,y,z]=[0,1,1],\ [1,1,1]\right)$

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Error, (in Student:-MultivariateCalculus:-DirectionalDerivative)
include points for only the x and y coordinates
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Příklad 5: Najděte kvadratickou aproximaci (Taylorův polynom druhého řádu)v bodě [0; 0] pro funkce

a) f(x,y): $f := (x,y) \rightarrow \exp(x\,y) TaylorApproximation\left(f(x,y), [x,y]=[0,0], 2\right);$

$$2\,x + y - 1$$

$a := unapply(\%, x, y)$

$$(x,y) \rightarrow 2\,x + y - 1$$

$evalf(a(.1, .1))$

$$-0.7$$

$f(.1, .1)$

$$-0.7$$

b) f(x,y): $f := (x,y) \rightarrow \exp(x^2 - y^2)$

$$(x,y) \rightarrow e^{x^2 + VectorCalculus:\text{-}`\text{-}`(y^2)}$$

$TaylorApproximation\left(\exp(x^2 - y^2), [x,y]=[0,0], 2\right);$

$$x^2 - y^2 + 1$$

$a := unapply(\%, x, y)$

$$(x,y) \rightarrow x^2 - y^2 + 1$$

$evalf(a(.1, .1))$

$$1.$$

$f(.1, .1)$

$$1.$$

c) f(x,y): $f := (x, y) \to \ln(1 + x^2 y)$

$$(x, y) \to \ln(1 + x^2 y)$$

$TaylorApproximation(\ln(1 + x^2 y), [x, y] = [0, 0], 2);$

$$0$$

$a := unapply(\%, x, y)$

$$(x, y) \to 0$$

$evalf(a(.1, .1))$

$$0.$$

$f(.1, .1)$

$$0.000999500333$$

d) f(x,y): $f := (x, y) \to \ln(1 + x^2 + y^2)$

$$(x, y) \to \ln(1 + x^2 + y^2)$$

$TaylorApproximation(\ln(1 + x^2 + y^2), [x, y] = [0, 0], 2);$

$$x^2 + y^2$$

$a := unapply(\%, x, y)$

$$(x, y) \to x^2 + y^2$$

$evalf(a(.1, .1))$

$$0.02$$

$f(.1, .1)$

$$0.0198026273$$