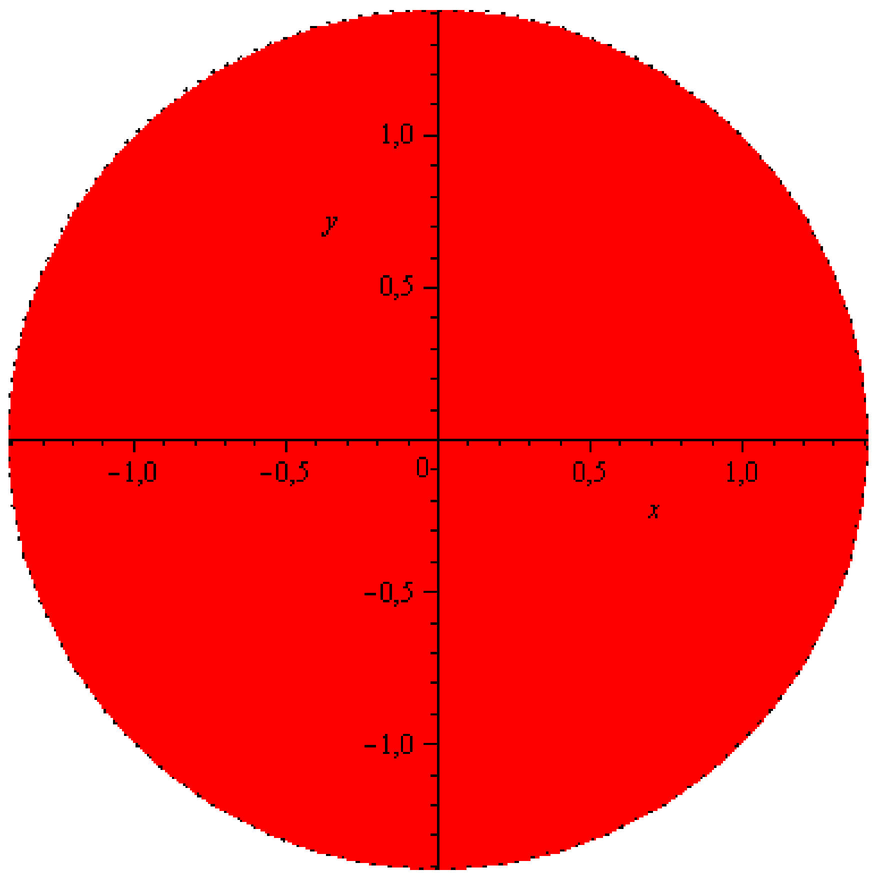
Seminář 4 **Příklad 1: Načrtněte uvedené množiny a rozhodněte, zda jsou konvexní:**

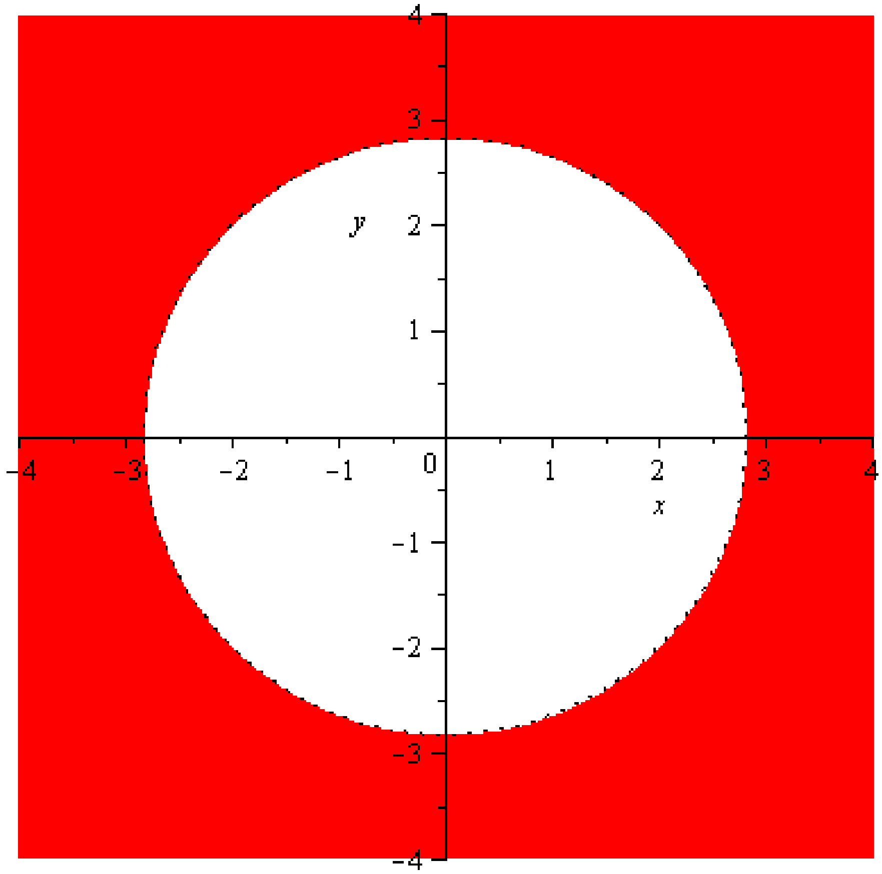
a) { (x,y): x^2+y^2<2}

$with(plots) : implicitplot(x^2 + y^2 < 2, x = -2..2, y = -2..2, filled = true)$



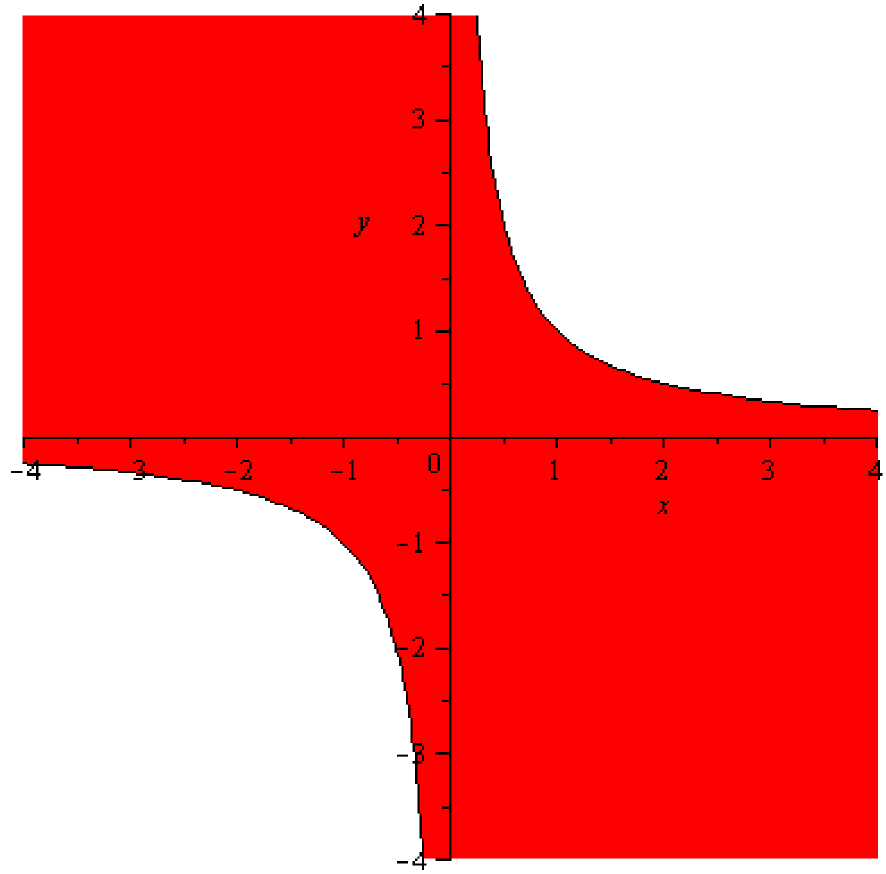
b) { (x,y): x^2+y^2>8}

$implicitplot(x^2 + y^2 > 8, x = -4..4, y = -4..4, filled = true)$



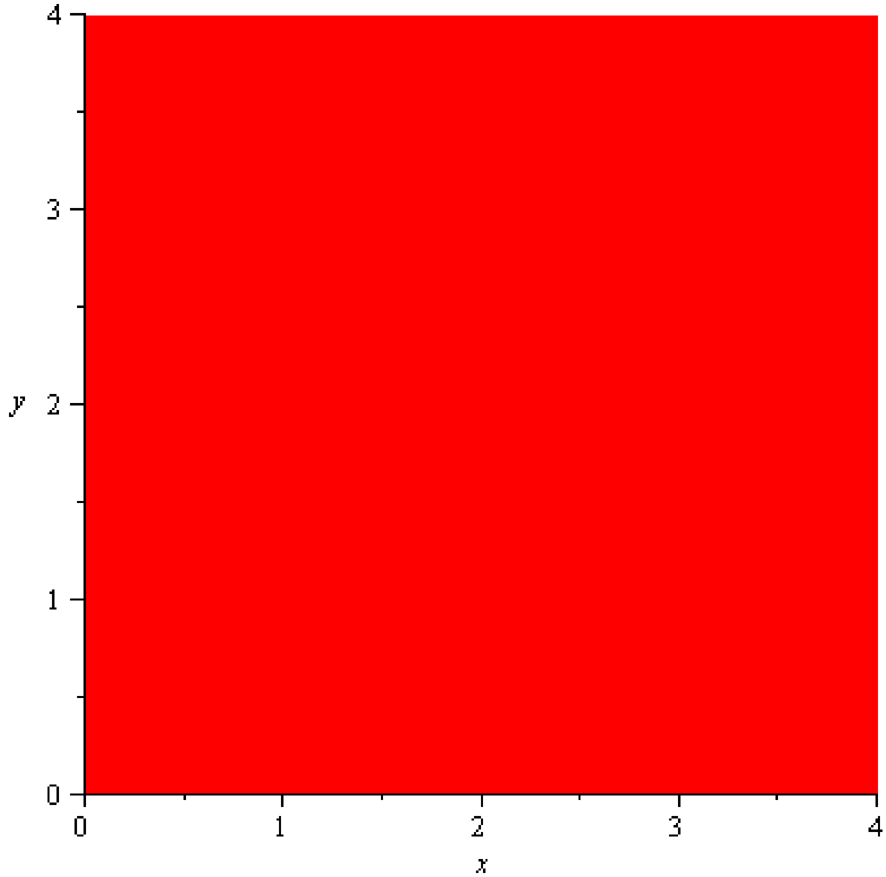
c){ (x,y): x y <= 1}

implicitplot$(x\, y \leq 1, x = -4..4, y = -4..4, filled = true)$



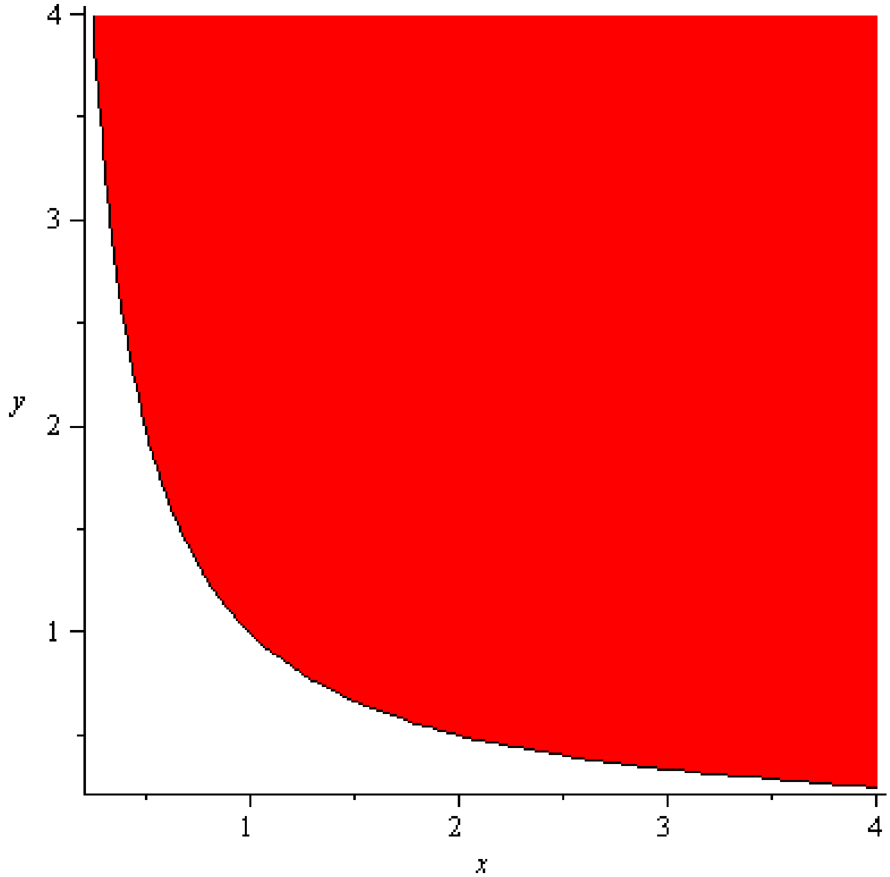
d){ (x,y): x>=0, y >= 0}

$implicitplot(y \geq 0, x = 0..4, y = 0..4, filled = true)$



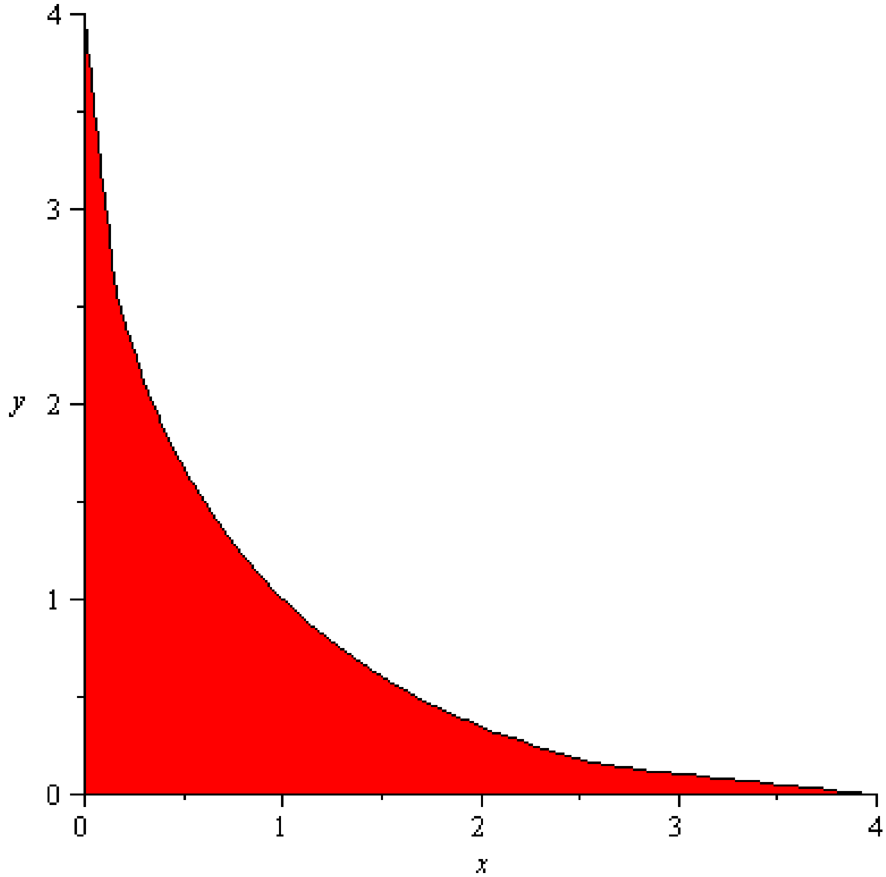
e){ (x,y): x>=0, y >= 0, x y >=1}

implicitplot$(x\ y \geq 1, x = 0\ ..4, y = 0\ ..4, \mathit{filled} = \mathit{true})$



f){ (x,y): sqrt(x)+sqrt(y) <= 2}

implicitplot(sqrt(*x*) + sqrt(*y*) ≤ 2, *x* = 0 ..4, *y* = 0 ..4, *filled* = *true*)



Příklad 2: Rozhodněte o konvexitě/konkávnosti uvedených funkcí:

a) $z := (x, y) \rightarrow x + y - \exp(x) - \exp(x + y)$

$$(x, y) \rightarrow x + y - e^{x} - e^{x+y}$$

$H := VectorCalculus[Hessian](z(x, y), [x, y]);$

$$\begin{bmatrix} -e^{x} - e^{x+y} & -e^{x+y} \\ -e^{x+y} & -e^{x+y} \end{bmatrix}$$

$with(LinearAlgebra) : Determinant(H)$

$$e^{x+y}\, e^{x}$$

b) $z := (x, y) \rightarrow \exp(x + y) + \exp(x - y) - \frac{y}{2}$

$$(x, y) \rightarrow e^{x+y} + e^{x-y} - \frac{1}{2}\, y$$

$H := VectorCalculus[Hessian]\left(\exp(x+y) + \exp(x-y) - \frac{y}{2}, [x, y]\right);$

$$\begin{bmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{bmatrix}$$

$Determinant(H)$

$$4\, e^{x+y}\, e^{x-y}$$

c) $w := (x, y, z) \rightarrow (x + 2y + 3z)\hat{}2$

$$(x, y, z) \rightarrow (x + 2y + 3z)^2$$

$H := VectorCalculus[Hessian]((x + 2y + 3z)\hat{}2, [x, y, z]);$

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \\ 6 & 12 & 18 \end{bmatrix}$$

$H[1, 1], Determinant(H[1..2, 1..2]), Determinant(H)$

$$2, 0, 0$$

Příklad 3: Ukažte, že funkce g(x, y)= x^3 + y^2 - 3x - 2y**definovaná pro** x>0, y>0**je ryze konvexní a najděte její minimum.**

$g := (x, y) \rightarrow x\hat{}3 + y\hat{}2 - 3x - 2y$

$$(x, y) \rightarrow x^3 + y^2 - 3x - 2y$$

$with(Student[MultivariateCalculus]) : Gradient(g(x, y), [x, y])$

$$\begin{bmatrix} 3x^2 - 3 \\ 2y - 2 \end{bmatrix}$$

$solve\left(\left[\frac{d}{dx} g(x, y) = 0, \frac{d}{dy} g(x, y) = 0\right], [x, y]\right)$

$$[[x = 1, y = 1], [x = -1, y = 1]]$$

$H := VectorCalculus[Hessian](g(x, y), [x, y]);$

$$\begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix}$$

Příklad 4: Funkce f(x_1, x_2, x_3)= x_1^2 + x_2^2 + 3x_3^2 - x_1x_2 + 2x_1x_3 + x_2x_3**má**

jeden stacionární bod. Ukažte, že tento stacionární bod je bodem lokálního minima.

$f := (x\_1, x\_2, x\_3) \rightarrow x\_1\text{^}2 + x\_2\text{^}2 + 3\,x\_3\text{^}2 - x\_1x\_2 + 2\,x\_1x\_3 + x\_2x\_3$

$$(x\_1, x\_2, x\_3) \rightarrow x\_1^2 + x\_2^2 + 3\,x\_3^2 - x\_1x\_2 + 2\,x\_1x\_3 + x\_2x\_3$$

$Gradient(f(x\_1, x\_2, x\_3), [x\_1, x\_2, x\_3])$

$$\begin{bmatrix} 2\,x\_1 \\ 2\,x\_2 \\ 6\,x\_3 \end{bmatrix}$$

$H := VectorCalculus[Hessian](f(x\_1, x\_2, x\_3), [x\_1, x\_2, x\_3]);$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Příklad 5: Nechť $f := (x, y) \rightarrow x\text{^}3 + y\text{^}3 - 3\,x\,y$

$$(x, y) \rightarrow x^3 + y^3 - 3\,x\,y$$

je funkce definovaná pro každé reálné x, y.

a) Ukažte, že body [0, 0] a [1, 1] jsou jediné reálné stacionární body.

$solve\left(\left[\frac{d}{dx} f(x, y) = 0, \frac{d}{dy} f(x, y) = 0\right], [x, y]\right)$

$$[[x = 0, y = 0], [x = 1, y = 1], [x = -1 - \mathrm{RootOf}(\_Z^2 + \_Z + 1, label = \_L2), y = \mathrm{RootOf}(\_Z^2 + \_Z + 1, label = \_L2)]]$$

b) Ověřte definitnost Hessovy matice v těchto bodech

$H := VectorCalculus[Hessian](f(x, y), [x, y]);$

$$\begin{bmatrix} 6\,x & -3 \\ -3 & 6\,y \end{bmatrix}$$

$h := unapply(\%, x, y)$

$$(x, y) \rightarrow rtable(1..2, 1..2, \{(1, 2) = -3, (2, 1) = -3, (1, 1) = 6\,x, (2, 2) = 6\,y\}, datatype = anything, subtype = Matrix, storage = rectangular, order = Fortran\_order)$$

$h(1, 1)$

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

$h(0, 0)$

$$\begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$

c) Určete, jakého typu jednotlivé stacionární body jsou.

Příklad 6: Určete stacionární body následujících funkcí.

a) $f := (x, y, z) \rightarrow x\text{^}2 + x\text{^}2\, y + y\text{^}2\, z + y\text{^}2 + z\text{^}2 - 4\, z$

$$(x, y, z) \rightarrow x^2 + x^2 y + y^2 z + y^2 + z^2 - 4 z$$

$extrema(f(x, y, z), \{\,\}, \{x, y, z\}, 'bod');$

$$\{-4 z, 1 - 4 z\}$$

bod

$$\left\{\{z = 0, y = 0, x = 0\}, \left\{x = -1, y = -1, z = -\frac{1}{2}\right\}, \{y = \sqrt{2}, z = -1, x = 0\}, \{y = -\sqrt{2}, z = -1, x = 0\}, \left\{x = 1, y = -1, z = -\frac{1}{2}\right\}\right\}$$

$H := VectorCalculus[Hessian](f(x, y, z), [x, y, z]);$

$$\begin{bmatrix} 2 + 2 y & 2 x & 0 \\ 2 x & 2 z + 2 & 2 y \\ 0 & 2 y & 2 \end{bmatrix}$$

$h := unapply(\%, x, y, z)$

$$(x, y, z) \rightarrow rtable(1..3, 1..3, \{(3, 3) = 2, (3, 2) = 2 y, (1, 1) = 2 + 2 y, (1, 2) = 2 x, (2, 1) = 2 x, (2, 2) = 2 z + 2, (2, 3) = 2 y\}, datatype = anything, subtype = Matrix, storage = rectangular, order = Fortran\_order)$$

$h(0, 0, 0)$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$h\left(1, -1, -\frac{1}{2}\right)$

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$h\left(-1, -1, -\frac{1}{2}\right)$

$$\begin{bmatrix} 0 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$h(0, \mathrm{sqrt}(2), -1)$

$$\begin{bmatrix} 2+2\sqrt{2} & 0 & 0 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 2 \end{bmatrix}$$

$h(0, -\mathrm{sqrt}(2), -1)$

$$\begin{bmatrix} 2-2\sqrt{2} & 0 & 0 \\ 0 & 0 & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 2 \end{bmatrix}$$

b) $f := (x\_1, x\_2, x\_3, x\_4) \to 20\,x\_2 + 48\,x\_3 + 6\,x\_4 + 8\,x\_1x\_2$
$- 4\,x\_1\text{^}2 - 12\,x\_3\text{^}2 - x\_4\text{^}2 - 4\,x\_2\text{^}3$

$$(x\_1, x\_2, x\_3, x\_4) \to 20\,x\_2 + 48\,x\_3 + 6\,x\_4 + 8\,x\_1x\_2 - 4\,x\_1^2 - 12\,x\_3^2 - x\_4^2 - 4\,x\_2^3$$

;
$extrema(f(x\_1, x\_2, x\_3, x\_4), \{\,\}, \{x\_1, x\_2, x\_3, x\_4\}, 'bod')$

$$\left\{-\frac{40}{9}\sqrt{15} + 57 + 8\,x\_1x\_2, \frac{40}{9}\sqrt{15} + 57 + 8\,x\_1x\_2\right\}$$

bod

$$\left\{\left\{x\_2 = -\frac{1}{3}\sqrt{15}, x\_4 = 3, x\_3 = 2, x\_1 = 0\right\}, \left\{x\_2 = \frac{1}{3}\sqrt{15}, x\_4 = 3, x\_3 = 2, x\_1 = 0\right\}\right\}$$

$H := VectorCalculus[Hessian](f(x\_1, x\_2, x\_3, x\_4), [x\_1, x\_2, x\_3, x\_4]);$

$$\begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & -24\,x\_2 & 0 & 0 \\ 0 & 0 & -24 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Příklad 7: Firma produkuje dva výrobky, označme je A a B. Náklady na den jsou

$C := (x, y) \rightarrow 0.04\,x\text{^}2 - 0.01\,x\,y + y\text{^}2 + 4\,x + 2\,y + 500$

$$(x, y) \rightarrow 0.04\,x^2 + (-1) \cdot 0.01\,x\,y + y^2 + 4\,x + 2\,y + 500$$

kde x **je počet jednotek A a** y **je počet jednotek B** (x>0, y>0). **Firma prodává výrobek A za 13 Kč a**
výrobek B za 8 Kč. Najděte funkci zisku pi(x, y) **a hodnoty** x , y **pro které nastává maximální zisk.**

$\mathrm{pi} := (x, y) \rightarrow 13\,x + 8\,y - C(x, y)$

$$(x, y) \rightarrow 13\,x + 8\,y - C(x, y)$$

extrema(pi(x, y), { }, {x, y} ,'*bod*');

$$\{18.94934320\}$$

bod

$$\{\{x = 112.9455910, y = 3.564727955\}\}$$

$solve\left(\left[\frac{d}{dx}\,\mathrm{pi}(x, y) = 0, \frac{d}{dy}\,\mathrm{pi}(x, y) = 0\right], [x, y]\right)$

$$[[x = 112.9455910, y = 3.564727955]]$$