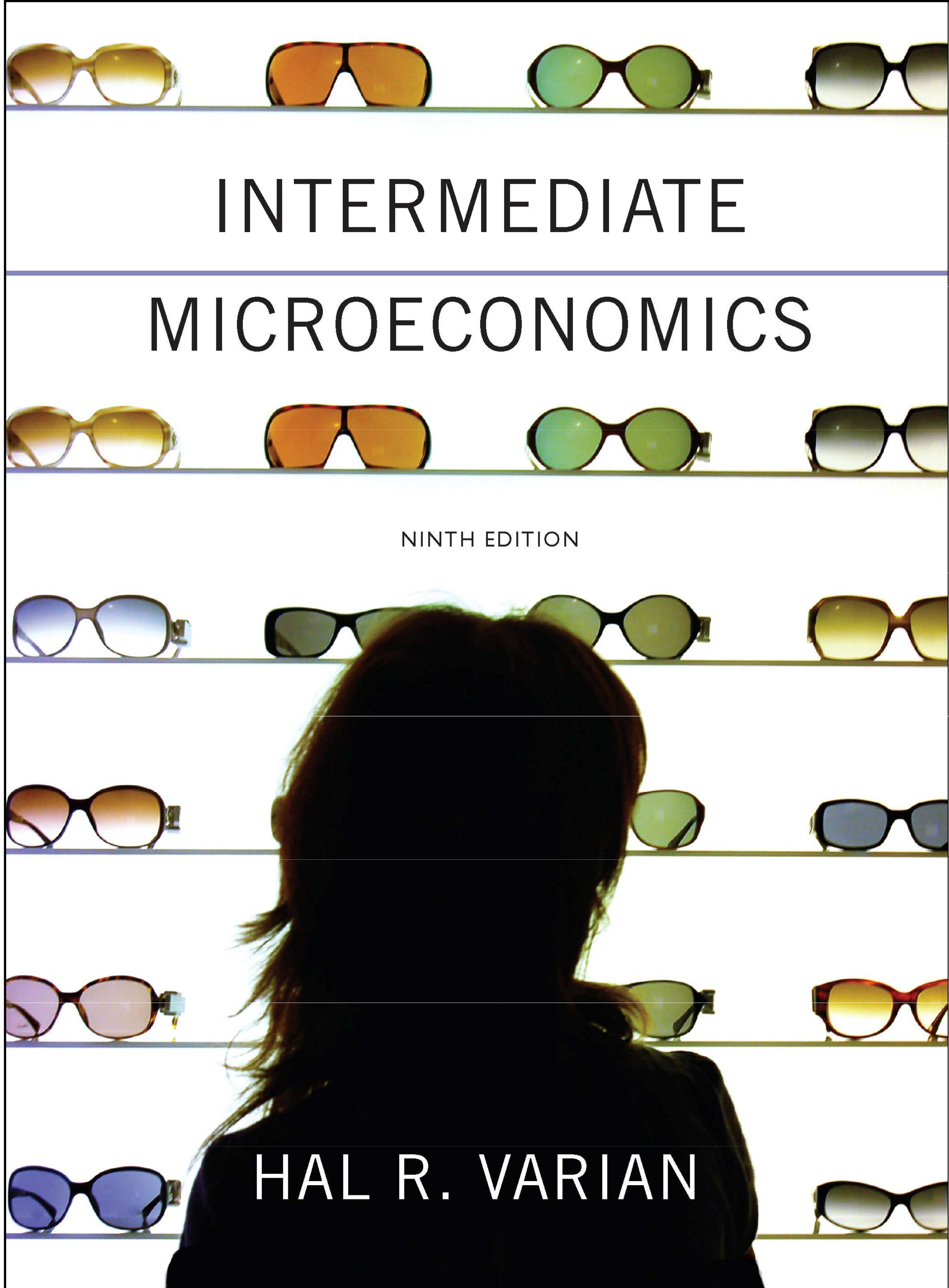


Chapter 4

Utility

Preferences - A Reminder

- **x $\succ$ y: x is preferred strictly to y.**
- **x $\sim$ y: x and y are equally preferred.**
- **x $\succsim$ y: x is preferred at least as much as is y.**

Preferences - A Reminder

- **Completeness: For any two bundles x and y it is always possible to state either that**

$$x \succsim y$$

or that

$$y \succsim x.$$

Preferences - A Reminder

- **Reflexivity: Any bundle x is always at least as preferred as itself; *i.e.***

$$\mathbf{x} \succsim \mathbf{x}.$$

Preferences - A Reminder

- **Transitivity: If**
  x is at least as preferred as y, and
  y is at least as preferred as z, then
  x is at least as preferred as z; *i.e.*

  $$x \succsim y \text{ and } y \succsim z \Rightarrow x \succsim z.$$

Utility Functions

- **A preference relation that is complete, reflexive, transitive and continuous can be represented by a continuous utility function.**
- **Continuity means that small changes to a consumption bundle cause only small changes to the preference level.**

Utility Functions

- **A utility function U(x) represents a preference relation $\succsim$ if and only if:**

x' $\succ$ x'' $\Longleftrightarrow$ U(x') > U(x'')

x' $\prec$ x'' $\Longleftrightarrow$ U(x') < U(x'')

x' $\sim$ x'' $\Longleftrightarrow$ U(x') = U(x'').

Utility Functions

- **Utility is an ordinal (i.e. ordering) concept.**
- ***E.g*. if U(x) = 6 and U(y) = 2 then bundle x is strictly preferred to bundle y. But x is not preferred three times as much as is y.**

Utility Functions & Indiff. Curves

- **Consider the bundles (4,1), (2,3) and (2,2).**
- **Suppose (2,3) $\succ$ (4,1) $\sim$ (2,2).**
- **Assign to these bundles any numbers that preserve the preference ordering;**
  e.g. **U(2,3) = 6 > U(4,1) = U(2,2) = 4.**
- **Call these numbers utility levels.**

Utility Functions & Indiff. Curves

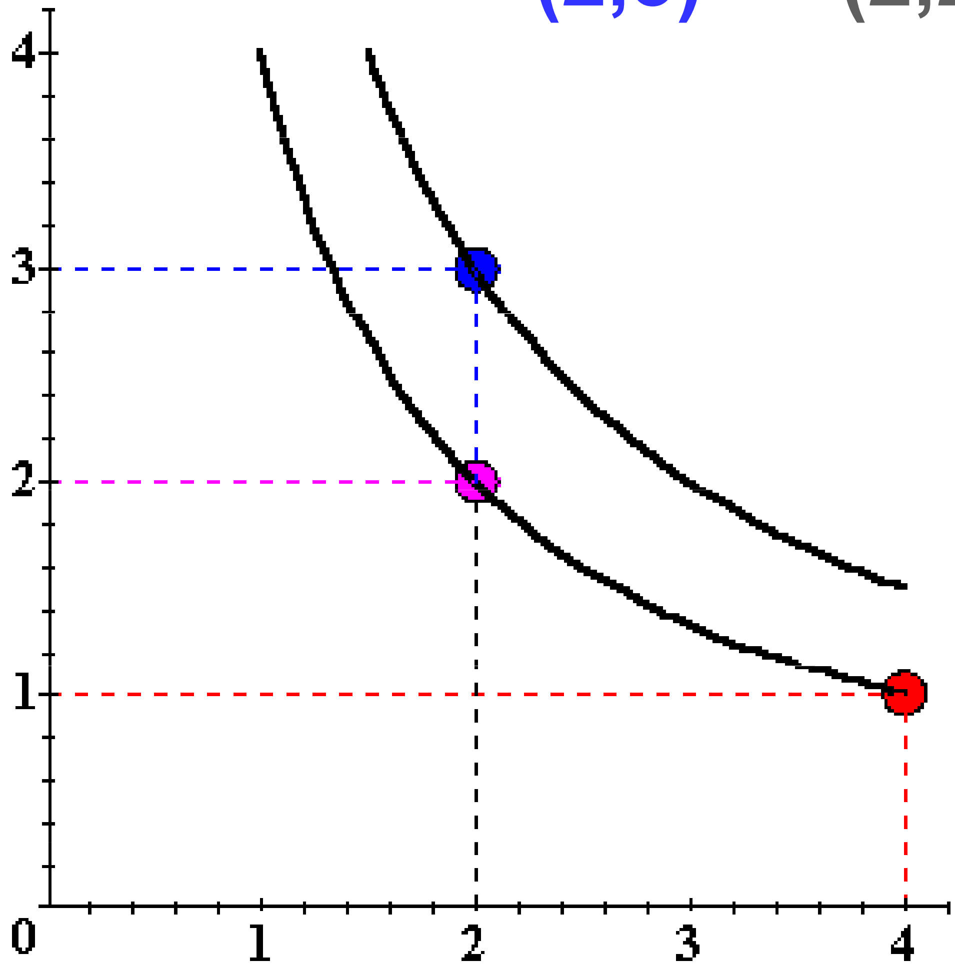
- **An indifference curve contains equally preferred bundles.**
- **Equal preference $\Rightarrow$ same utility level.**
- **Therefore, all bundles in an indifference curve have the same utility level.**

Utility Functions & Indiff. Curves

- **So the bundles (4,1) and (2,2) are in the indiff. curve with utility level U** $\equiv$ 4
- **But the bundle (2,3) is in the indiff. curve with utility level U $\equiv$ 6.**
- **On an indifference curve diagram, this preference information looks as follows:**

Utility Functions & Indiff. Curves

(2,3) (2,2) (4,1)

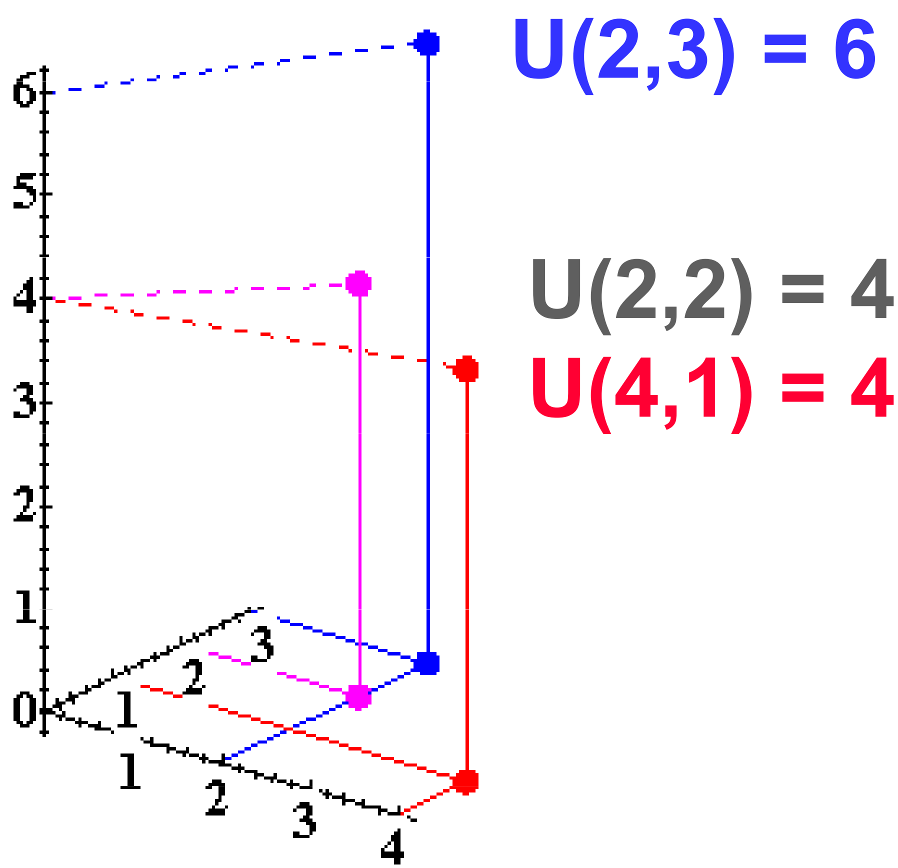


Utility Functions & Indiff. Curves

- **Another way to visualize this same information is to plot the utility level on a vertical axis.**

Utility Functions & Indiff. Curves

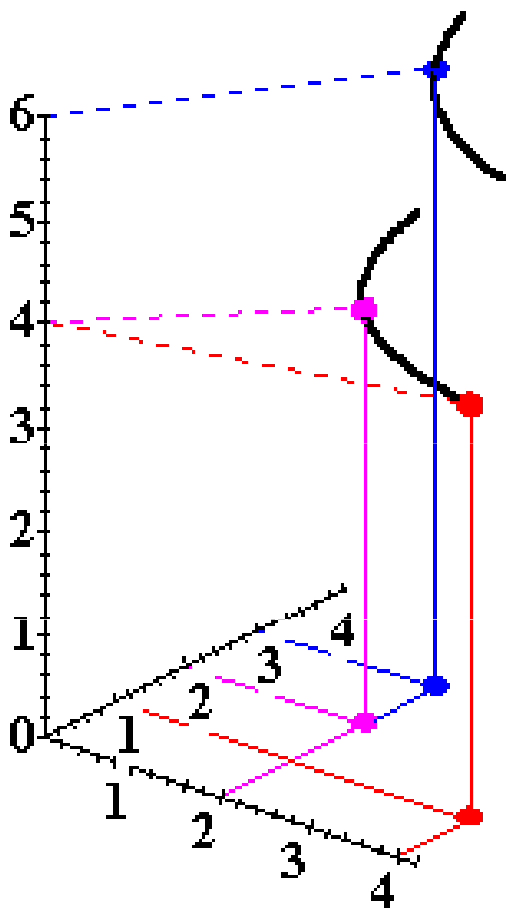
3D plot of consumption & utility levels for 3 bundles



Utility Functions & Indiff. Curves

- **This 3D visualization of preferences can be made more informative by adding into it the two indifference curves.**

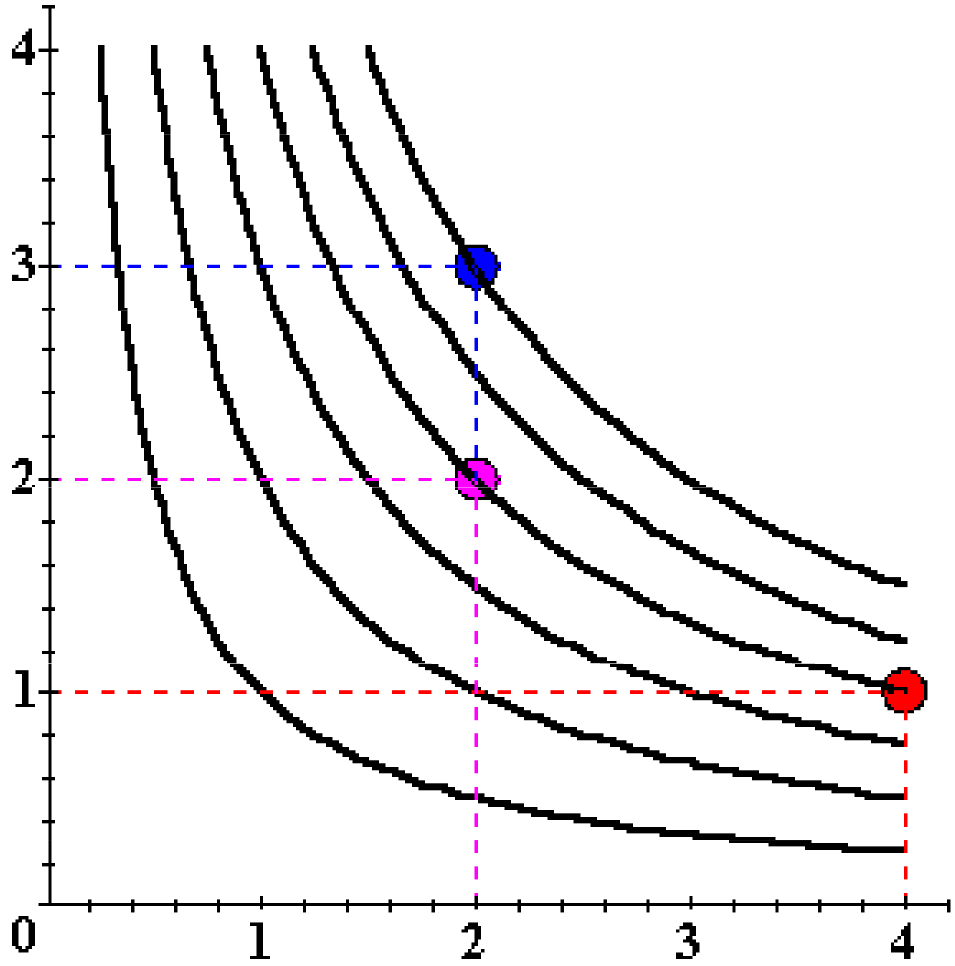
Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves

- **Comparing more bundles will create a larger collection of all indifference curves and a better description of the consumer's preferences.**

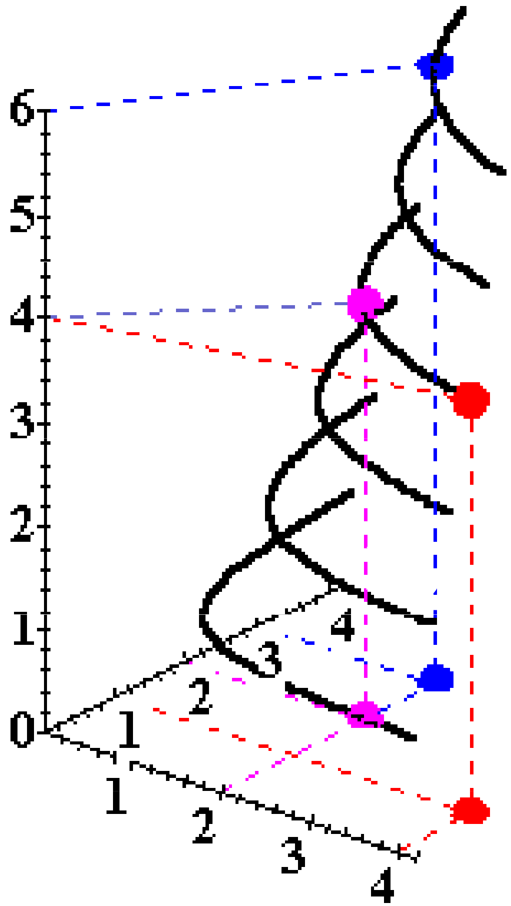
Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves

- **As before, this can be visualized in 3D by plotting each indifference curve at the height of its utility index.**

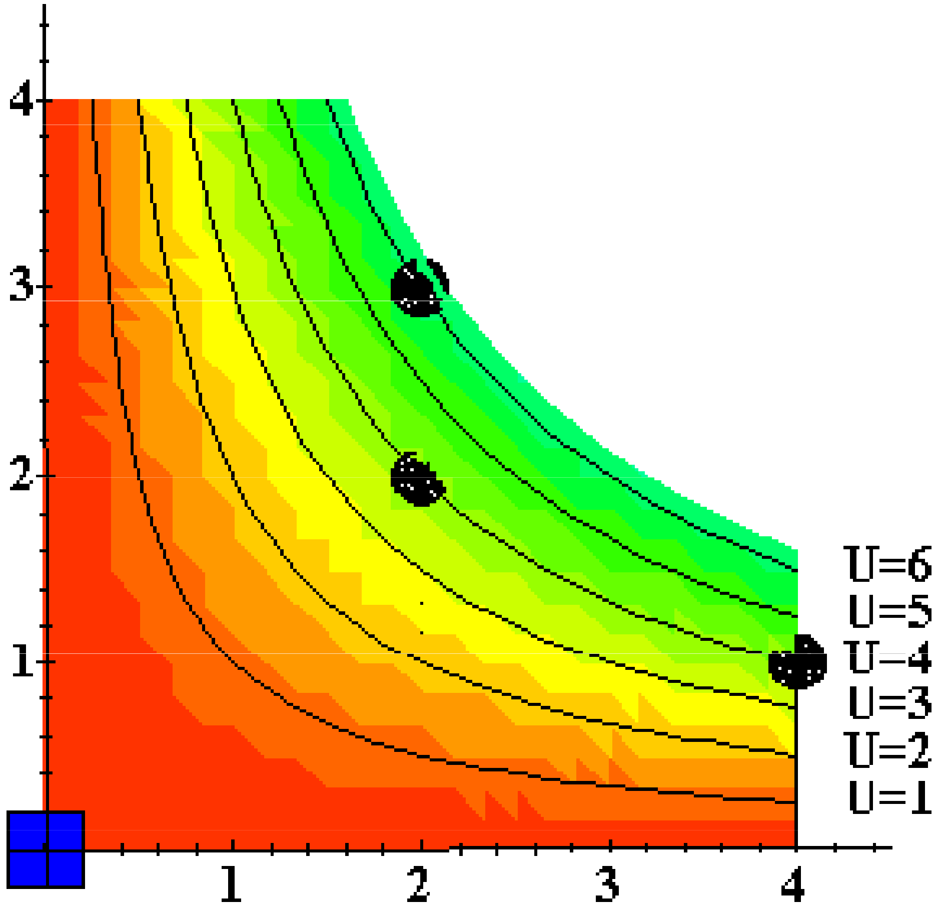
Utility Functions & Indiff. Curves



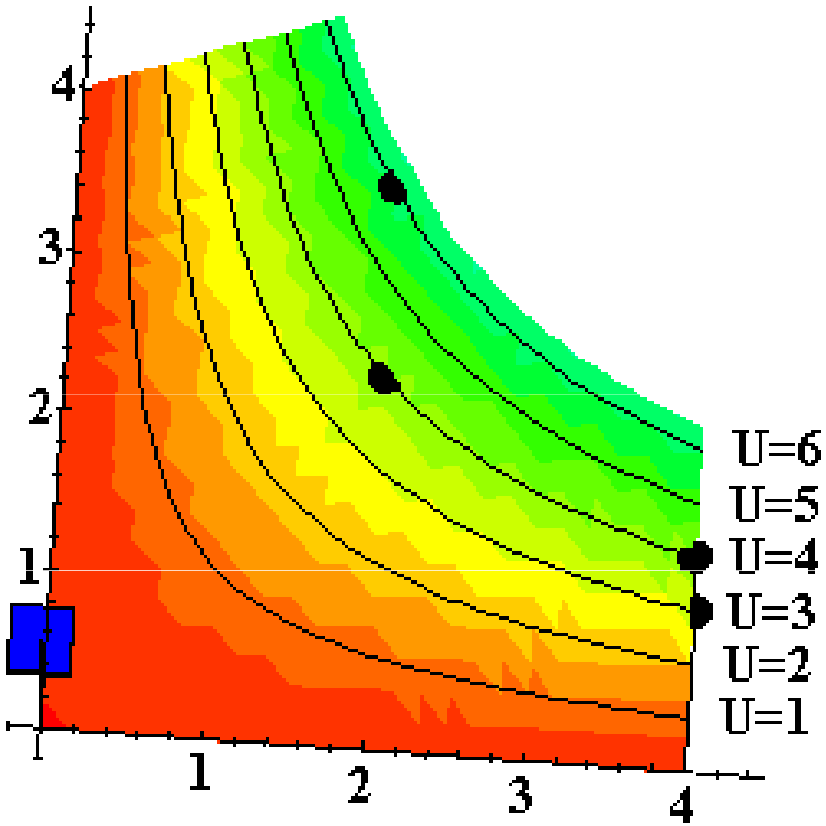
Utility Functions & Indiff. Curves

- **Comparing all possible consumption bundles gives the complete collection of the consumer's indifference curves, each with its assigned utility level.**
- **This complete collection of indifference curves completely represents the consumer's preferences.**

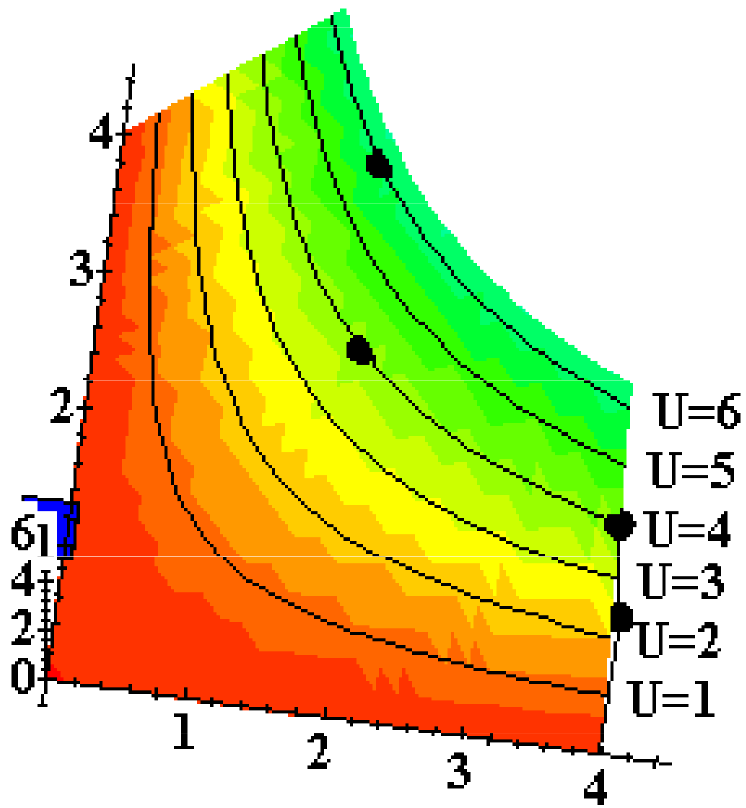
Utility Functions & Indiff. Curves



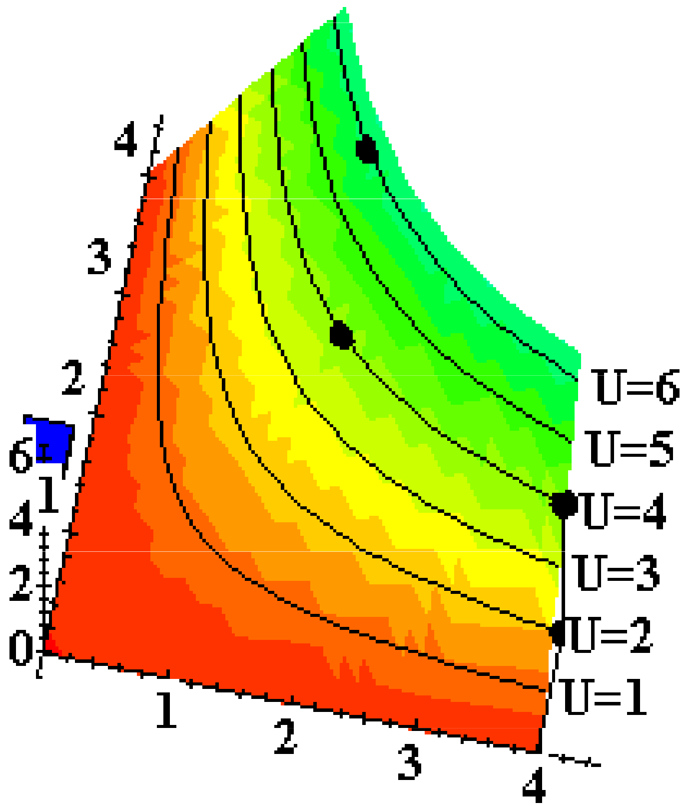
Utility Functions & Indiff. Curves



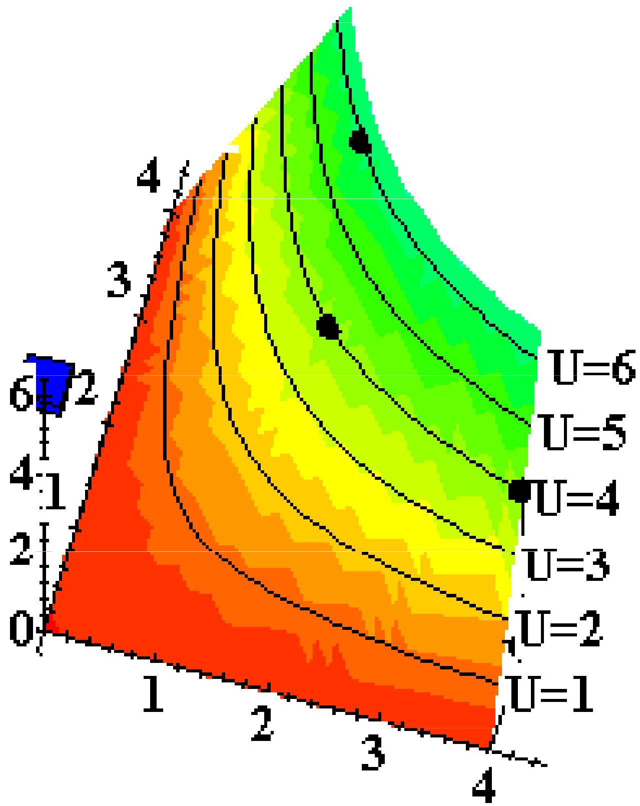
Utility Functions & Indiff. Curves



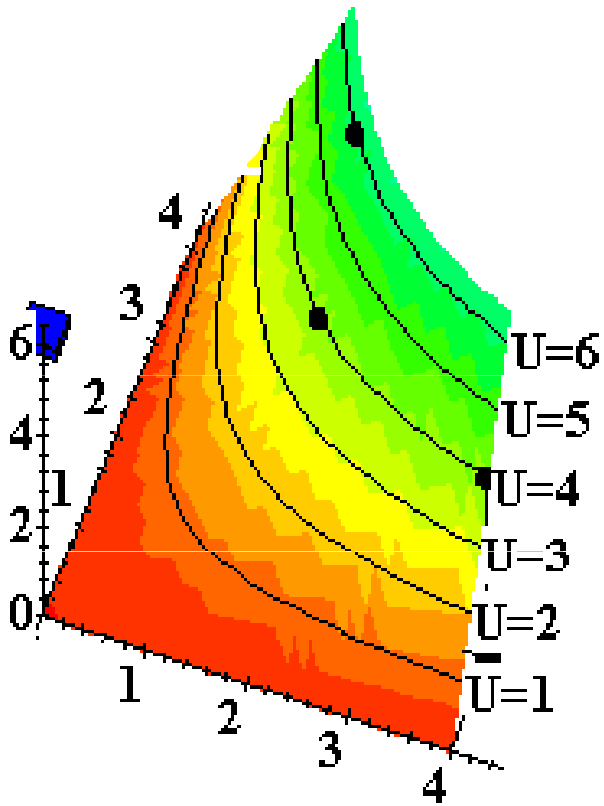
Utility Functions & Indiff. Curves



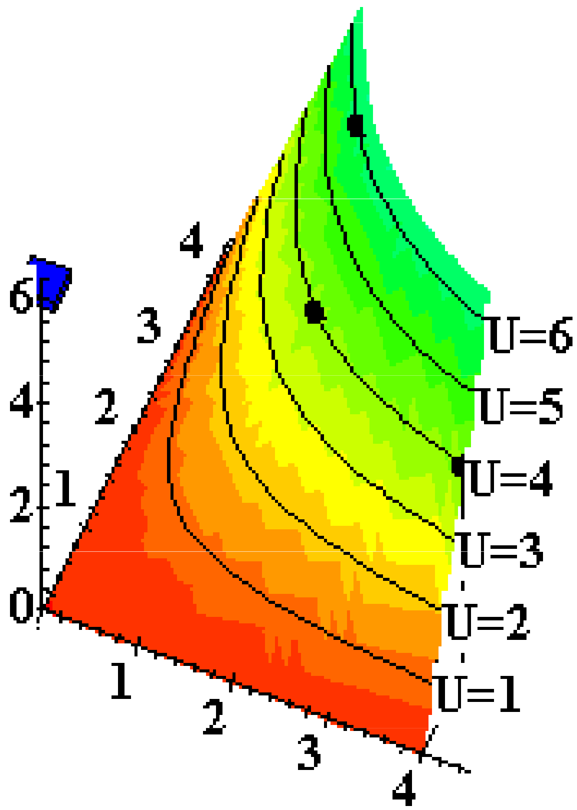
Utility Functions & Indiff. Curves



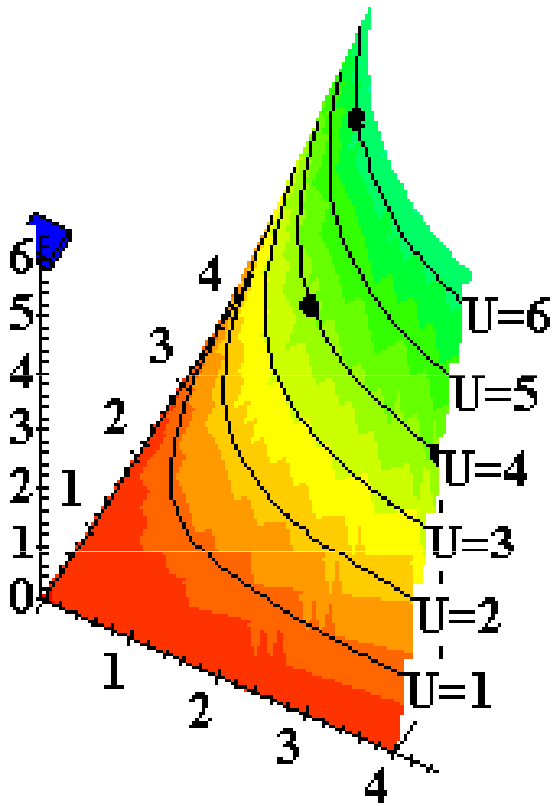
Utility Functions & Indiff. Curves



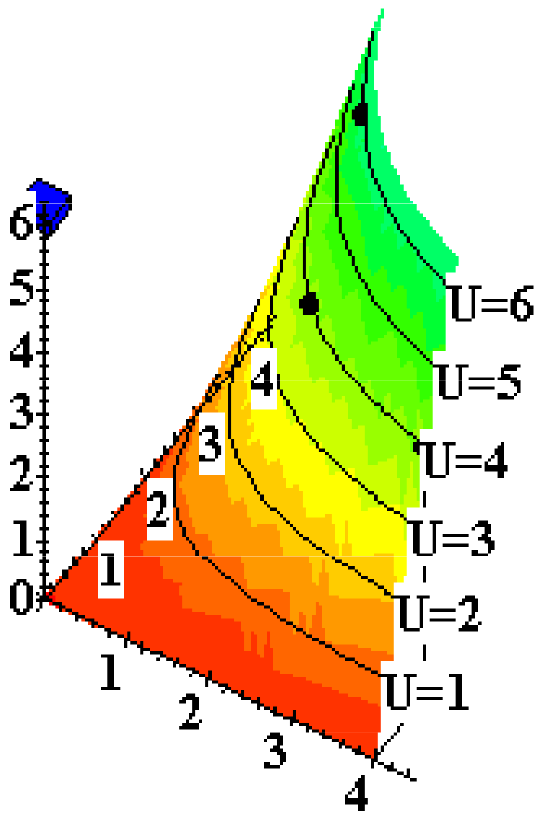
Utility Functions & Indiff. Curves



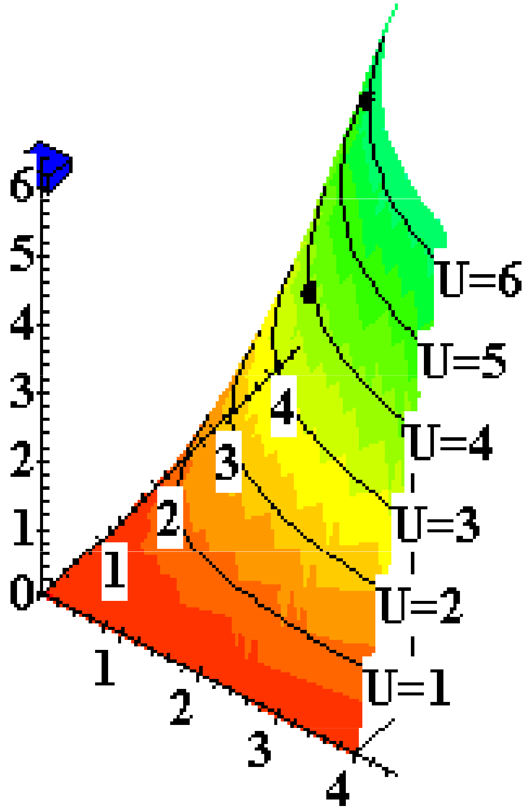
Utility Functions & Indiff. Curves



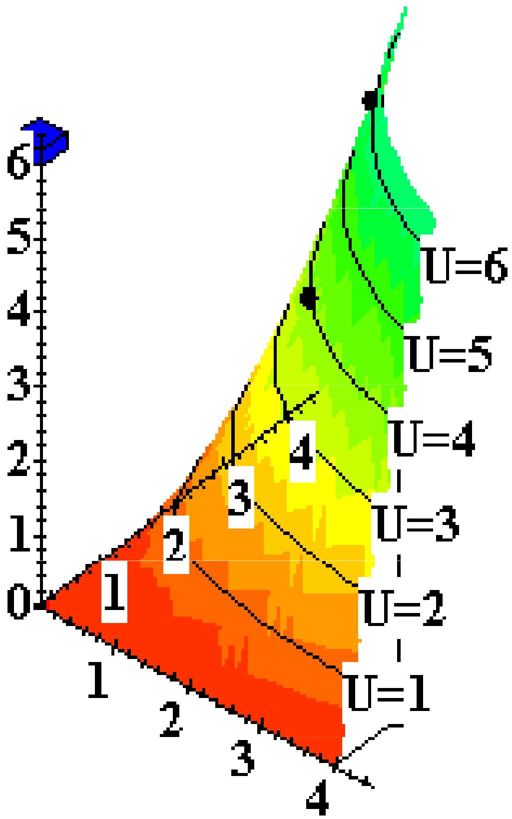
Utility Functions & Indiff. Curves



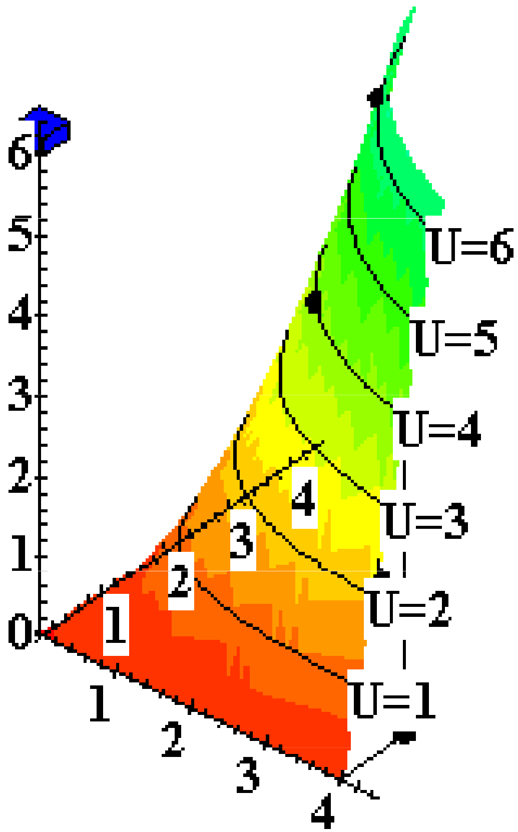
Utility Functions & Indiff. Curves



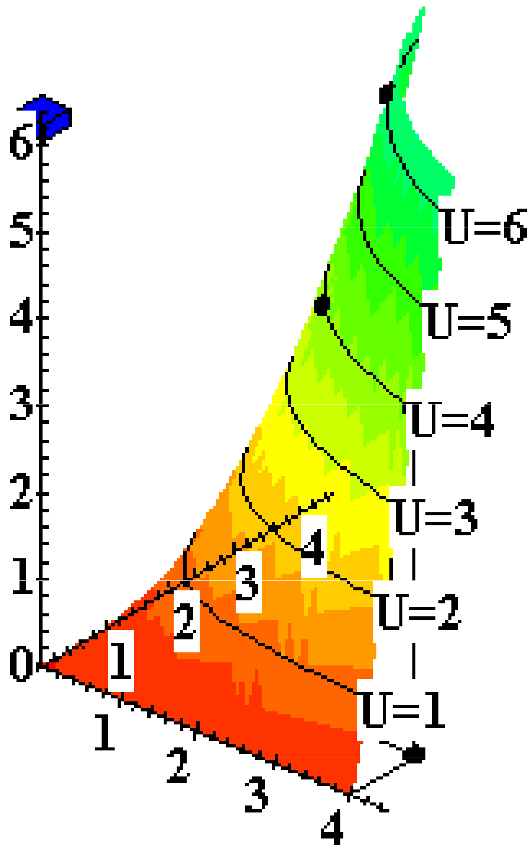
Utility Functions & Indiff. Curves



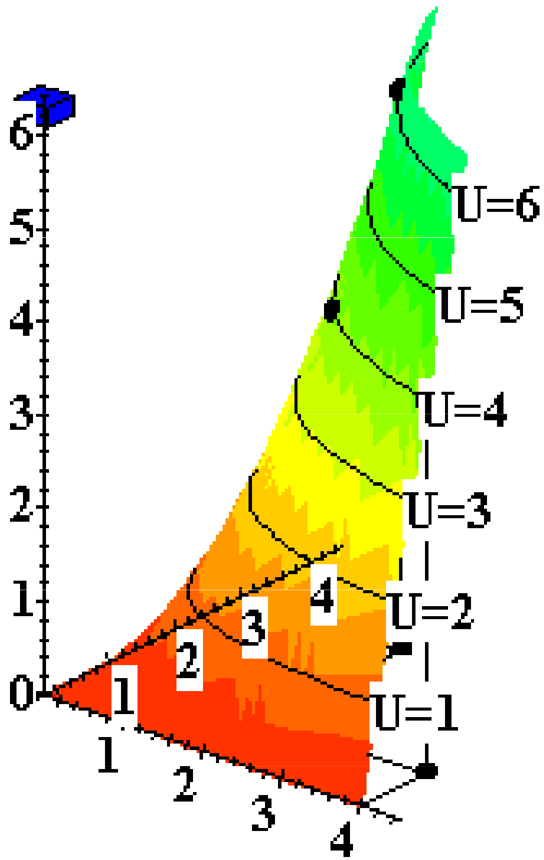
Utility Functions & Indiff. Curves



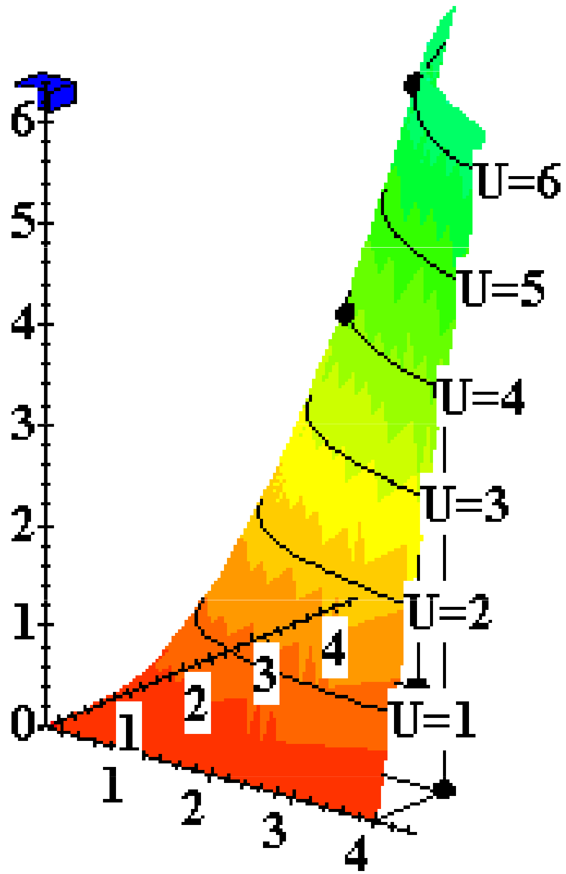
Utility Functions & Indiff. Curves



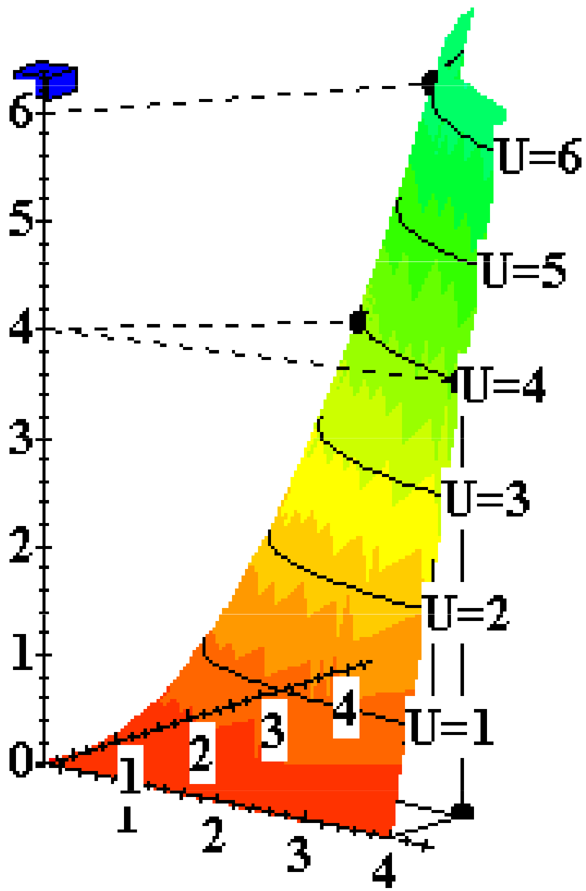
Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves



Utility Functions & Indiff. Curves

- **The collection of all indifference curves for a given preference relation is an indifference map.**
- **An indifference map is equivalent to a utility function; each is the other.**

Utility Functions

- **There is no unique utility function representation of a preference relation.**
- **Suppose $U(x_1,x_2) = x_1x_2$ represents a preference relation.**
- **Again consider the bundles (4,1), (2,3) and (2,2).**

Utility Functions

- **$U(x_1,x_2) = x_1x_2$, so**

  $U(2,3) = 6 > U(4,1) = U(2,2) = 4$;

  that is, $(2,3) \succ (4,1) \sim (2,2)$.

Utility Functions

- $U(x_1,x_2) = x_1x_2$ ⟹ $(2,3) \succ (4,1) \sim (2,2)$.
- Define $V = U^2$.

Utility Functions

- $U(x_1,x_2) = x_1x_2$ ⟹ $(2,3) \succ (4,1) \sim (2,2)$.
- Define $V = U^2$.
- Then $V(x_1,x_2) = x_1^2x_2^2$ and
  $V(2,3) = 36 > V(4,1) = V(2,2) = 16$
  so again
  $(2,3) \succ (4,1) \sim (2,2)$.
- V preserves the same order as U and so represents the same preferences.

Utility Functions

- **$U(x_1,x_2) = x_1 x_2$ ➡ $(2,3) \succ (4,1) \sim (2,2)$.**
- **Define $W = 2U + 10$.**

Utility Functions

- $U(x_1,x_2) = x_1x_2$ ⟹ $(2,3) \succ (4,1) \sim (2,2)$.
- Define W = 2U + 10.
- Then $W(x_1,x_2) = 2x_1x_2+10$ so
  W(2,3) = 22 > W(4,1) = W(2,2) = 18.
  Again,
  $(2,3) \succ (4,1) \sim (2,2)$.
- W preserves the same order as U and V and so represents the same preferences.

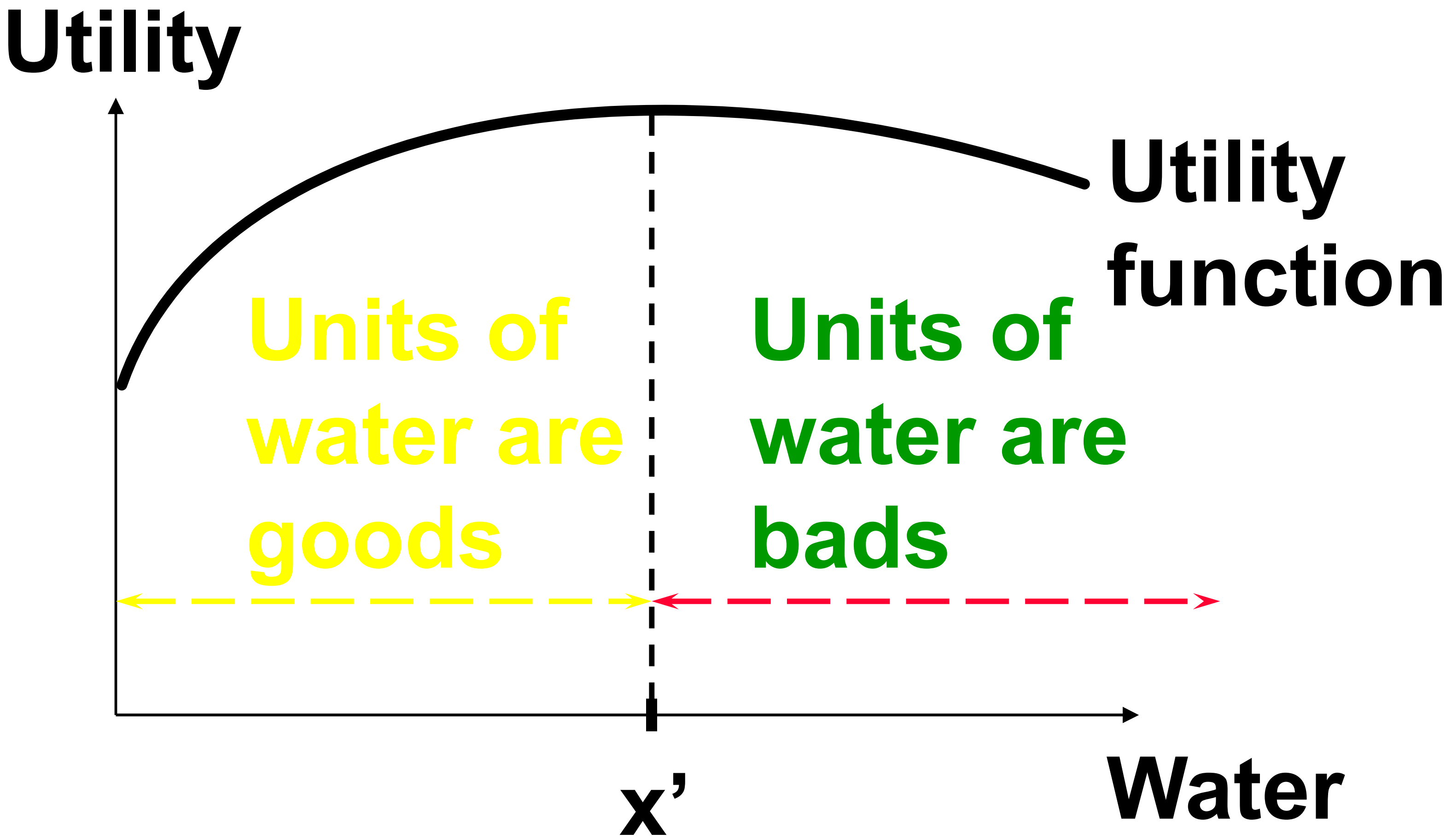
Utility Functions

- **If**
  - **U is a utility function that represents a preference relation $\succsim$ and**
  - **f is a strictly increasing function,**
- **then V = f(U) is also a utility function representing $\succsim$.**

Goods, Bads and Neutrals

- **A good is a commodity unit which increases utility (gives a more preferred bundle).**
- **A bad is a commodity unit which decreases utility (gives a less preferred bundle).**
- **A neutral is a commodity unit which does not change utility (gives an equally preferred bundle).**

Goods, Bads and Neutrals



Around x' units, a little extra water is a neutral.

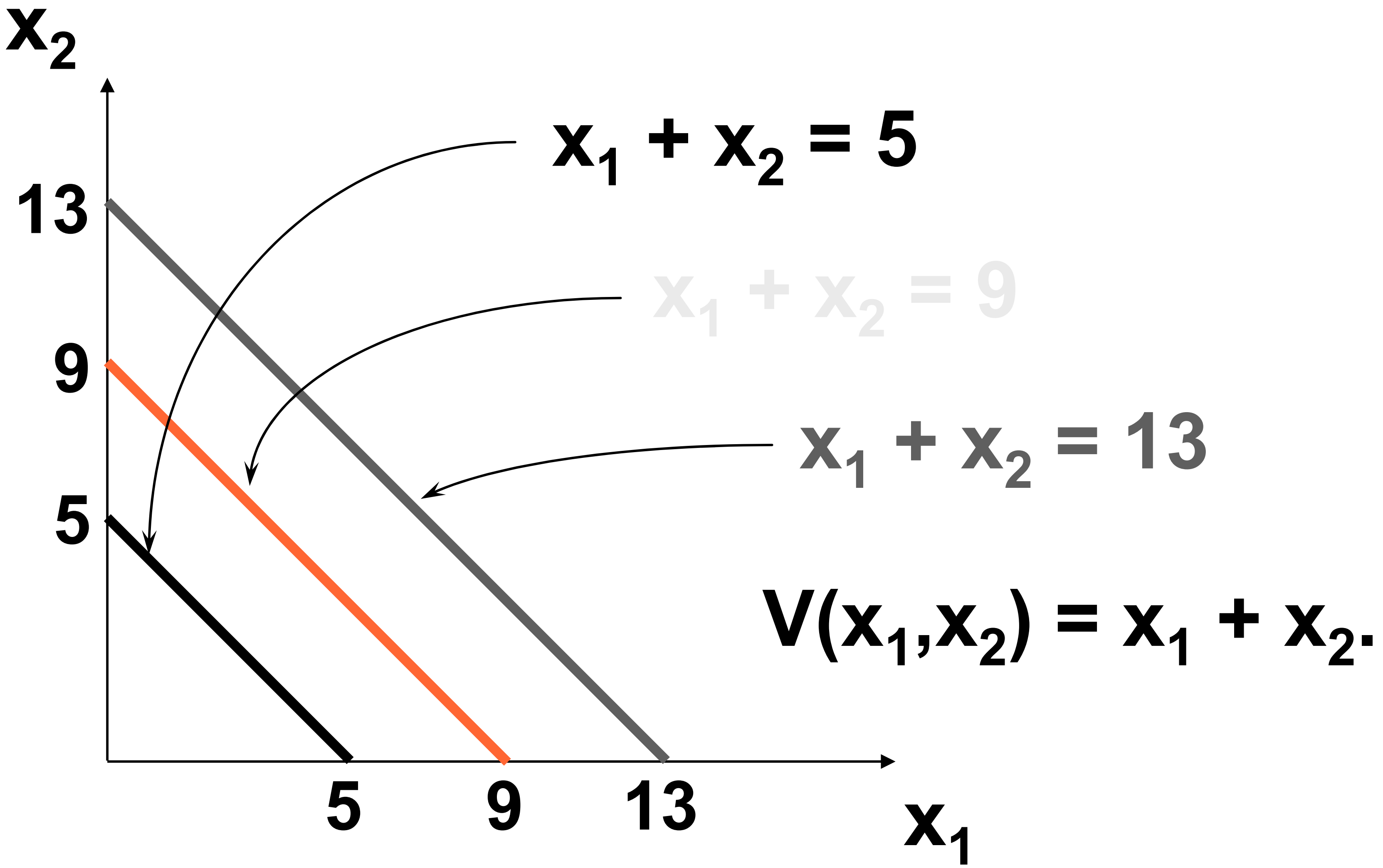
Some Other Utility Functions and Their Indifference Curves

- **Instead of $U(x_1,x_2) = x_1x_2$ consider**

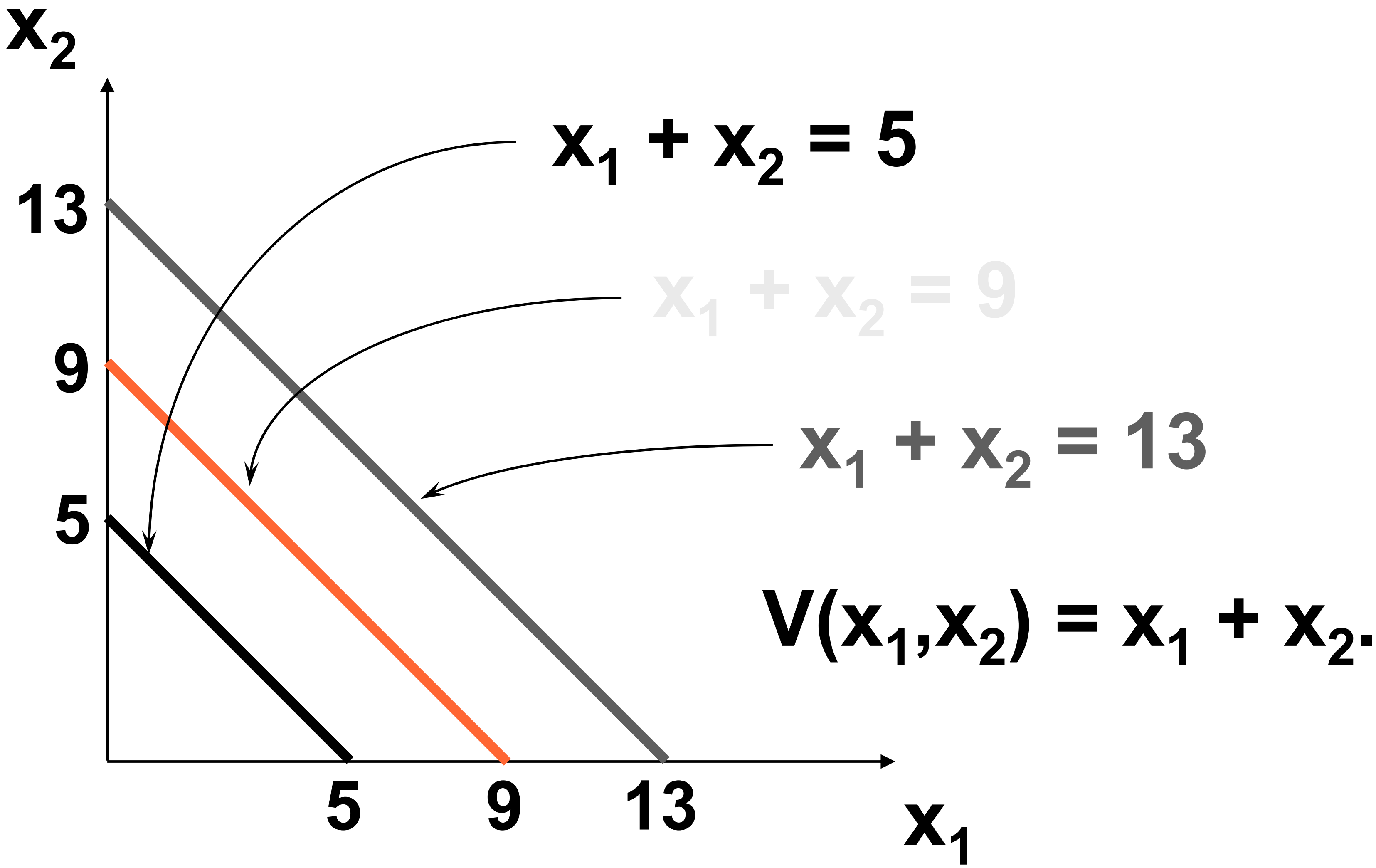
$$V(x_1,x_2) = x_1 + x_2.$$

What do the indifference curves for this "perfect substitution" utility function look like?

Perfect Substitution Indifference Curves



Perfect Substitution Indifference Curves



All are linear and parallel.

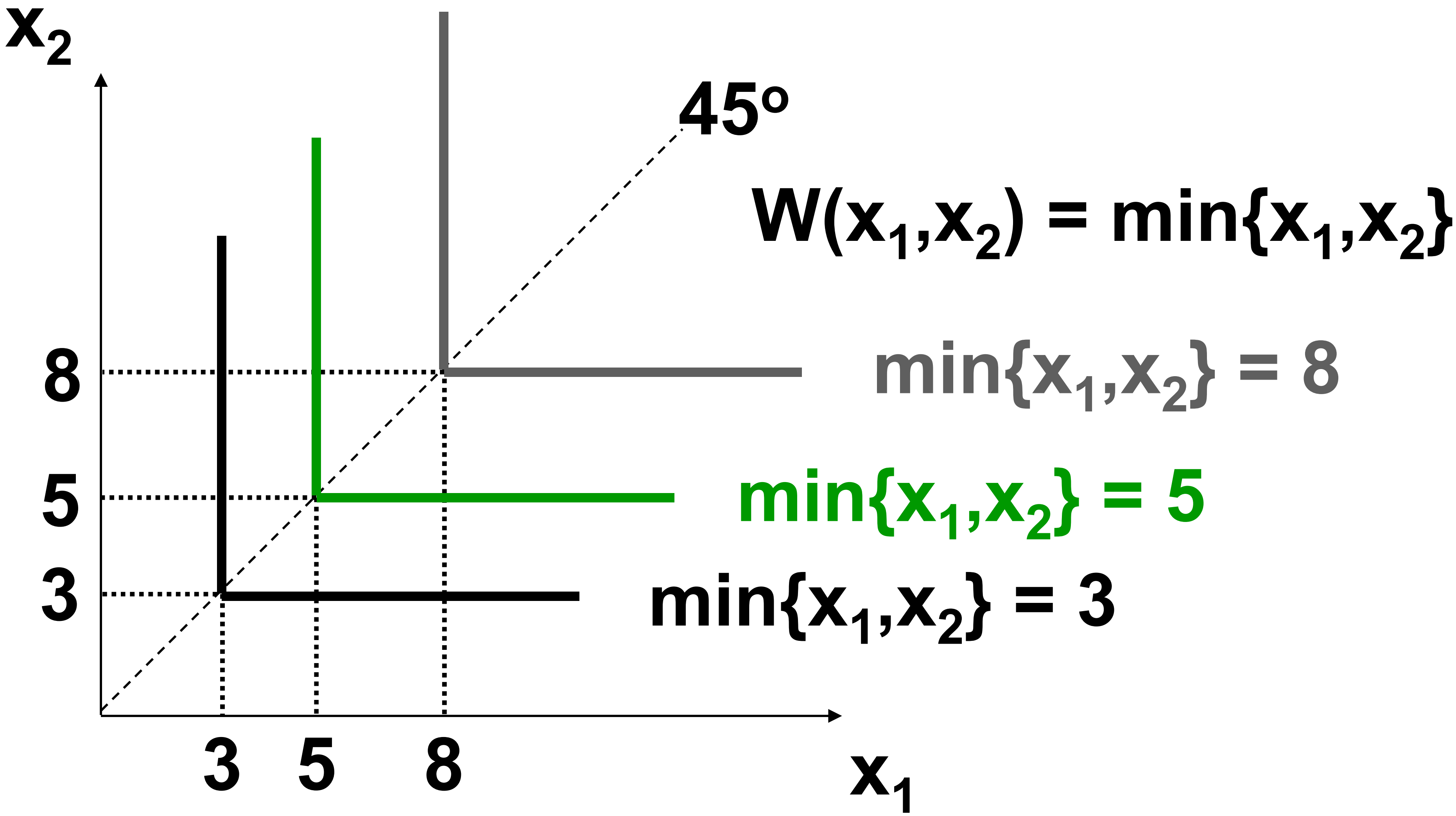
Some Other Utility Functions and Their Indifference Curves

- **Instead of $U(x_1,x_2) = x_1x_2$ or $V(x_1,x_2) = x_1 + x_2$, consider**

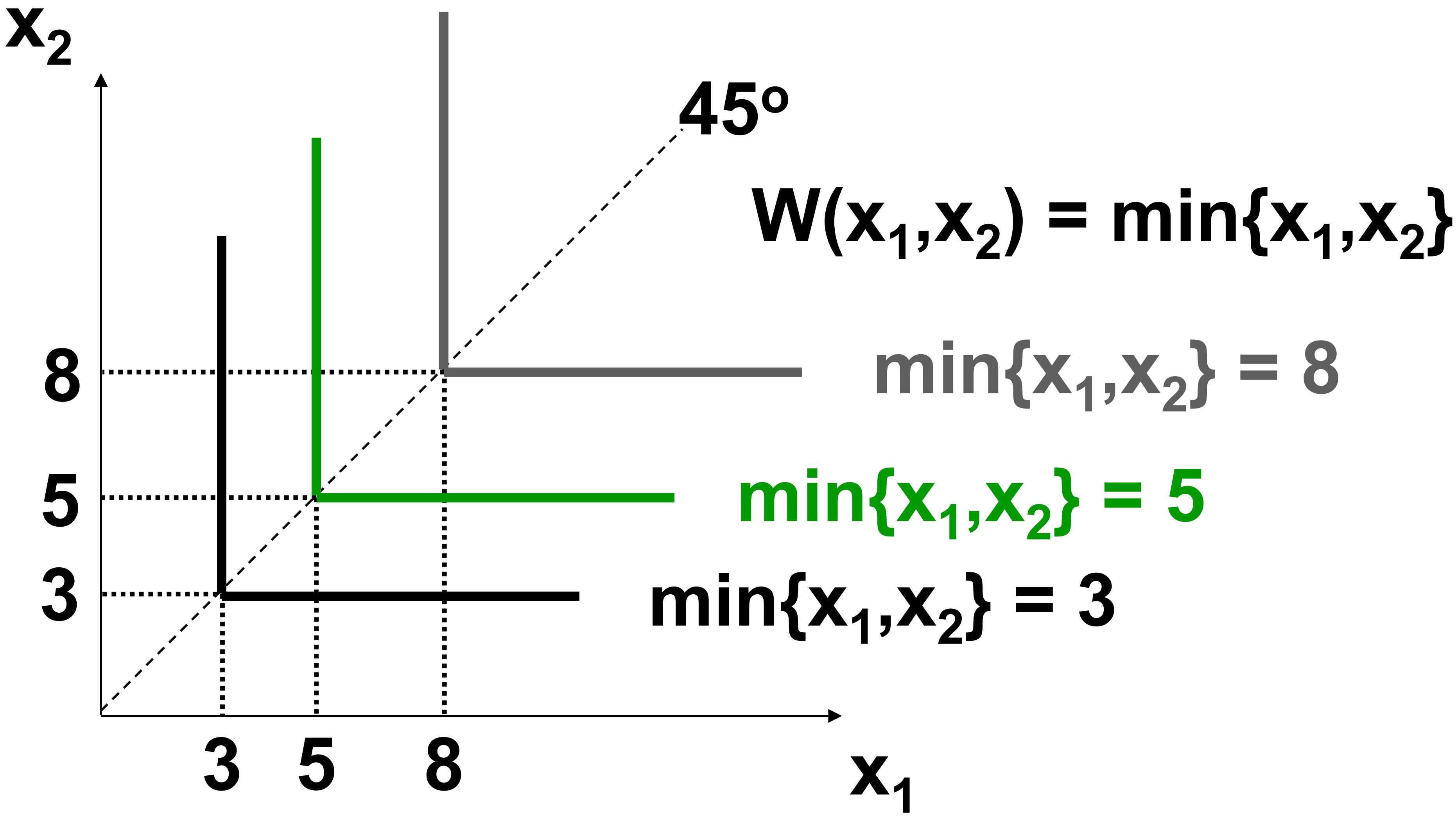
$$W(x_1,x_2) = \min\{x_1,x_2\}.$$

What do the indifference curves for this "perfect complementarity" utility function look like?

Perfect Complementarity Indifference Curves



Perfect Complementarity Indifference Curves



All are right-angled with vertices on a ray from the origin.

Some Other Utility Functions and Their Indifference Curves

- **A utility function of the form**

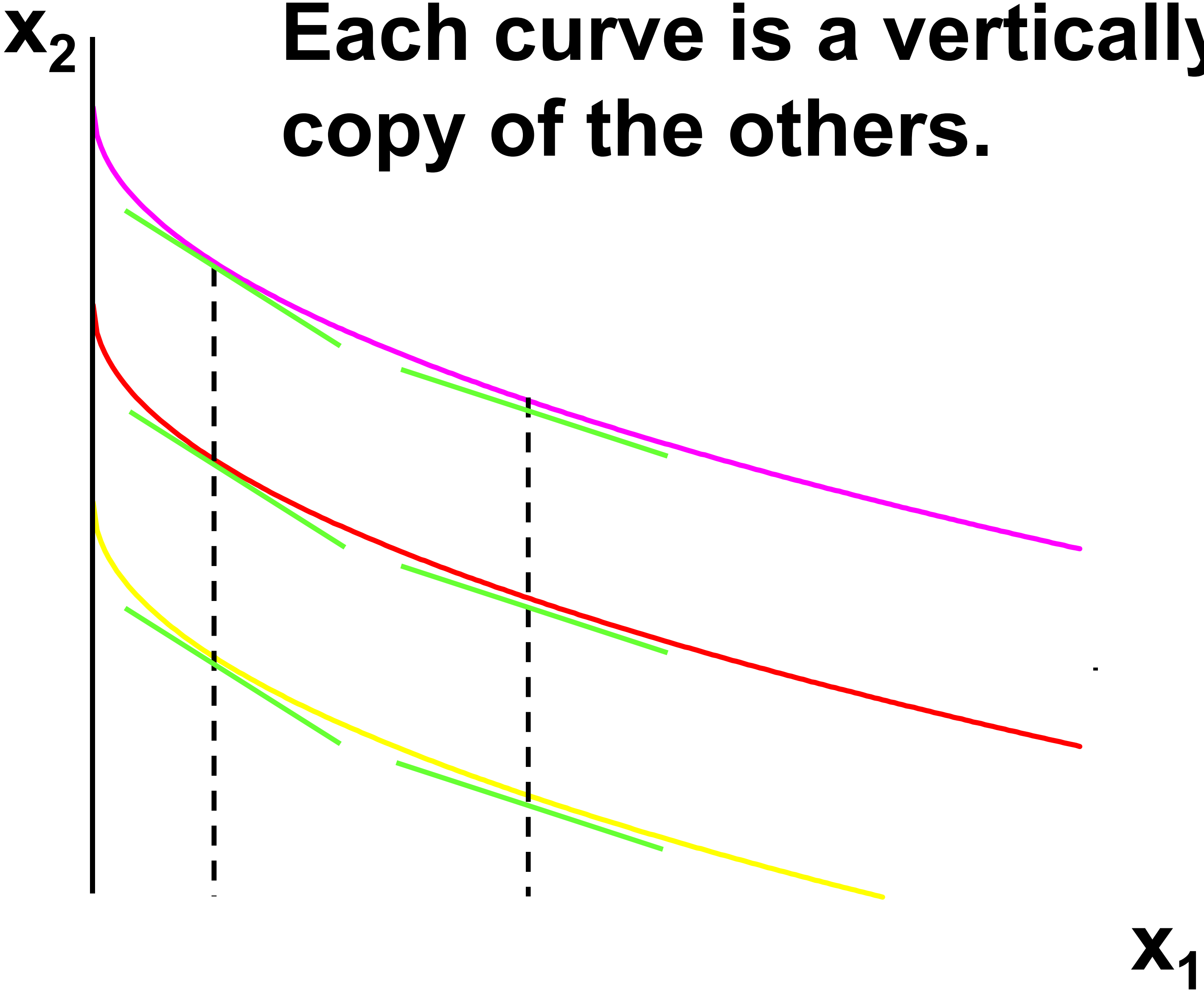
$$U(x_1,x_2) = f(x_1) + x_2$$

is linear in just x_2 and is called quasi-linear.

- ***E.g.*** $U(x_1,x_2) = 2x_1^{1/2} + x_2$.

Quasi-linear Indifference Curves

Each curve is a vertically shifted copy of the others.



Some Other Utility Functions and Their Indifference Curves

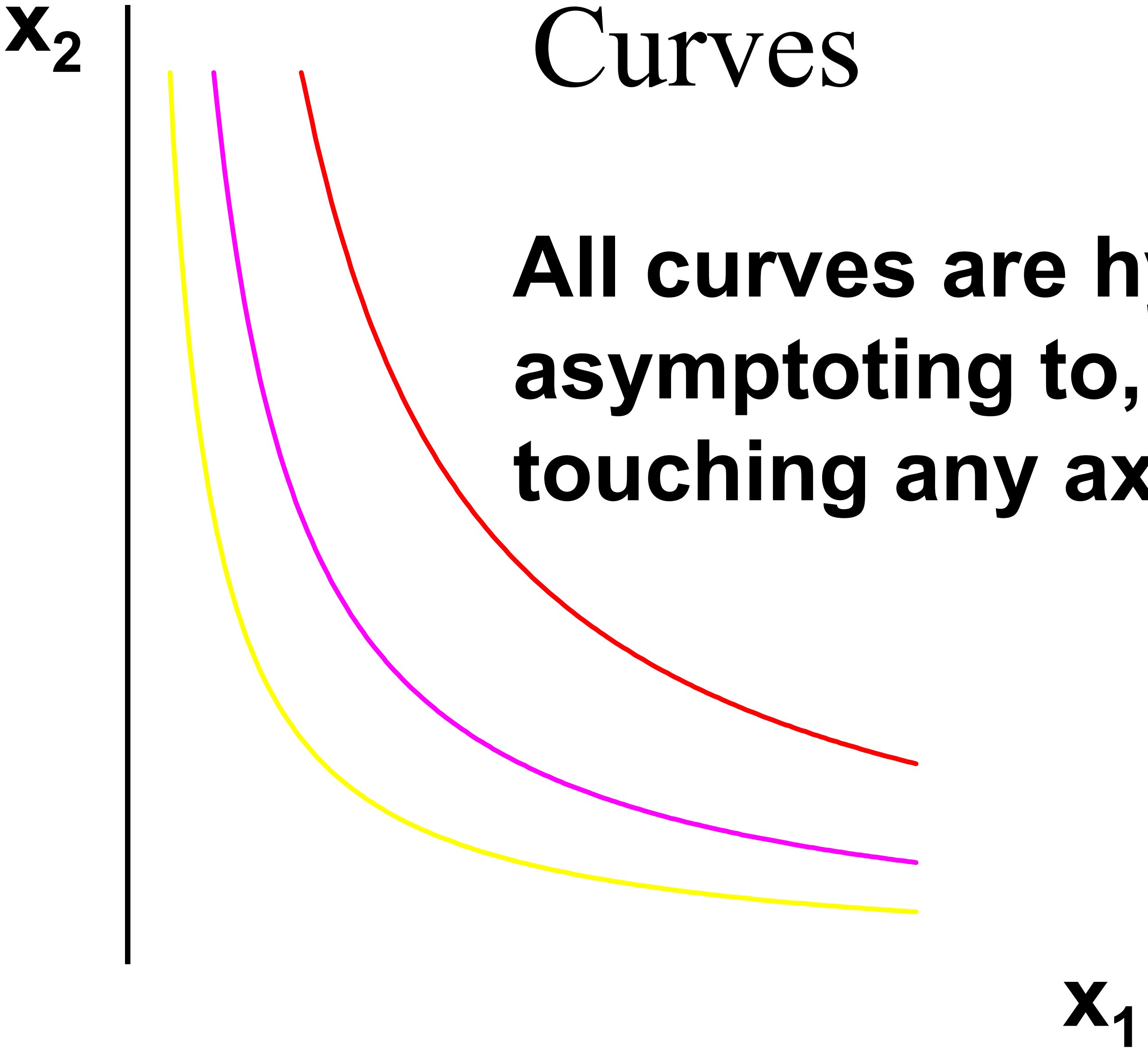
- Any utility function of the form

$$U(x_1,x_2) = x_1^a x_2^b$$

with a > 0 and b > 0 is called a Cobb-Douglas utility function.

- *E.g.* $U(x_1,x_2) = x_1^{1/2} x_2^{1/2}$ (a = b = 1/2)
  $V(x_1,x_2) = x_1 x_2^3$ (a = 1, b = 3)

Cobb-Douglas Indifference Curves



All curves are hyperbolic, asymptoting to, but never touching any axis.

Marginal Utilities

- **Marginal means "incremental".**
- **The marginal utility of commodity i is the rate-of-change of total utility as the quantity of commodity i consumed changes; *i.e.***

$$MU_i = \frac{\partial U}{\partial x_i}$$

Marginal Utilities

- *E.g.* if $U(x_1,x_2) = x_1^{1/2} x_2^2$ then

$$MU_1 = \frac{\partial U}{\partial x_1} = \frac{1}{2} x_1^{-1/2} x_2^2$$

Marginal Utilities

- *E.g.* if $U(x_1,x_2) = x_1^{1/2} x_2^2$ then

$$MU_1 = \frac{\partial U}{\partial x_1} = \frac{1}{2} x_1^{-1/2} x_2^2$$

Marginal Utilities

- *E.g.* if $U(x_1,x_2) = x_1^{1/2} x_2^2$ then

$$MU_2 = \frac{\partial U}{\partial x_2} = 2x_1^{1/2} x_2$$

Marginal Utilities

- *E.g.* if $U(x_1,x_2) = x_1^{1/2} x_2^2$ then

$$MU_2 = \frac{\partial U}{\partial x_2} = 2x_1^{1/2}x_2$$

Marginal Utilities

- *So*, if $U(x_1,x_2) = x_1^{1/2} x_2^2$ then

$$MU_1 = \frac{\partial U}{\partial x_1} = \frac{1}{2} x_1^{-1/2} x_2^2$$

$$MU_2 = \frac{\partial U}{\partial x_2} = 2 x_1^{1/2} x_2$$

Marginal Utilities and Marginal Rates-of-Substitution

- **The general equation for an indifference curve is**

  $U(x_1,x_2) \equiv k$, a constant.

  Totally differentiating this identity gives

$$\frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 = 0$$

Marginal Utilities and Marginal Rates-of-Substitution

$$\frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 = 0$$

rearranged is

$$\frac{\partial U}{\partial x_2}dx_2 = -\frac{\partial U}{\partial x_1}dx_1$$

Marginal Utilities and Marginal Rates-of-Substitution

And $$\frac{\partial U}{\partial x_2}dx_2 = -\frac{\partial U}{\partial x_1}dx_1$$

rearranged is

$$\frac{dx_2}{dx_1} = -\frac{\partial U / \partial x_1}{\partial U / \partial x_2}.$$

This is the MRS.

Marg. Utilities & Marg. Rates-of-Substitution; An example

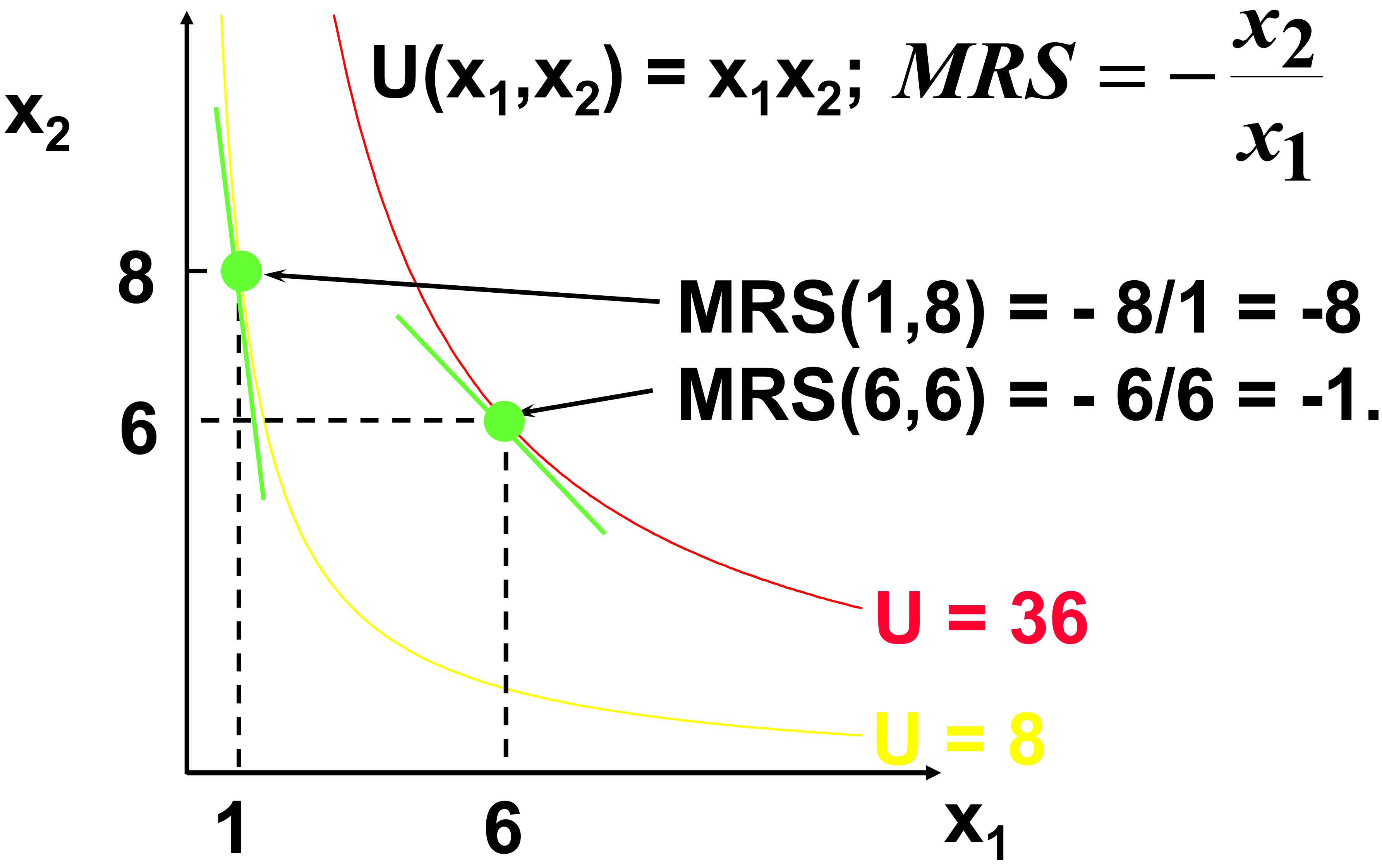
- **Suppose $U(x_1,x_2) = x_1x_2$. Then**

$$\frac{\partial U}{\partial x_1} = (1)(x_2) = x_2$$

$$\frac{\partial U}{\partial x_2} = (x_1)(1) = x_1$$

so $$MRS = \frac{dx_2}{dx_1} = -\frac{\partial U / \partial x_1}{\partial U / \partial x_2} = -\frac{x_2}{x_1}.$$

Marg. Utilities & Marg. Rates-of-Substitution; An example



Marg. Rates-of-Substitution for Quasi-linear Utility Functions

- **A quasi-linear utility function is of the form $U(x_1,x_2) = f(x_1) + x_2$.**

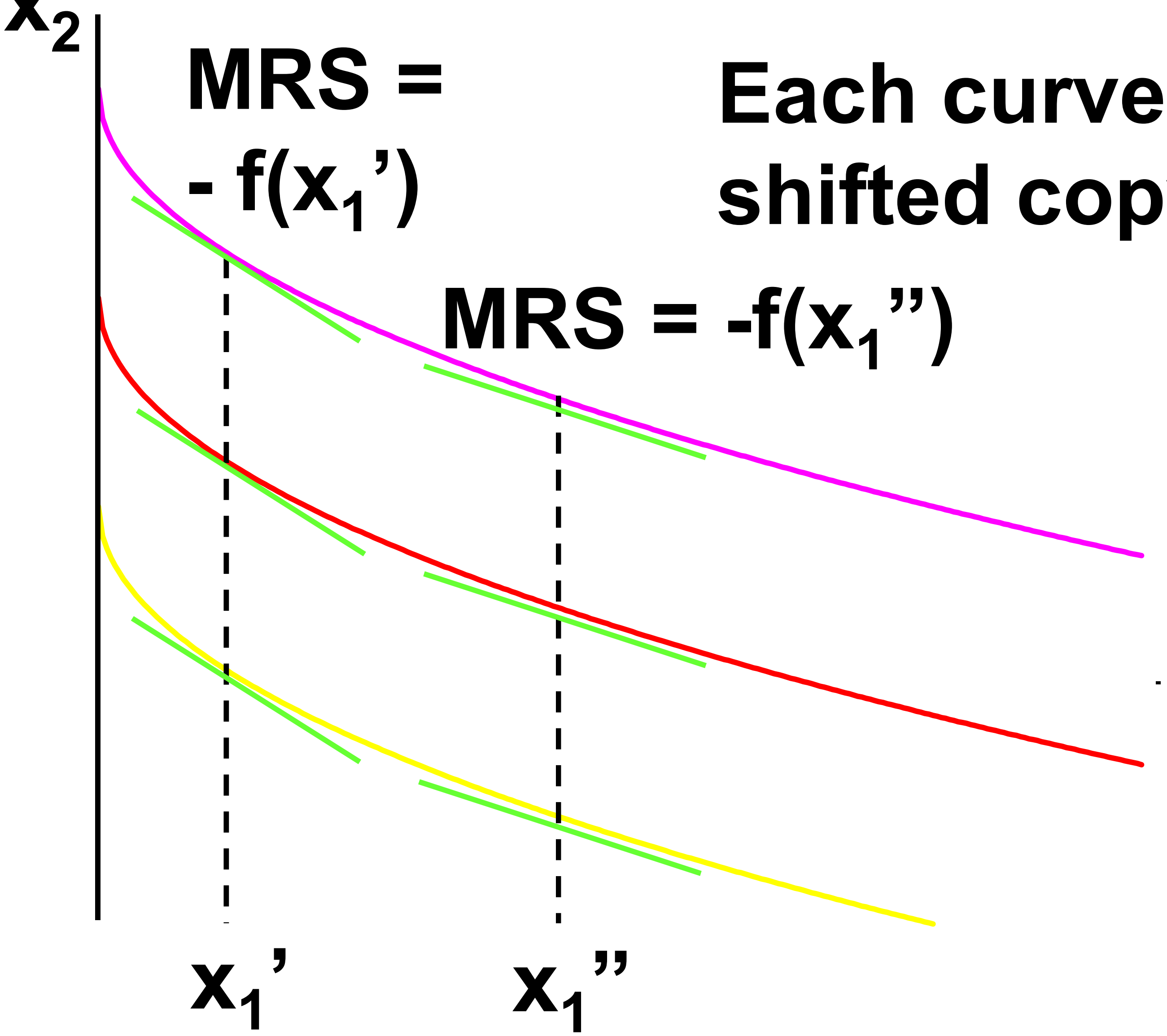
$$\frac{\partial U}{\partial x_1} = f'(x_1) \qquad \frac{\partial U}{\partial x_2} = 1$$

so $$MRS = \frac{dx_2}{dx_1} = -\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = -f'(x_1).$$

Marg. Rates-of-Substitution for Quasi-linear Utility Functions

- **MRS = - $f'(x_1)$ does not depend upon x_2 so the slope of indifference curves for a quasi-linear utility function is constant along any line for which x_1 is constant. What does that make the indifference map for a quasi-linear utility function look like?**

Marg. Rates-of-Substitution for Quasi-linear Utility Functions



Each curve is a vertically shifted copy of the others.

MRS is a constant along any line for which x_1 is constant.

Monotonic Transformations & Marginal Rates-of-Substitution

- **Applying a monotonic transformation to a utility function representing a preference relation simply creates another utility function representing the same preference relation.**
- **What happens to marginal rates-of-substitution when a monotonic transformation is applied?**

Monotonic Transformations & Marginal Rates-of-Substitution

- **For $U(x_1,x_2) = x_1x_2$ the MRS = $-x_2/x_1$.**
- **Create $V = U^2$; *i.e.* $V(x_1,x_2) = x_1^2x_2^2$. What is the MRS for V?**

$$MRS = -\frac{\partial V / \partial x_1}{\partial V / \partial x_2} = -\frac{2x_1x_2^2}{2x_1^2x_2} = -\frac{x_2}{x_1}$$

which is the same as the MRS for U.

Monotonic Transformations & Marginal Rates-of-Substitution

- **More generally, if V = f(U) where f is a strictly increasing function, then**

$$MRS = -\frac{\partial V / \partial x_1}{\partial V / \partial x_2} = -\frac{f'(U) \times \partial U / \partial x_1}{f'(U) \times \partial U / \partial x_2}$$

$$= -\frac{\partial U / \partial x_1}{\partial U / \partial x_2}.$$

So MRS is unchanged by a positive monotonic transformation.