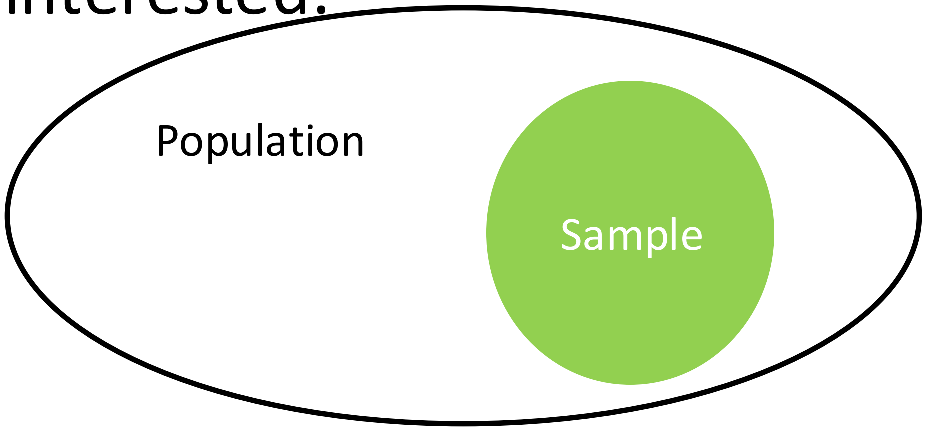
Statistical concepts and market returns

Populations and samples

- The subset of data used in statistical inference is known as a **sample** and the larger body of data is known as the **population.**
  - The **population** is defined as all members of the group in which we are interested.



Parameters and Sample Statistics

A population has parameters, and a sample has statistics.

- Descriptive statistics that characterize population values are called **parameters**.
  - Examples: mean, median, mode, variance, skewness, kurtosis
- Descriptive statistics that characterize samples are known as **sample statistics**.
  - Examples: sample mean, sample median, sample variance
- By convention, we often omit the term “sample” in front of sample statistics, a practice that can lead to confusion when discussing both the sample and the population.

Measurement Scales

Statistical inference is affected by the type of data we are trying to analyze.

Weak Scales

- **Nominal scales** categorize data but do not rank them.
  - Examples: fund style, country of origin, manager gender
- **Ordinal scales** sort data into categories that are ordered with respect to the characteristic along which the scale is measured.
  - Examples: “star” rankings, class rank, credit rating
- **Interval scales** provide both the relative position (rank) and assurance that the differences between scale values are equal.
  - Example: temperature
- **Ratio scales** have all the characteristics of interval scales and a zero point at the origin.
  - Examples: rates of return, corporate profits, bond maturity

Strong Scales

Identifying Scales of Measurement

- **State the scale of measurement for each of the following:**
- Credit ratings for bond issues
- Cash dividends per share
- Hedge fund classification types
- Bond maturity in years

Identifying Scales of Measurement

- **Credit ratings for bond issues**: Ordinal scale
- Credit ratings (e.g., AAA, BB, etc.) rank bond issues in terms of creditworthiness, which involves an inherent order but not precise differences between ranks.
- **Cash dividends per share**: Ratio scale
- Cash dividends per share can have a meaningful zero value (e.g., no dividend paid) and allows for comparisons involving addition, subtraction, multiplication, and division.
- **Hedge fund classification types**: Nominal scale
- Classification types are categorical without an inherent order (e.g., equity hedge, event-driven, etc.).
- **Bond maturity in years**: Ratio scale
- Bond maturity in years has a true zero point (i.e., no maturity) and allows for all arithmetic operations.

Holding period returns

Holding period returns are a fundamental building block of the statistical analysis of investments.

- Holding period returns (HPR) are calculated as the price at the end of the period plus any cash distribution during the period minus the beginning of period price, all divided by the beginning period price.

$$R_t = \frac{P_t - P_{t-1} + D_t}{P_{t-1}}$$

- For this stock, which is nondividend paying, the HPRs are:

Time	Price	HPR	Time	Price	HPR
0	27.00	—	7	25.90	2.38%
1	25.77	–4.57%	8	27.01	4.28%
2	24.73	–4.04%	9	28.20	4.42%
3	24.32	–1.64%	10	29.52	4.68%
4	24.39	0.28%	11	31.63	7.16%
5	24.71	1.34%	12	35.25	11.43%
6	25.30	2.35%			

Frequency distributions

A tabular display of data summarized into intervals is known as a frequency distribution.

Constructing a frequency distribution:

1. Sort the data in ascending order.
2. Calculate the range of the data, defined as

   Range = Maximum value – Minimum value.
3. Decide on the number of intervals in the frequency distribution, *k*.
4. Determine interval width as Range/*k*.
5. Determine the intervals by successively adding the interval width to the minimum value to determine the ending points of intervals, stopping after reaching an interval that includes the maximum value.
6. Count the number of observations falling in each interval.
7. Construct a table of the intervals listed from smallest to largest that shows the number of observations falling in each interval.

Frequency Distributions

Focus on: Holding Period Returns

- Suppose we have 12 holding period return observations from a non-dividend-paying stock, sorted in ascending order:

  −4.57, −4.04, −1.64, 0.28, 1.34, 2.35, 2.38, 4.28, 4.42, 4.68, 7.16, and 11.43.
- Using k = 4, we have intervals with width of 4.
- The resulting frequency distribution is

Interval	Absolute Frequency
−4.57 ≤ observation < −0.57	3
−0.57 ≤ observation < 3.43	4
3.43 ≤ observation < 7.43	4
7.43 ≤ observation ≤ 11.43	1

Relative and cumulative frequency

Focus on: Holding Period Returns

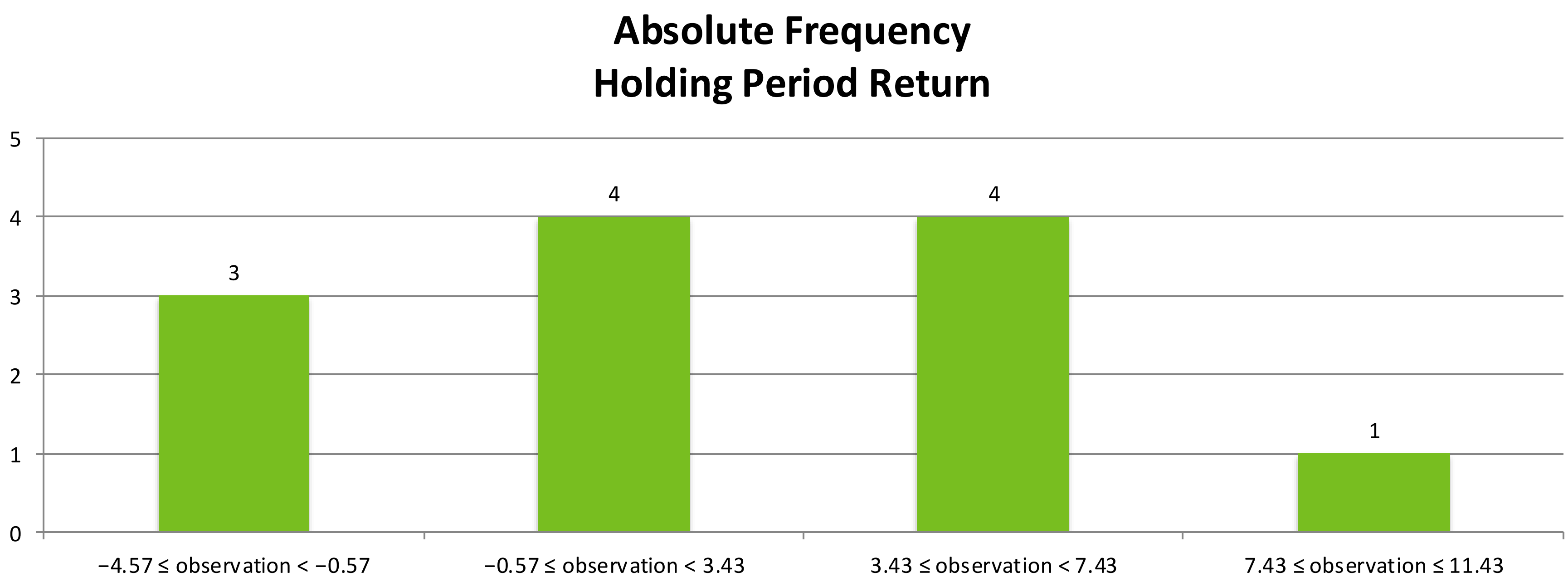
- **Relative frequency** is the absolute frequency divided by the total number of observations.
- **Cumulative (relative) frequency** is the relative frequency of all observations occurring before a given interval.

Interval	Absolute Frequency	Relative Frequency	Cumulative Frequency
$-4.57 \le$ observation < -0.57	3 $\xrightarrow{\div 12}$	0.250	0.250
$-0.57 \le$ observation < 3.43	4	0.333	0.583
$3.43 \le$ observation < 7.43	4	0.333	0.917
$7.43 \le$ observation ≤ 11.43	1	0.083	1.000

Histograms

Focus on: Holding Period Returns

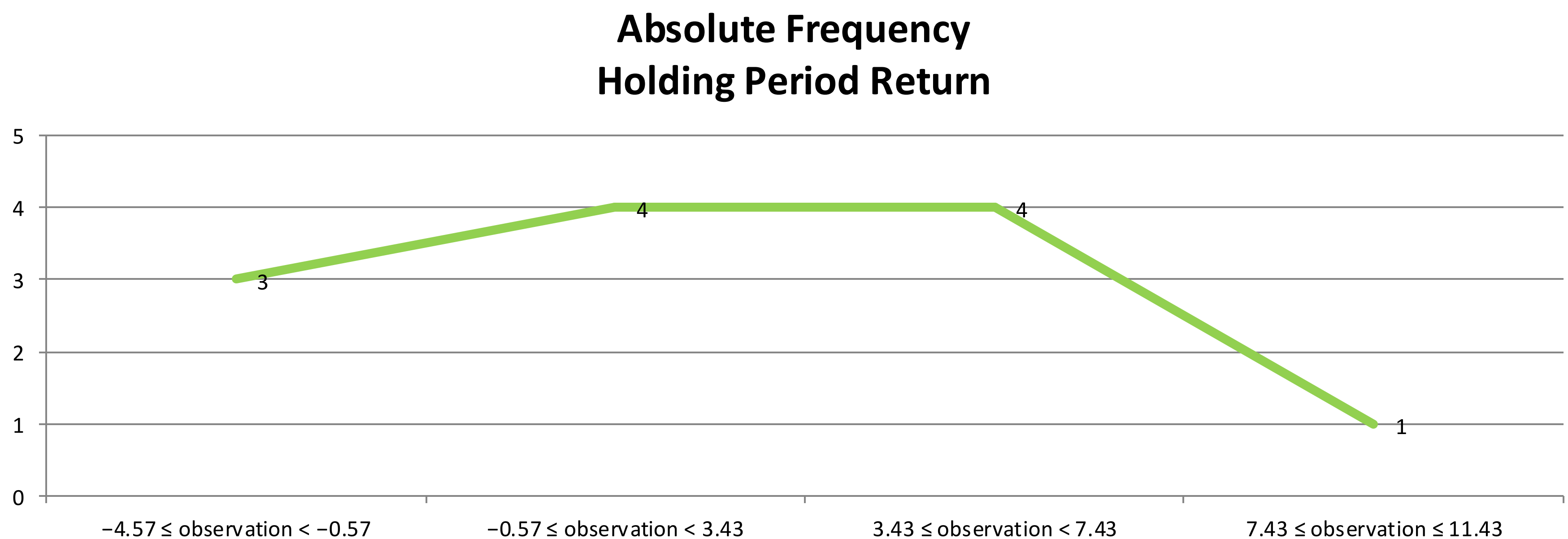
- Histograms are the graphical representation of a frequency distribution.



Frequency Polygon

Focus on: Holding Period Returns

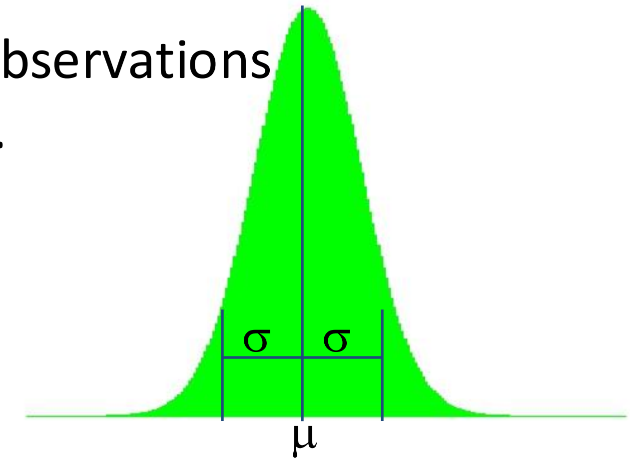
- Frequency polygons are often used to provide higher visual continuity than histograms.



Measures of central tendency

These measures describe where the data are centered.

- Arithmetic Mean
  - The **arithmetic mean** is the sum of the observations divided by the number of observations.



  - **Population mean →** $\mu = \frac{\sum_{i=1}^{N} X_i}{N}$
  - **Sample mean →** $\bar{X} = \frac{\sum_{i=1}^{N} X_i}{N}$
  - The sample mean is often interpreted as center of gravity, for a given set of data.
  - **Cross-sectional data** occur across different observation types at one point in time, and **time-series data** occur for the same unit of observation across time.

Measures of central tendency

Focus on: Cross-Sectional Sample Mean Return

Country	Return	Country	Return
Austria	–2.97%	Italy	–23.64%
Belgium	–29.71%	Netherlands	–34.27%
Denmark	–29.67%	Norway	–29.73%
Finland	–41.65%	Portugal	–28.29%
France	–33.99%	Spain	–29.47%
Germany	–44.05%	Sweden	–43.07%
Greece	–39.06%	Switzerland	–25.84%
Ireland	–38.97%	United Kingdom	–25.66%

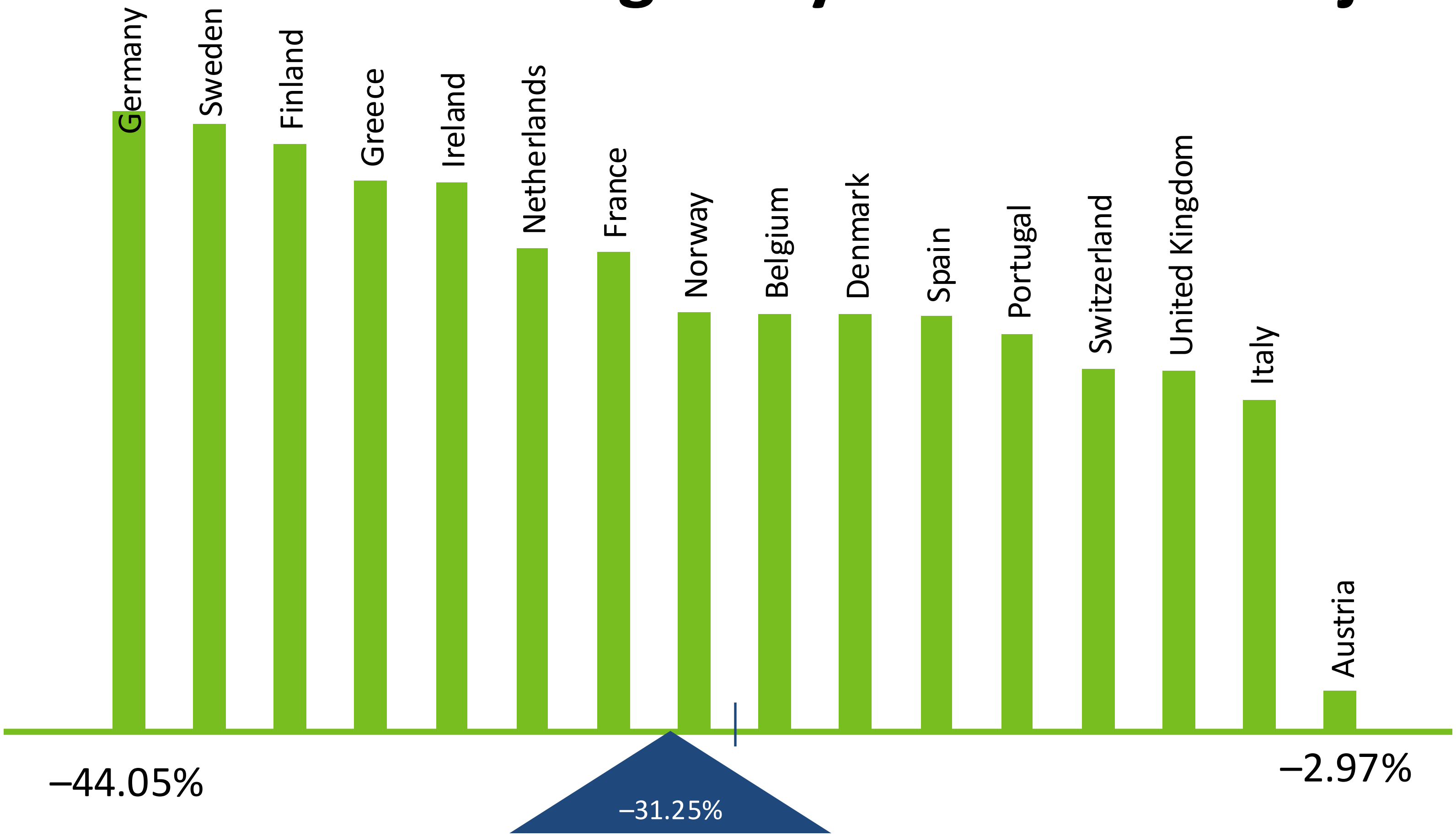
$$\bar{X} = \frac{\sum_{i=1}^{N} X_i}{N}$$

$$\bar{X} = \frac{-500.04}{16} = -31.25\%$$

Source: www.msci.com.

Measures of central tendency

Mean as a center of gravity for the data object



Measures of central tendency

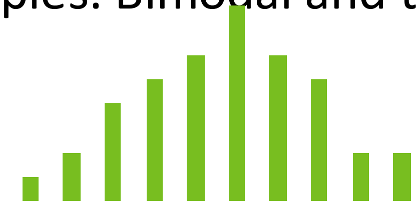
These measures also describe where the data are centered.

- Weighted Mean → $\bar{X}_W = \sum_{i=1}^{n} w_i X_i$
  - The sum of the observations times each observation's weight (proportional representation in the sample), where the weight is chosen to meet a statistical or financial goal. Example: Portfolio return
- Geometric Mean → $G = \sqrt[n]{\prod_{i=1}^{n} X_i}$
  - Represents the growth rate or compounded return on an investment when X is $1 + R$
- Harmonic Mean → $\bar{X}_H = {}^{n}\big/{\sum_{i=1}^{n} {}^{1}/_{X_i}}$
  - A weighted mean in which each observation's weight is inversely proportional to its magnitude. Example: Cost averaging

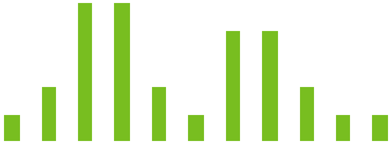
Measures of central tendency

These measures also describe where the data are centered.

- The **median** is the middle observation by rank.
  - When we have an odd number of observations, the median will be the closest to the middle. When we have an even number, the median will be the average of the two middle values.
- The **mode** is the most frequently occurring value in a distribution.
  - Distributions are unimodal when there is a single most frequently occurring value and multimodal if there is more than one frequently occurring value.
  - Examples: Bimodal and trimodal



Unimodal



Bimodal

Measures of central tendency

Focus on: Calculating a Median or Mode

$$\text{Median} = \frac{-29.73\% + (-29.71\%)}{2} = -29.72\%$$

Rank	Country	Return	Rank	Country	Return
1	Germany	–44.05%	**9**	**Belgium**	**–29.71%**
2	Sweden	–43.07%	10	Denmark	–29.67%
3	Finland	–41.65%	11	Spain	–29.47%
4	Greece	–39.06%	12	Portugal	–28.29%
5	Ireland	–38.97%	13	Switzerland	–25.84%
6	Netherlands	–34.27%	14	United Kingdom	–25.66%
7	France	–33.99%	15	Italy	–23.64%
8	**Norway**	**–29.73%**	16	Austria	–2.97%

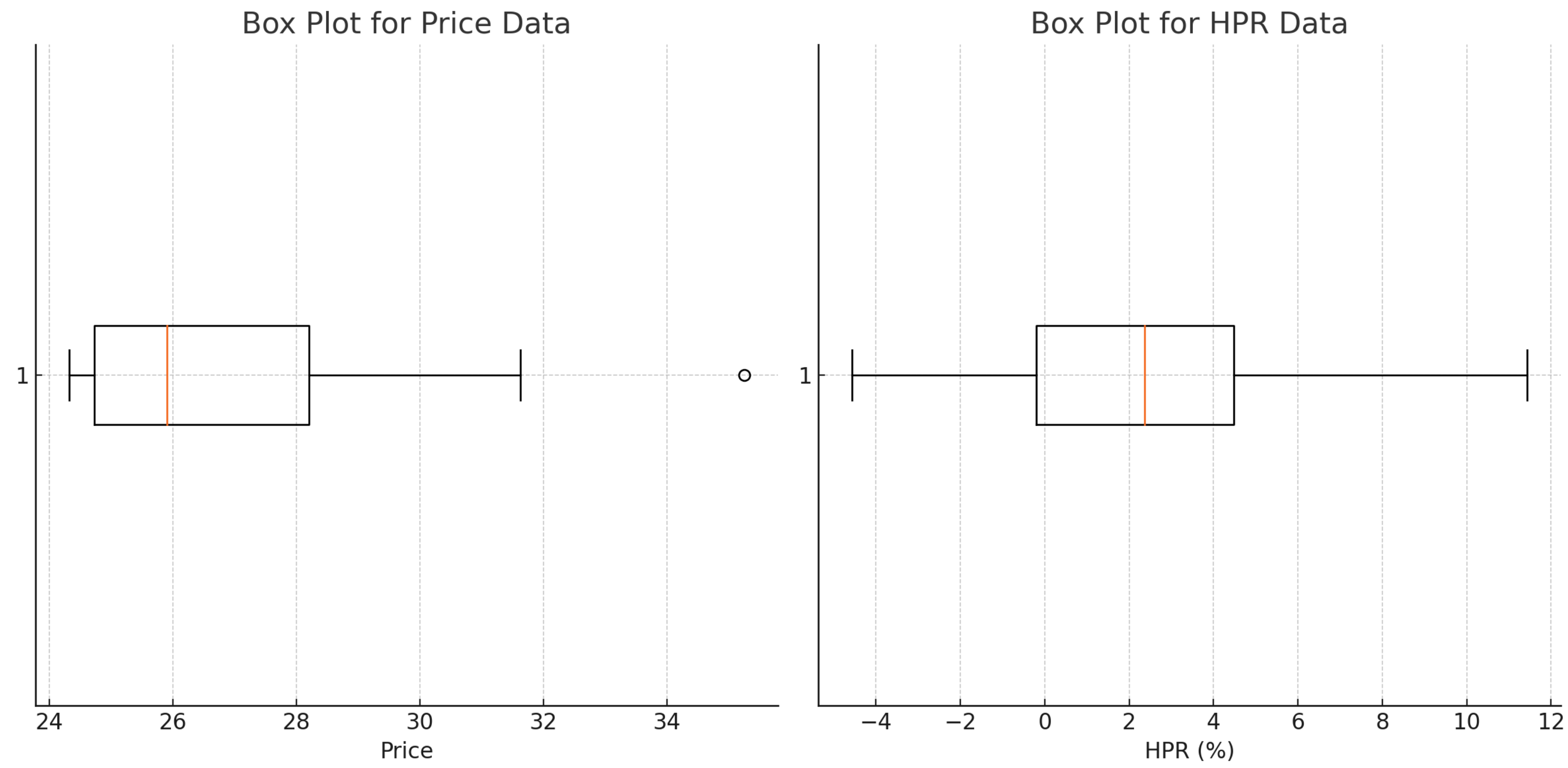
Median: The Case of the Price–Earnings Ratio

ble 10. P/Es for a Client Portfolio

Stock	Consensus Current EPS	Consensus Current P/E
Caterpillar, Inc.	6.34	13.15
Ford Motor Company	1.55	10.97
General Dynamics	6.96	12.15
Green Mountain Coffee Roasters	3.25	25.27
McDonald's Corporation	5.61	17.16
Qlik Technologies	0.17	204.82
Questcor Pharmaceuticals	4.79	13.94

e: Consensus current P/E was calculated as price as of 9 September 2013 divided by consensus EPS as of the same date.

Box plot


Box Plot for Price Data
1
24
26
28
30
32
34
Price
Box Plot for HPR Data
1
−4
−2
0
2
4
6
8
10
12
HPR (%)

Weighted average

Also known as a weighted mean, the most common application of this measure in investments is the weighted mean return to a portfolio.

- Consider again the country-level data. You have constructed a portfolio that has 50% of its weight in Portugal, Ireland, Greece, and Spain and 50% of its weight in Germany and the UK. Each of the first four countries is equally weighted within the 50%, as are Germany and the UK within their 50%. What is the weighted average return to the portfolio?

$$\bar{X}_W = \sum_{i=1}^{n} w_i X_i$$

Country	Weight	Return	Component Return
Portugal	12.50%	–28.29%	–3.54%
Ireland	12.50%	–23.64%	–2.96%
Greece	12.50%	–39.06%	–4.88%
Spain	12.50%	–29.47%	–3.68%
Germany	25.00%	–44.05%	–1.01%
UK	25.00%	–25.66%	–6.42%
Sum	100%	Weighted Mean =	–32.49%

Measures of dispersion

Dispersion measures variability around a measure of central tendency.
If mean return represents reward, then dispersion represents risk.

- Mean Absolute Deviation (MAD) → $\text{MAD} = \frac{\sum_{i=1}^{n}|X_i - \bar{X}|}{n}$
  - The arithmetic average of the absolute value of deviations from the mean.

Measures of dispersion

Dispersion measures variability around a measure of central tendency. If mean return represents reward, then dispersion represents risk.

- Variance is the average squared deviation from the mean.
  - Population variance →$\sigma^2 = \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{n}$
  - Sample variance → $s^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$
- Sample variance is "penalized" by dividing by $n - 1$ instead of n to account for the fact that the measure of central tendency used, $\bar{X}$, is an estimate of the true population parameter, μ, and so has some uncertainty associated with it.
- Standard deviation is the square root of variance.

Measures of dispersion

Focus on: Sample Standard Deviation

Country	Return	Squared Deviation from Mean
Germany	–44.05%	0.016384
Sweden	–43.07%	0.013971
Finland	–41.65%	0.010816
Greece	–39.06%	0.00610
…	…	…
Austria	–2.97%	0.0780
	Sum=	0.1486
	s^2=	0.0099
	s =	9.95%

Annualizing Standard Deviation of Returns

- Formula for Annualizing Standard Deviation

$$\sigma_{\text{annual}} = \sigma_{\text{period}} \times \sqrt{N}$$

- **Where:**
- σannual: Annualized standard deviation.
- σperiod: Standard deviation of returns for a given period (e.g., daily, weekly, monthly).
- N: Number of periods in a year.
  - For daily returns: N=252 (trading days)
  - For weekly returns: N=52
  - For monthly returns: N=12
- **Example:**
- Daily standard deviation: 1.5%
- Annualized: 1.5%×252=23.85%

Semivariance

We are often concerned with measures of risk that focus on the "downside" of the possible outcomes—in other words, the losses.

- Semivariance is the average squared deviation below the mean.
  - Semideviation is the square root of semivariance.
  - Both are a measure of dispersion focusing only on those observations below the mean.

$$\sum_{for\ all\ Xi<\bar{X}} \frac{(X_i - \bar{X})^2}{n^* - 1}$$

  - Target semivariance, by analogy, is the average squared deviation below some specified target rate, *B*, and represents the "downside" risk of being below the target, *B.*

$$\sum_{for\ all\ Xi<B} \frac{(X_i - B)^2}{n^* - 1}$$

Beta coeficient

- **Beta**: Measures systematic risk (market risk) by comparing the volatility of an asset relative to a benchmark index (e.g., S&P 500). A beta greater than 1 implies the asset is more volatile than the market.

Beta coeficient

The formula for calculating the beta (β) of a security or portfolio, which measures its sensitivity to market movements, is:

$$\beta = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)}$$

Where:

- β: Beta of the security or portfolio.
- $\mathrm{Cov}(R_i, R_m)$: Covariance between the return of the security/portfolio (R_i) and the return of the market (R_m).
- $\mathrm{Var}(R_m)$: Variance of the market return (R_m).

Normal Distribution Function

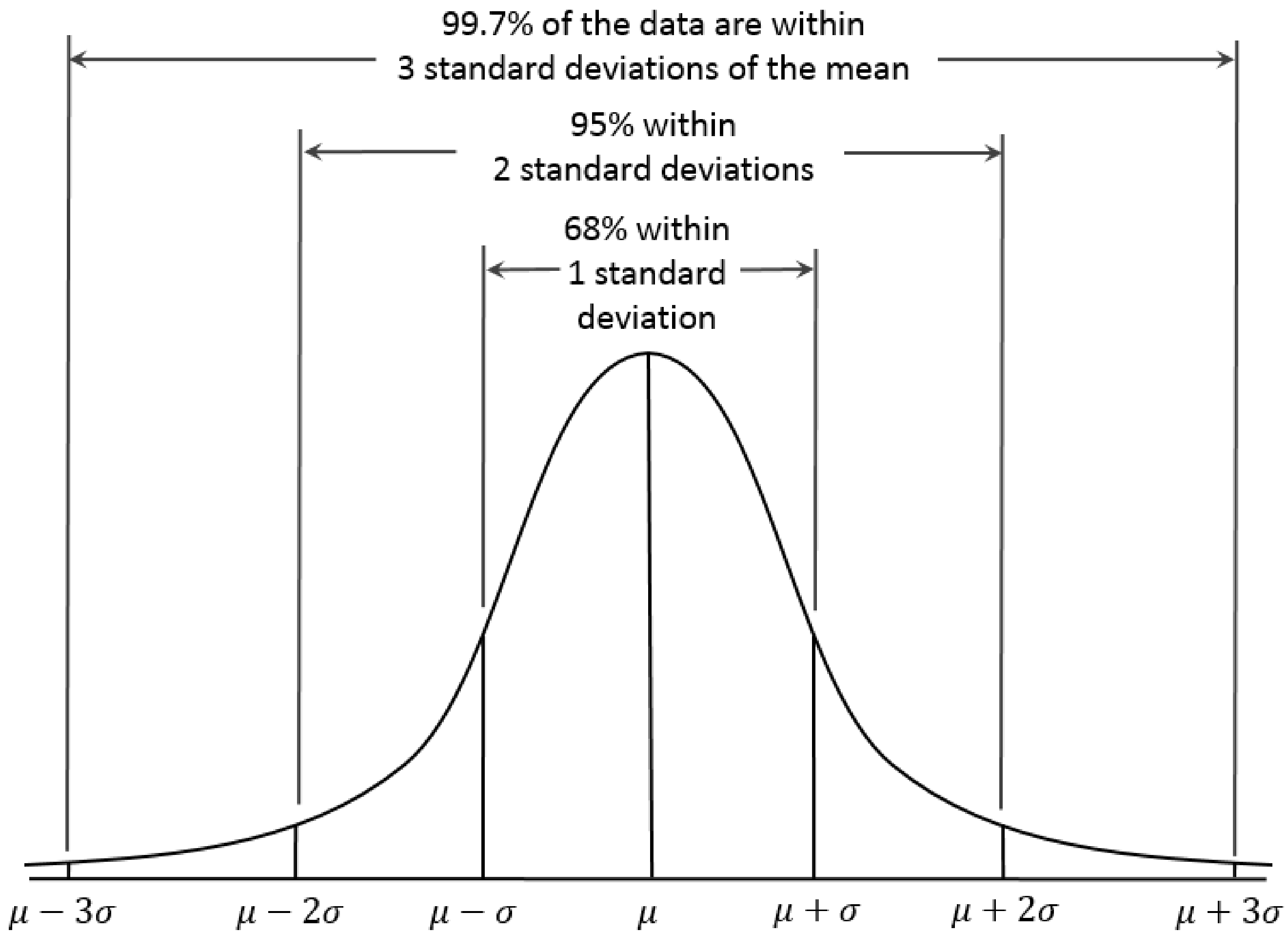


EXHIBIT 5-9 Histogram of U.S. Large Company Stock Returns, 1926-2008

						2006				
						2004				
				2000	2007	1988	2003	1997		
				1990	2005	1986	1999	1995		
				1981	1994	1979	1998	1991		
				1977	1993	1972	1996	1989		
				1969	1992	1971	1983	1985		
				1962	1987	1968	1982	1980		
				1953	1984	1965	1976	1975		
				1946	1978	1964	1967	1955		
			2001	1940	1970	1959	1963	1950		
			1973	1939	1960	1952	1961	1945		
		2002	1966	1934	1956	1949	1951	1938	1958	
	2008	1974	1957	1932	1948	1944	1943	1936	1935	1954
1931	1937	1939	1941	1929	1947	1926	1942	1927	1928	1933

–60 –50 –40 –30 –20 –10 0 10 20 30 40 50 60 70

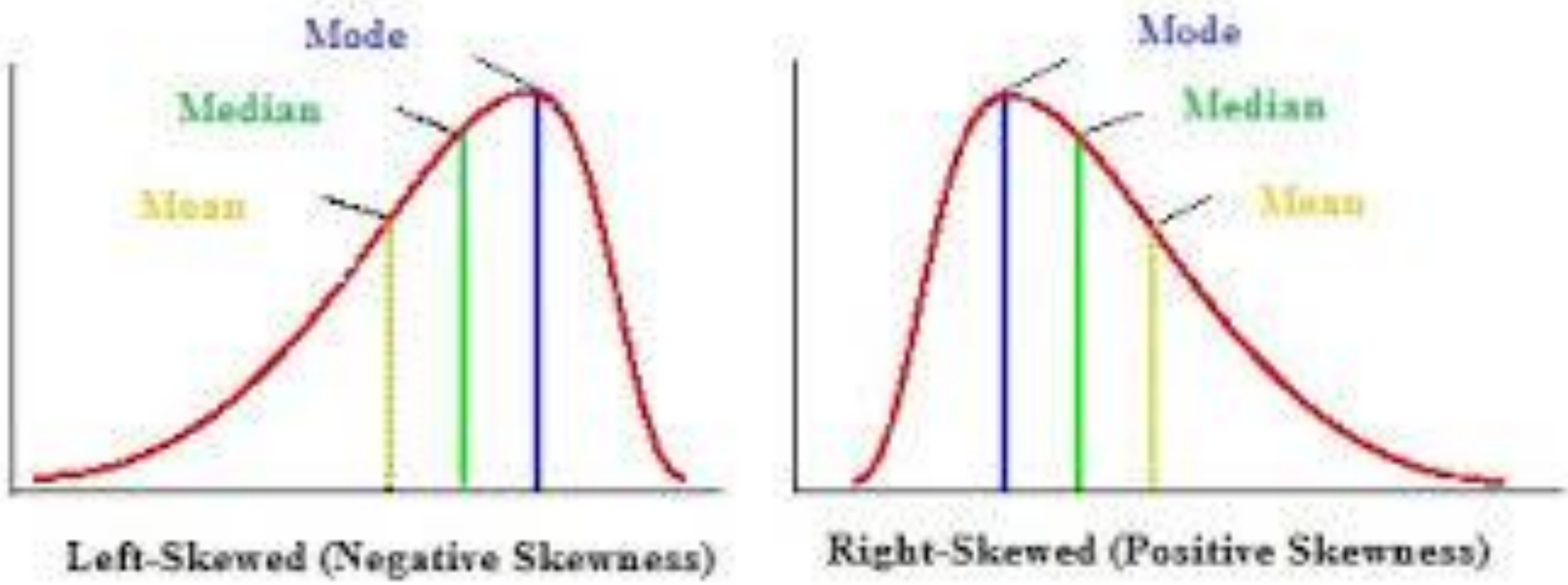
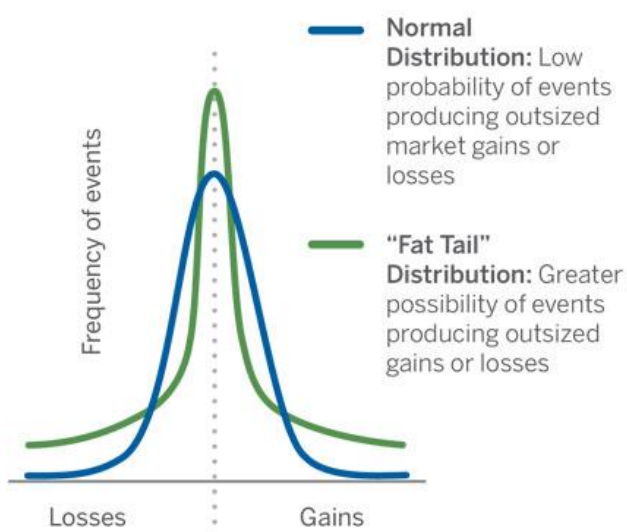


EXHIBIT 1: ARE TAILS GETTING "FATTER?"



Source: Brown Advisory

Chebyshev's inequality

This expression gives the minimum proportion of values, *p*, within *k* standard deviations of the mean for any distribution whenever *k* > 1.

$$p \geq 1 - \frac{1}{k^2}$$

k	Interval around the Mean	*p*
1.25	$\bar{X} \pm 1.25s$	0.36
1.50	$\bar{X} \pm 1.50s$	0.56
2.00	$\bar{X} \pm 2.00s$	0.75
2.50	$\bar{X} \pm 2.50s$	0.84
3.00	$\bar{X} \pm 3.00s$	0.89
4.00	$\bar{X} \pm 4.00s$	0.94

Chebyshev's inequality

Focus on: Calculating Proportions Using Chebyshev's Inequality

- For our country data, the mean is –31.25% and the sample standard deviation is 9.95%.
- Lower cutoff at 1.25 standard deviations:

  –31.25% – 1.25 (9.95%) = – 43.6875%
- Upper cutoff at 1.25 standard deviations:

  –31.25% + 1.25 (9.95%) = – 18.8125%

k	Lower Cutoff	Upper Cutoff	Actual *p*	Chebyshev's *p*
1.25	–43.69%	–18.81%	0.875	0.36
1.50	–46.18%	–16.32%	0.938	0.56
2.00	–51.16%	–11.34%	0.95	0.75
2.50	–56.13%	–6.37%	0.97	0.84
3.00	–61.11%	–1.39%	0.997	0.89
4.00	–71.07%	8.57%	1.000	0.94

Chebyshev’s inequality

Applying Chebyshev’s Inequality

According to Table 22, the arithmetic mean monthly return and standard deviation of monthly returns on the S&P 500 were 0.94 percent and 5.50 percent, respectively, during the 1926–2012 period, totaling 1,044 monthly observations. Using this information, address the following:

1. Calculate the endpoints of the interval that must contain at least 75 percent of monthly returns according to Chebyshev’s inequality.

Risk-Adjusted Return Metrics

- These metrics compare risk and return together to assess how well an investment compensates for the risk taken:
- **Sharpe Ratio**: Measures the return per unit of risk (excess return over the risk-free rate divided by the standard deviation of returns). A higher Sharpe ratio indicates better risk-adjusted performance.
- **Treynor Ratio**: Similar to the Sharpe ratio but uses **Beta** as the risk measure, focusing on systematic risk. This metric is useful for evaluating assets that are part of a diversified portfolio.
- **Jensen's Alpha (Alpha)**: Represents the excess return of an investment compared to the expected return predicted by the Capital Asset Pricing Model (CAPM). A positive alpha suggests that the asset has outperformed the market given its level of risk.
- **Sortino Ratio**: Similar to the Sharpe ratio, but only considers downside risk. This metric is more relevant for investors who are more concerned with minimizing losses rather than total volatility.
- **Information Ratio**: Measures the return of an asset in excess of a benchmark, adjusted for the tracking error (standard deviation of the difference between the asset and benchmark returns).

Sharpe Ratio:

- **Definition**: Measures the return per unit of total risk.
- **Formula**:
- (Return−Risk Free Rate)/Standard Deviation of Returns
- **Interpretation**:
  - A **higher Sharpe ratio** indicates better risk-adjusted returns.
  - Useful for comparing the performance of different investments or portfolios.
- **Strength**: Considers total risk (volatility) and is widely applicable.
- **Use Case**: Evaluating whether the additional risk taken has been rewarded with higher returns.

Treynor Ratio:

- **Definition**: Measures the return per unit of systematic risk.
- **Formula**: (Return−Risk-Free Rate)/Beta
- **Interpretation**: **Higher Treynor ratio** implies better compensation for taking on market risk.
- Useful when evaluating the performance of investments in a diversified portfolio.
- **Strength**: Focuses on systematic risk, which is relevant in well-diversified portfolios.
- **Use Case**: Suitable for investors looking to assess performance in relation to market exposure.

Jensen's Alpha (Alpha):

- **Definition**: Measures the excess return of an asset compared to its expected return, as predicted by the Capital Asset Pricing Model (CAPM).
- **Formula**: Alpha=Actual Return−(Risk-Free Rate+β(Market Return−Risk-Free Rate))
- **Interpretation**: **Positive Alpha** indicates the asset has outperformed expectations given its level of market risk.
- **Negative Alpha** suggests underperformance relative to market expectations.
- **Strength**: Shows whether an investment adds value beyond what is expected from its risk level.
- **Use Case**: Used by portfolio managers to demonstrate their ability to generate value beyond market returns.

Sortino Ratio:

- **Definition**: Similar to the Sharpe Ratio but only considers downside deviation (negative volatility).
- **Formula**: (Return−Risk-Free Rate)/Downside Deviation
- **Interpretation**: **Higher Sortino ratio** indicates better risk-adjusted returns with a focus on minimizing losses.
- Ignores upside volatility, focusing on what most concerns risk-averse investors—downside risk.
- **Strength**: Ideal for assessing risk-adjusted returns when downside risk is more important than total volatility.
- **Use Case**: Useful in evaluating funds or investments for investors who are particularly concerned with avoiding losses.

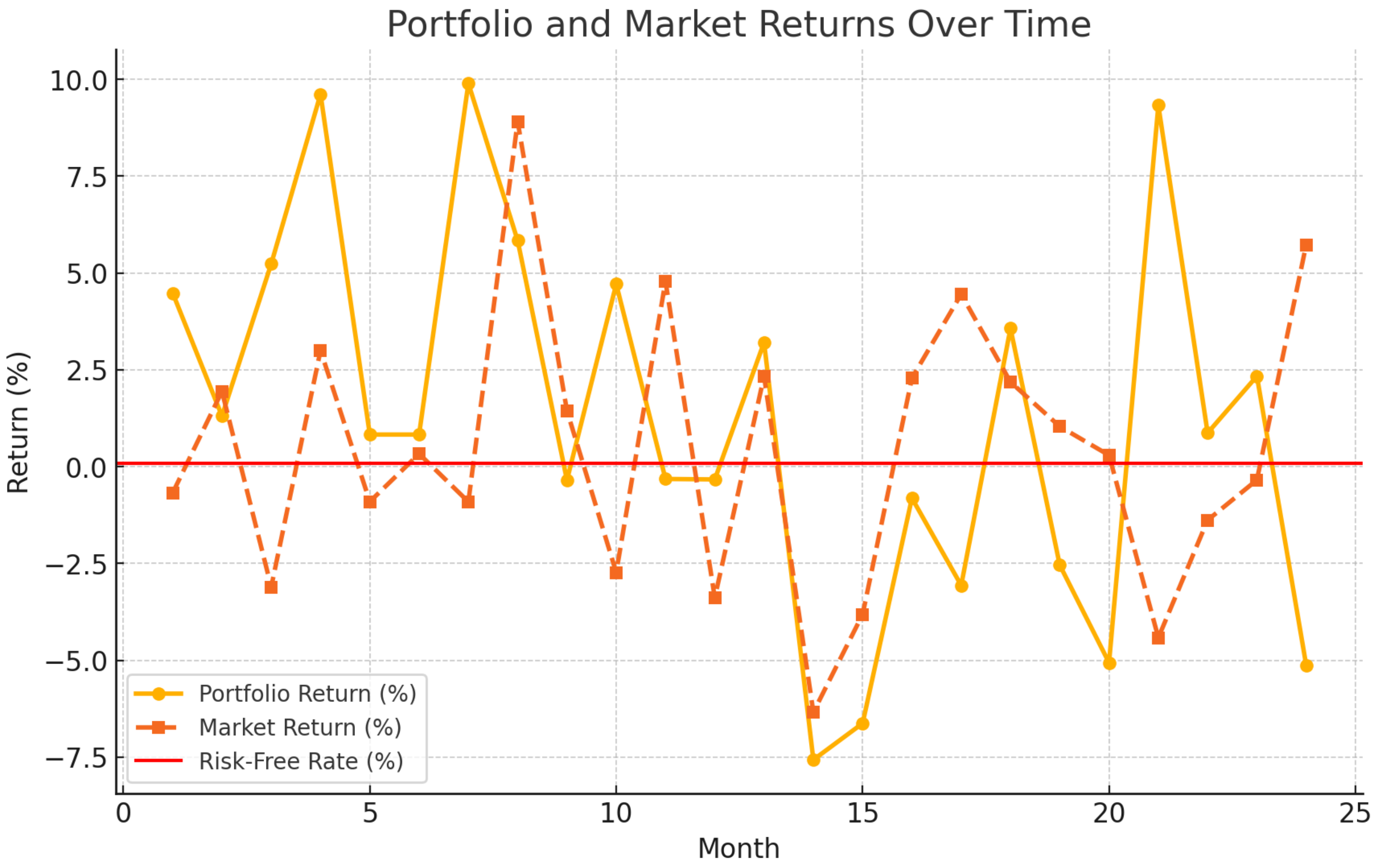
Information Ratio:

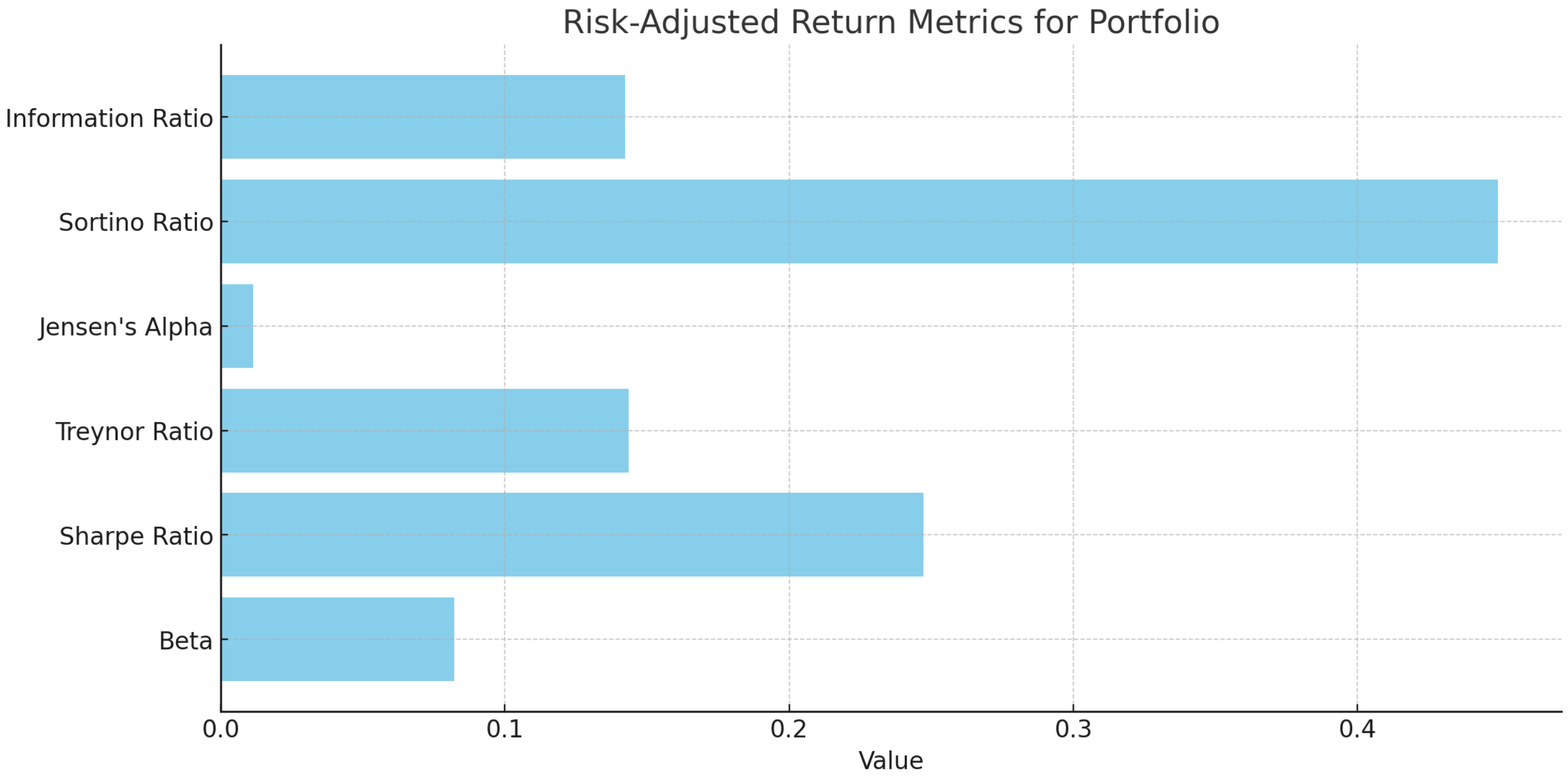
- **Definition**: Measures the return of an investment relative to a benchmark, adjusted for tracking error.
- **Formula**: (Portfolio Return−Benchmark Return)/Tracking Err or
- **Interpretation**: **Higher Information Ratio** indicates a portfolio manager is achieving consistent excess returns compared to the benchmark.
- Assesses both the return and the consistency of beating a benchmark.
- **Strength**: Useful for investors seeking active management with consistent benchmark outperformance.**Use Case**: Helps compare portfolio managers in terms of their efficiency in generating benchmark-beating returns.

Summary

- **Risk-Adjusted Metrics** offer critical insights into how well an investment compensates for risk.
- **Sharpe Ratio**: Measures return per unit of total risk.
- **Treynor Ratio**: Evaluates return in relation to market risk.
- **Jensen's Alpha**: Shows whether an investment beats expected market performance.
- **Sortino Ratio**: Focuses on downside risk, important for loss-averse investors.
- **Information Ratio**: Highlights the consistency and efficiency of beating a benchmark.

Example:

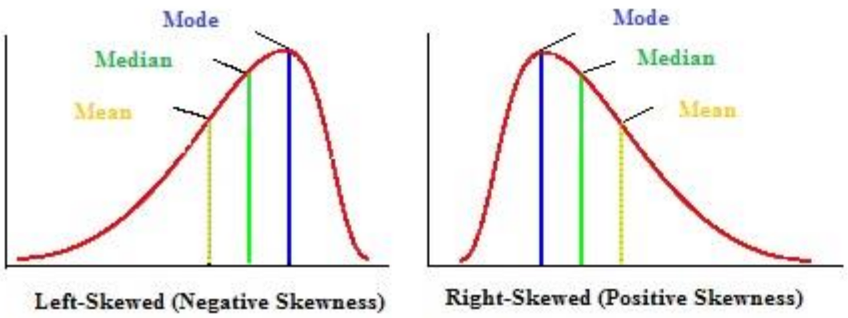

Portfolio and Market Returns Over Time
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
Return (%)
0
5
10
15
20
25
Month
Portfolio Return (%)
Market Return (%)
Risk-Free Rate (%)



- The **Sortino Ratio** and **Sharpe Ratio** are relatively higher, suggesting better performance in terms of returns adjusted for risk.
- **Jensen's Alpha** is positive, indicating the portfolio has performed slightly better than expected, based on its level of risk.
- The portfolio has a **very low beta**, which means it has **much lower sensitivity to market movements**. For every 1% move in the market, the portfolio is expected to move only about **0.082%** in the same direction.

Combining centrality, dispersion, and symmetry

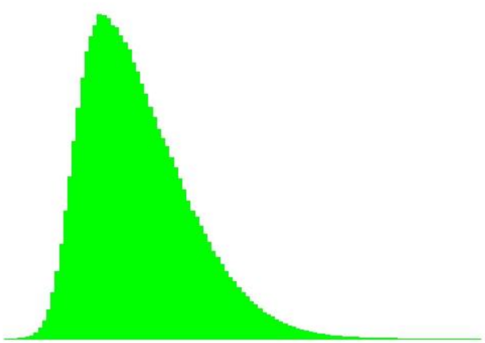
- For a symmetrical distribution, the mean, median, and mode (if it exists) will all be at the same location.
- If the distribution is positively skewed, then the mean will be greater than the median, which will be greater than the mode (if it exists).
- If the distribution is negatively skewed, then the mean will be less than the median, which will be less than the mode (if it exists).



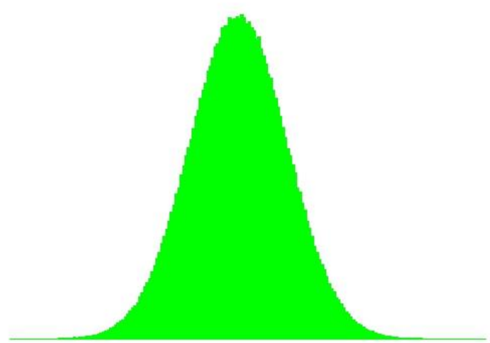
Skewness

The degree of symmetry in the dispersion of values around the mean is known as skewness.

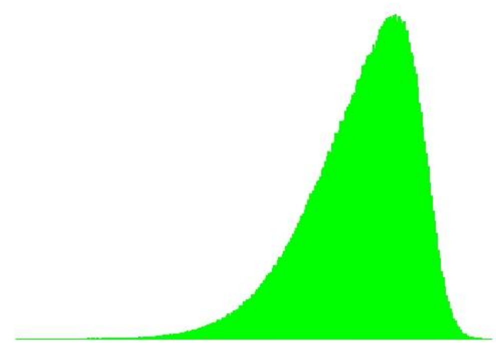
- If observations are equally dispersed around the mean, the distribution is said to be symmetrical.
- If the distribution has a long tail on one side and a “fatter” distribution on the other side, it is said to be skewed in the direction of the long tail.



Skew Right



No Skew



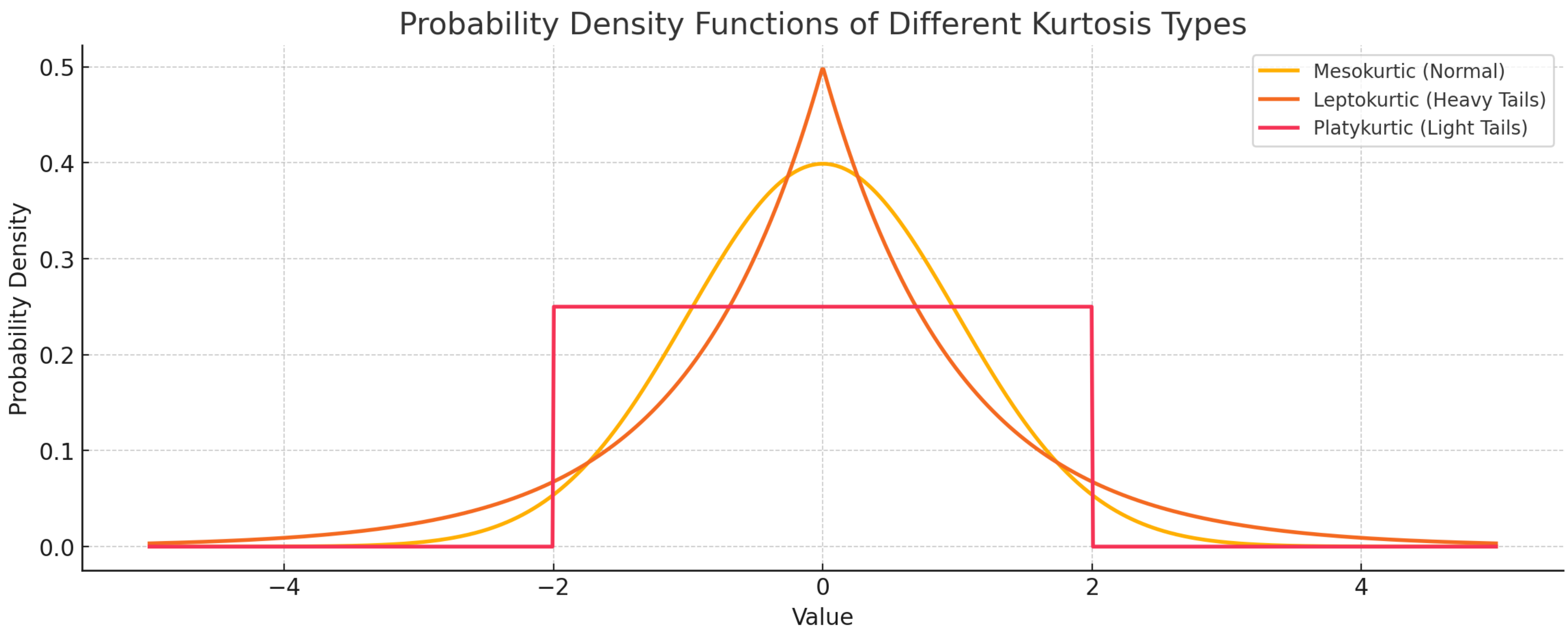
Skew Left

Skewness

- Left-skewed data, the **standard deviation** is influenced significantly by outliers on the left, and it may give the impression of a higher risk or variability even though the majority of the data lies to the right of the mean.
- Right-skewed data, the **standard deviation** is influenced significantly by outliers on the right, and it may give the impression of a higher risk or variability even though the majority of the data lies to the left of the mean.
- **Downside deviation** might be a more accurate measure of risk than standard deviation, since it excludes the extreme positive values.

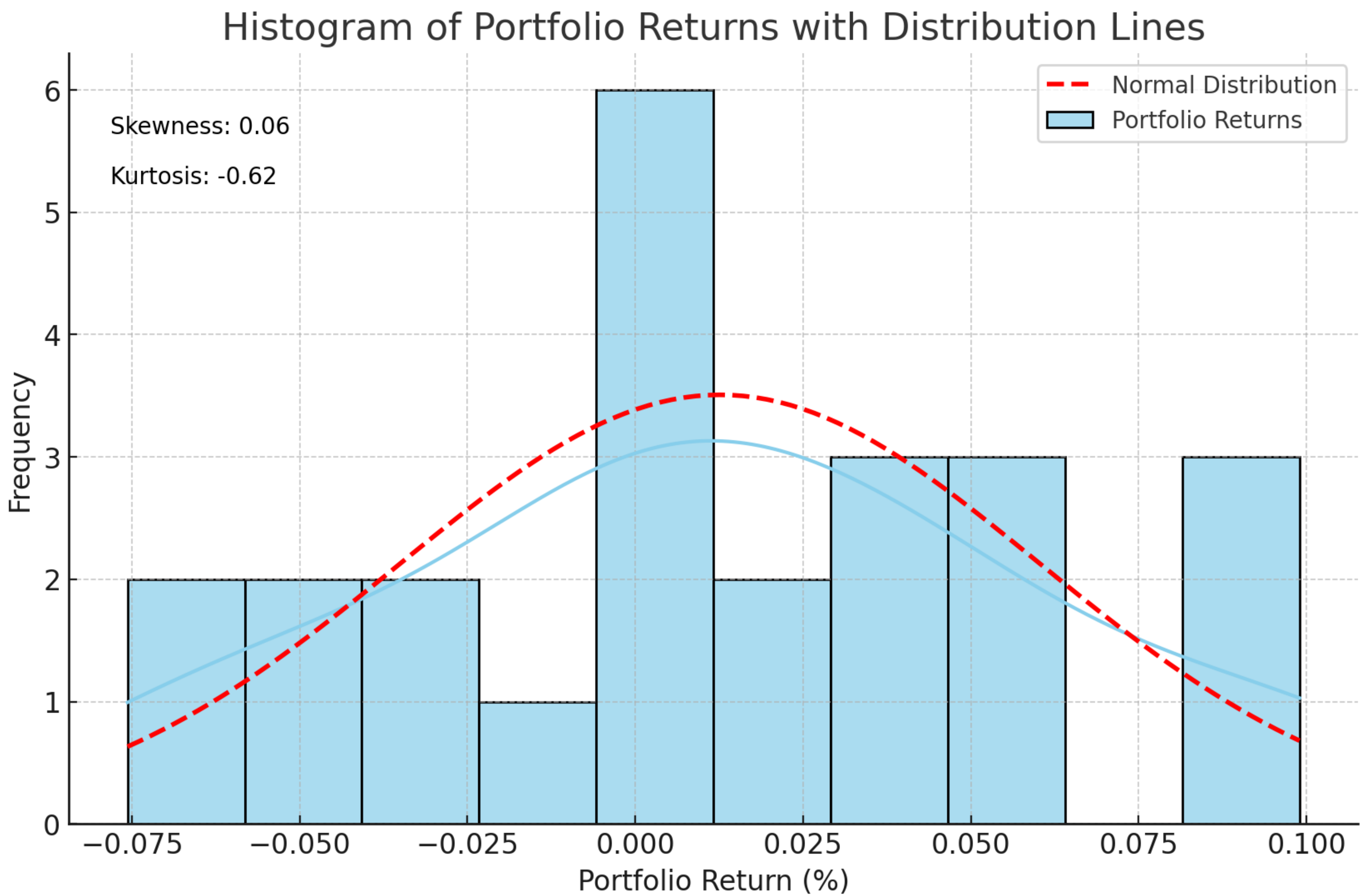
Kurtosis

- Kurtosis measures the relative amount of “peakedness” as compared with the normal distribution, which has a kurtosis of 3.
  – We typically express this measure in terms of excess kurtosis being the observed kurtosis minus 3.
  – Distributions are referred to as being
  1. Leptokurtic (more peaked than the normal; fatter tails)
  2. Platykurtic (less peaked than the normal; thinner tails) or
  3. Mesokurtic (equivalent to the normal).


Probability Density Functions of Different Kurtosis Types
Mesokurtic (Normal)
Leptokurtic (Heavy Tails)
Platykurtic (Light Tails)
Probability Density
0.5
0.4
0.3
0.2
0.1
0.0
−4
−2
0
2
4
Value

Portfolio Returns

- **Skewness**: **0.06**
- The skewness value is **close to zero**, indicating that the distribution of portfolio returns is approximately **symmetrical**. There is no significant skew towards the positive or negative side, suggesting that the returns are evenly distributed around the mean.
- **Kurtosis**: **-0.62** (Excess Kurtosis)
- The negative kurtosis value (less than zero) suggests that the return distribution is **platykurtic**, meaning it has **flatter tails** compared to a normal distribution. This implies fewer extreme values (both positive and negative), suggesting a lower likelihood of extreme returns (less risk of outliers).


Histogram of Portfolio Returns with Distribution Lines
Skewness: 0.06
Kurtosis: -0.62
Normal Distribution
Portfolio Returns
Frequency
0
1
2
3
4
5
6
−0.075
−0.050
−0.025
0.000
0.025
0.050
0.075
0.100
Portfolio Return (%)

Summary

- The underlying foundation of statistically based quantitative analysis lies with the concepts of a sample versus a population.
  - We use sample statistics to describe the sample and to infer information about its associated population.
  - Descriptive statistics for samples and populations include measures of centrality, location, and dispersion, such as mean, range, and variance, respectively.
  - We can combine traditional measures of return (such as mean) and risk (such as standard deviation) to measure the combined effects of risk and return using the coefficient of variation and the Sharpe Ratio.
- The normal distribution is of central importance in investments, and as a result, we often compare statistical properties, such as skewness and kurtosis, with those of the normal distribution.