1. cvičení

1. Polynomy a interpolace

polynom: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $a_0, \dots, a_n \in \mathbb{R}$ koeficienty

n – stupeň polynomu ($a_n \neq 0$)

počet kořenů v $\mathbb{C}$ … n

př: $2x^3 + 4x - 2$ $\qquad 2x - 4$

Interpolace – najít polynom procházející danými body

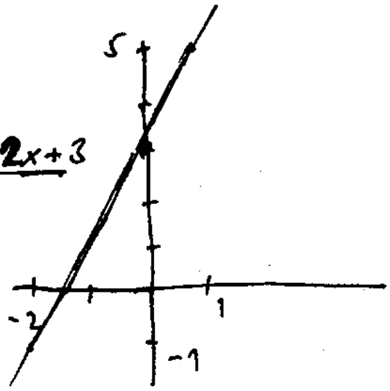
př1 Najděte polynom procházející body:

a) $[-2,-1]$, $[1,5]$

$f(x) = ax + b$

$$\begin{aligned} -1 &= a\cdot(-2) + b \\ 5 &= a\cdot 1 + b \end{aligned} \quad \Rightarrow \quad 6 = 3a \quad b = 3, \quad a = 2$$

$f(x) = 2x + 3$



b) $[-2,4]$, $[0,-2]$, $[1,-2]$

$f(x) = ax^2 + bx + c$

$$\begin{aligned} 4 &= a\cdot 4 + b\cdot(-2) + c \\ -2 &= a\cdot 0 + b\cdot 0 + c \\ -2 &= a\cdot 1 + b\cdot 1 + c \end{aligned}$$

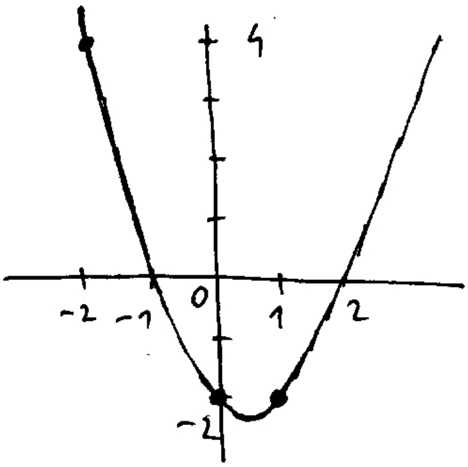
$$\left(\begin{array}{ccc|c} 4 & -2 & 1 & 4 \\ 0 & 0 & 1 & -2 \\ 1 & 1 & 1 & -2 \end{array}\right)\cdot(-4) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 12 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$c = -2$

$b = -1$

$a = 1$

$f(x) = x^2 - x - 2$



c) $[-1,1],\ [0,1],\ [1,5],\ [2,31]$

$f(x) = ax^3 + bx^2 + cx + d$

$$\begin{aligned}
1 &= a\cdot(-1) + b\cdot 1 + c\cdot(-1) + d\\
1 &= a\cdot 0 + b\cdot 0 + c\cdot 0 + d\\
5 &= a\cdot 1 + b\cdot 1 + c\cdot 1 + d\\
31 &= a\cdot 8 + b\cdot 4 + c\cdot 2 + d
\end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5\\ -1 & 1 & -1 & 1 & 1\\ 8 & 4 & 2 & 1 & 31\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)\cdot(-8) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5\\ 0 & 2 & 0 & 2 & 6\\ 0 & +4 & +6 & +7 & +9\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)\cdot(-2)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5\\ 0 & 2 & 0 & 2 & 6\\ 0 & 0 & 6 & 3 & -3\\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$d = 1$, $c = -1$, $b = 2$, $a = 3$

$f(x) = 3x^3 + 2x^2 - x + 1$

nebo: Lagrangeovým interpolačním polynomem:

$$f(x) = 1\,\frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} + 1\cdot\frac{(x-(-1))(x-1)(x-2)}{(0-(-1))(0-1)(0-2)} + 5\,\frac{(x-(-1))(x-0)(x-2)}{(1-(-1))(1-0)(1-2)} +$$

$$+ 31\,\frac{(x-(-1))(x-0)(x-1)}{(2-(-1))(2-0)(2-1)} = -\frac{1}{6}x(x^2-3x+2) + \frac{1}{2}(x^2-1)(x-2) - \frac{5}{2}x(x^2-x-2) +$$

$$+ \frac{31}{6}x(x^2-1) = \frac{1}{6}(-x^3+3x^2-2x) + \frac{1}{2}(x^3-2x^2-x+2) - \frac{5}{2}(x^3-x^2-2x) + \frac{31}{6}(x^3-x) =$$

$$= 3x^3 + 2x^2 - x + 1$$

d) $[-1,2],\ [1,4],\ [2,8],\ [3,22]$ ... $f(-1) = 2,\ f(1) = 4,\ f(2) = 8,\ f(3) = 22$.

$$f(x) = 2\cdot\frac{(x-1)(x-2)(x-3)}{(-1-1)(-1-2)(-1-3)} + 4\cdot\frac{(x-(-1))(x-2)(x-3)}{(1-(-1))(1-2)(1-3)} + 8\cdot\frac{(x-(-1))(x-1)(x-3)}{(2-(-1))(2-1)(2-3)} + 22\,\frac{(x-(-1))(x-1)(x-2)}{(3-(-1))(3-1)(3-2)} =$$

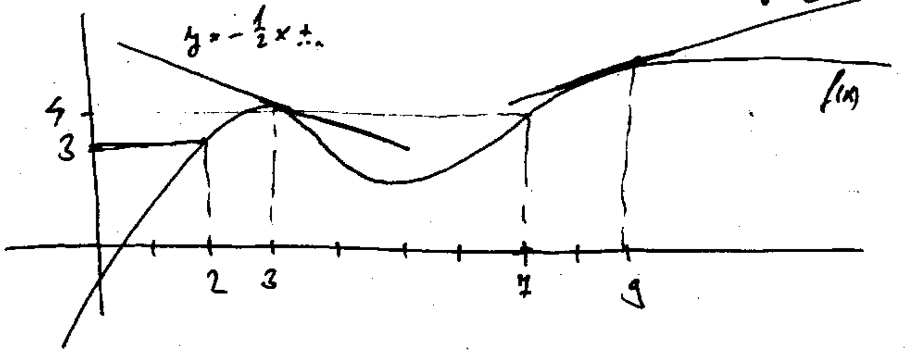
$=$

$=$

$= x^3 - x^2 + 4$

Hermiteův interpolační polynom

dány body + derivace v některých bodech



$f(2) = 3$

$f(7) = 4$

$f'(3) = -\frac{1}{2}$

$f'(9) = \frac{1}{3}x$

Př 2 Najdete polynom $f(x)$ splňující podmínky: a) $f(-1) = 0$, $f(1) = 2$, $f'(\frac{1}{2}) = 3$

$f(x) = ax^2 + bx + c$ $\qquad$ $f'(x) = 2ax + b$

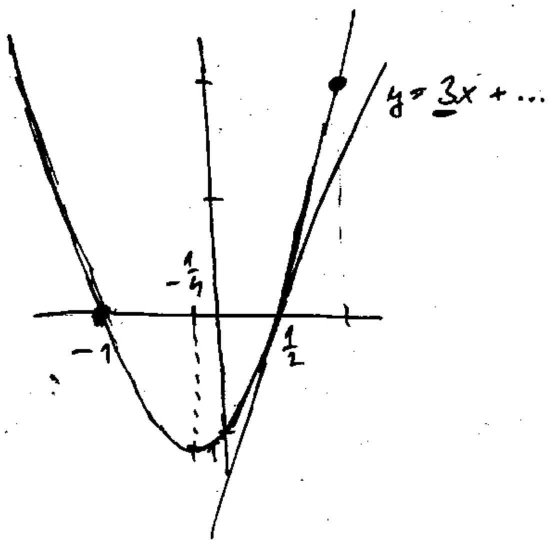
$0 = a\cdot 1 + b\cdot(-1) + c$ $\qquad$ $3 = 2a\cdot\frac{1}{2} + b$

$2 = a\cdot 1 + b\cdot 1 + c$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{array}\right)^{\cdot(-1)} \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & -1 & 3 \end{array}\right)$$

$b = 1$, $c = -1$, $a = 2$

$\underline{f(x) = 2x^2 + x - 1}$



b) $f(1) = -1$, $f(0) = -2$, $f'(2) = 12$, $f'(1) = 3$

$f(x) = ax^3 + bx^2 + cx + d$ $\qquad$ $f'(x) = 3ax^2 + 2bx + c$

$-1 = a\cdot 1 + b\cdot 1 + c\cdot 1 + d$ $\qquad$ $12 = 3a\cdot 4 + 2b\cdot 2 + c$

$-2 = a\cdot 0 + b\cdot 0 + c\cdot 0 + d$ $\qquad$ $3 = 3a\cdot 1 + 2b\cdot 1 + c$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 3 & 2 & 1 & 0 & 3 \\ 12 & 4 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)^{\cdot(-3)\ \cdot(-12)} \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & -1 & -2 & -3 & 6 \\ 0 & -8 & -11 & -12 & 24 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)^{\cdot(-8)} \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 3 & -6 \\ 0 & 0 & 5 & 12 & -24 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$d = -2$ $\qquad$ $a = 1$

$c = 0$

$b = 0$

$\underline{f(x) = x^3 - 2}$

rozklad ryze lomenné racionální fce na parciální zlomky

$\frac{P(x)}{Q(x)}$ st $P <$ st Q ... ryze lomenná rac. fce

parc. zlomky – zlomky tvaru : $\frac{A}{(Bx+C)^n}$, $\frac{Ax+B}{(Cx^2+Dx+E)^n}$ ← nerozložitelný

(př 3) Rozložte na parc. zlomky :

a) $\frac{x-3}{(x-1)(x-2)}$

$$\frac{x-3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \quad / \cdot (x-1)(x-2)$$

$$x-3 = A(x-2) + B(x-1)$$

$$x-3 = Ax - 2A + Bx - B$$

$x: \quad 1 = A + B$

$k: \quad -3 = -2A - B$

$-2 = -A$

$A = 2 \, , \; B = -1$

$$\frac{x-3}{(x-1)(x-2)} = \frac{2}{x-1} - \frac{1}{x-2}$$

b) $\frac{5x+11}{(x+2)(x+3)^2}$

$$\frac{5x+11}{(x+2)(x+3)^2} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{(x+3)^2} \quad / \cdot (x+2)(x+3)^2$$

$$= A(x+3)^2 + B(x+2)(x+3) + C(x+2)$$

$$= A(x^2+6x+9) + B(x^2+5x+6) + C(x+2)$$

$$= Ax^2 + 6Ax + 9A + Bx^2 + 5Bx + 6B + Cx + 2C$$

$x^2: \quad 0 = A + B \quad \to A = -B$

$x: \quad 5 = 6A + 5B + C$

$k: \quad 11 = 9A + 6B + 2C$

$5 = -B + C \quad / \cdot (-3)$

$11 = -3B + 2C$

$-4 = -C$

$C = 4$

$B = -1$

$A = 1$

$$\frac{5x+11}{(x+2)(x+3)^2} = \frac{1}{x+2} - \frac{1}{x+3} + \frac{4}{(x+3)^2}$$

c) $\dfrac{5x^2-2x+3}{x^3-1}$ $\qquad D = 1^2-4\cdot1\cdot1 = -3 < 0 \Rightarrow x^2+x+1$ je nerozložitelný

$$\frac{5x^2-2x+3}{x^3-1} = \frac{5x^2-2x+3}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \quad \Big/ \cdot (x-1)(x^2+x+1)$$

$$= A(x^2+x+1) + (Bx+C)(x-1)$$

$$= Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$$+\left|\begin{array}{l} x^2:\ 5 = A+B \\ x:\ -2 = A-B+C \\ k:\ 3 = A-C \end{array}\right.$$

$$6 = 3A \qquad B = 3$$

$$A = 2 \qquad C = -1$$

$$\frac{5x^2-2x+3}{x^3-1} = \frac{2}{x-1} + \frac{3x-1}{x^2+x+1}.$$

d) $\dfrac{-3x^2+10x-6}{x^4-3x^3+2x^2}$

$$\frac{-3x^2+10x-6}{x^4-3x^3+2x^2} = \frac{-3x^2+10x-6}{x^2(x^2-3x+2)} = \frac{-3x^2+10x-6}{x^2(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x-2} \quad \Big/ \cdot x^2(x-1)(x-2)$$

$$= Ax\overbrace{(x-1)(x-2)}^{x^2-3x+2} + B\overbrace{(x-1)(x-2)}^{x^2-3x+2} + Cx^2(x-2) + Dx^2(x-1)$$

$$= A(x^3-3x^2+2x) + B(x^2-3x+2) + C(x^3-2x^2) + D(x^3-x^2)$$

$$\left.\begin{array}{l} x^3:\ 0 = A+C+D \\ x^2:\ -3 = -3A+B-2C-D \\ x:\ 10 = 2A-3B \\ k:\ -6 = 2B \end{array}\right\} \Rightarrow$$

$$\Rightarrow B = -3, \quad A = \frac{1}{2}$$

$$0 = \frac{1}{2} + C + D$$

$$-3 = -\frac{3}{2} - 3 - 2C - D$$

$$-\frac{1}{2} = C + D$$

$$\frac{3}{2} = -2C - D$$

$$1 = -C \qquad D = \frac{1}{2}$$

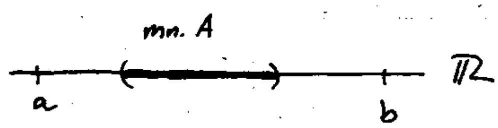
$$C = -1$$

$$\frac{3x^2+10x-6}{x^4-3x^3+2x^2} = \frac{1}{2x} - \frac{3}{x^2} - \frac{1}{x-1} + \frac{1}{2(x-2)}$$

2. cvičení

2. Diferenciální počet

- supremum a infimum



dolní závora a $\quad a \leq x \quad \forall x \in A$

horní závora b $\quad b \geq x \quad \forall x \in A$

supremum = nejmenší horní závora

infimum = největší dolní závora

př 1 Určete supremum a infimum množin.

a) $A = (-2, 3]$

$\inf A = -2$

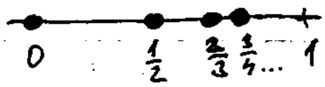
$\sup A = 3$

b) $B = [1, 6) \cup \{7\}$

$\inf B = 1$

$\sup B = 7$

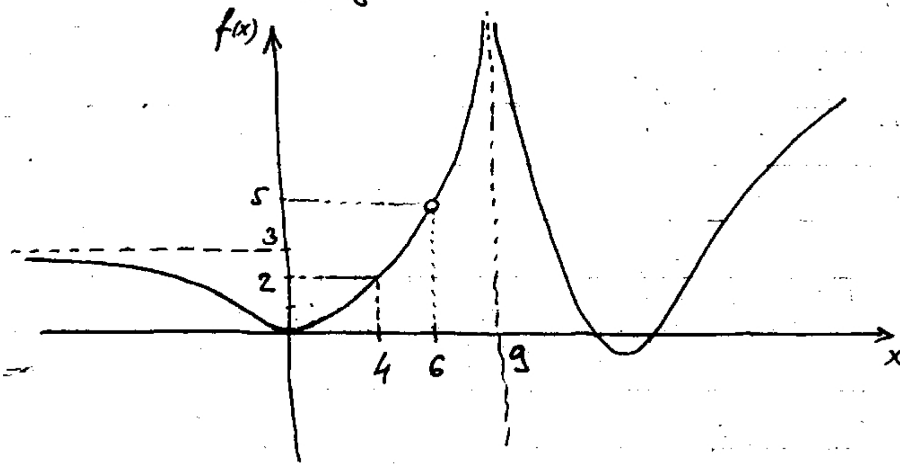
c) $C = \{\frac{n-1}{n}, n \in \mathbb{N}\}$ $\quad 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$



$\inf C = 0$

$\sup C = 1$

- limita (aneb „když se něco někam blíží")



$\lim_{x \to -\infty} f(x) = 3$ vl. limita v nevl. bodě

$\lim_{x \to 4} f(x) = 2$, $\lim_{x \to 6} f(x) = 5$ } vl. limita ve vl. bodě

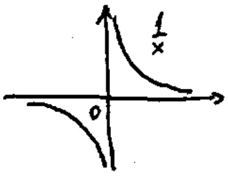
$\lim_{x \to 9} f(x) = \infty$ nevl. limita ve vl. bodě

$\lim_{x \to \infty} f(x) = \infty$ nevl. limita v nevl. bodě pozn.: neurčité výrazy $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, \infty^0, 1^\infty$

Spočtěte limity

pr 2 $\lim\limits_{x\to 1}(-2x+5) = \underline{\underline{3}}$ $\qquad \lim\limits_{x\to\infty}(-2x+5) = \underline{\underline{-\infty}}$

pr 3 $\lim\limits_{x\to 3}\frac{1}{x} = \underline{\underline{\frac{1}{3}}}$ $\qquad \lim\limits_{x\to 0}\frac{1}{x} \ \dots\ \not\exists$



$\lim\limits_{x\to 0^+}\frac{1}{x} = \infty$

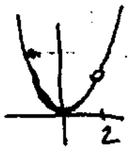
$\lim\limits_{x\to 0^-}\frac{1}{x} = -\infty$

jednostranné limity

pozn. $\lim\ \exists \iff \exists$ obě jednostranné limity a rovnají se

pr 4 $\lim\limits_{x\to 0}\frac{1}{x^2} = \underline{\underline{\infty}}$

pr 5 $\lim\limits_{x\to 2}\frac{(x-2)x^2}{x-2} = \lim\limits_{x\to 2}x^2 = \underline{\underline{4}}$



pr 6 $\lim\limits_{x\to -\infty}e^x = 0$

pr 7 $\lim\limits_{x\to 1}\frac{x^2-1}{x^2-3x+2} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 1}\frac{(x+1)(x-1)}{(x-2)(x-1)} = \lim\limits_{x\to 1}\frac{x+1}{x-2} = \frac{1+1}{1-2} = \underline{\underline{-2}}$

pr 8 $\lim\limits_{x\to\infty}\frac{x^2-7}{x^2+3x-5}\ \begin{matrix}/:x^2\\ /:x^2\end{matrix} = \left\|\frac{\infty}{\infty}\right\| = \lim\limits_{x\to\infty}\frac{1-\frac{7}{x^2}}{1+\frac{3}{x}-\frac{5}{x^2}} = \frac{1-0}{1+0-0} = \underline{\underline{1}}$

pr 9 $\lim\limits_{x\to\infty}\frac{-8x}{x^3+5}\ \begin{matrix}/:x^3\\ /:x^3\end{matrix} = \left\|\frac{\infty}{\infty}\right\| = \lim\limits_{x\to\infty}\frac{-\frac{8}{x^2}}{1+\frac{5}{x^3}} = \frac{0}{1+0} = \underline{\underline{0}}$

pr 10 $\lim\limits_{x\to -\infty}\frac{x^2-2x+1}{5x}\ \begin{matrix}/:x\\ /:x\end{matrix} = \left\|\frac{\infty}{-\infty}\right\| = \lim\limits_{x\to -\infty}\frac{x-2+\frac{1}{x}}{5} = \left\|\frac{-\infty}{5}\right\| = \underline{\underline{-\infty}}$

pr 11 $\lim\limits_{x\to\infty}\frac{\sqrt{x^2+9}\ /:x}{2x+3\ /:x} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to\infty}\frac{\sqrt{1+\frac{9}{x^2}}}{2+\frac{3}{x}} = \frac{\sqrt{1}}{2} = \underline{\underline{\frac{1}{2}}}$

pr 12 $\lim\limits_{x\to -2}\frac{\sqrt{6+x}-2}{x+2} = \left\|\frac{0}{0}\right\| = \lim\limits_{x\to -2}\frac{\sqrt{6+x}-2}{x+2}\cdot\frac{\sqrt{6+x}+2}{\sqrt{6+x}+2} = \lim\limits_{x\to -2}\frac{\overbrace{6+x-4}^{x+2}}{\cancel{(x+2)}(\sqrt{6+x}+2)} =$

$= \lim\limits_{x\to -2}\frac{1}{\sqrt{6+x}+2} = \frac{1}{\sqrt{6-2}+2} = \underline{\underline{\frac{1}{4}}}$

pr 13 $\lim\limits_{x\to\infty}\left(\sqrt{x^2+4x}-x\right) = \|\infty-\infty\| = \lim\limits_{x\to\infty}\frac{\sqrt{x^2+4x}-x}{1}\cdot\frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x} = \lim\limits_{x\to\infty}\frac{x^2+4x-x^2}{\sqrt{x^2+4x}+x} =$

$= \lim\limits_{x\to\infty}\frac{4x}{\sqrt{x^2+4x}+x}\ \begin{matrix}/:x\\ /:x\end{matrix} = \left\|\frac{\infty}{\infty}\right\| = \lim\limits_{x\to\infty}\frac{4}{\sqrt{1+\frac{4}{x}}+1} = \frac{4}{\sqrt{1}+1} = \underline{\underline{2}}$

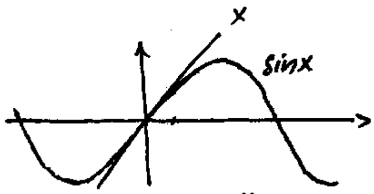
(př 14) $\lim\limits_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \left\| \frac{0}{0} \right\| = \lim\limits_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \cdot \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \lim\limits_{x\to 0} \frac{1+x-(1-x)}{x\cdot(\sqrt{1+x}+\sqrt{1-x})} =$

$= \lim\limits_{x\to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})} = \lim\limits_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{1+1} = \underline{\underline{1}}$

(př 15) $\lim\limits_{x\to \infty} \frac{\sin x}{x} = \left\| \frac{[-1,1]}{\infty} \right\| = \underline{\underline{0}}$

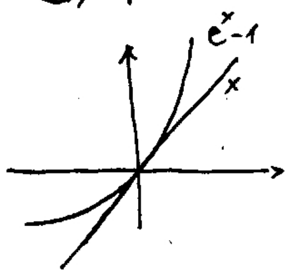
(př 16) $\lim\limits_{x\to \infty} \frac{1}{x}\cdot \sin x = \| 0 \cdot [-1,1] \| = \underline{\underline{0}}$

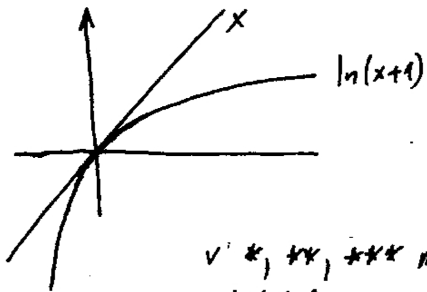
víme: * $\boxed{\lim\limits_{x\to 0} \frac{\sin x}{x} = 1}$



*** $\boxed{\lim\limits_{x\to 0} \frac{\ln(x+1)}{x} = 1}$

** $\boxed{\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1}$





v *, **, *** může být jakákoli fce $g(x)$ taková, že $g(x) \to 0$ pro $x \to 0$

(př 17) $\lim\limits_{x\to 0} \frac{tg\, x}{x} = \lim\limits_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot \frac{1}{1} = \underline{\underline{1}}$

(př 18) $\lim\limits_{x\to 0} \frac{e^x-1}{\sin x} = \lim\limits_{x\to 0} \frac{\frac{e^x-1}{x}}{\frac{\sin x}{x}} = \frac{1}{1} = \underline{\underline{1}}$

(př 19) $\lim\limits_{x\to 0} \frac{\sin x}{x^3} = \lim\limits_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{x^2} = \| 1 \cdot \infty \| = \underline{\underline{\infty}}$

(př 20) $\lim\limits_{x\to 0} \frac{\sin 2x}{x} = 2 \cdot \lim\limits_{x\to 0} \frac{\sin 2x}{2x} = 2 \cdot 1 = \underline{\underline{2}}$

(př 21) $\lim\limits_{x\to 0} \frac{\sin 5x}{\sin 7x} = \lim\limits_{x\to 0} \frac{\frac{\sin 5x}{5x}}{\frac{\sin 7x}{7x}} \cdot \frac{5}{7} = \frac{1}{1} \cdot \frac{5}{7} = \underline{\underline{\frac{5}{7}}}$

(př 22) $\lim\limits_{x\to 0} \frac{e^x - e^{-x}}{\sin 2x} = \lim\limits_{x\to 0} \frac{e^x - 1 - e^{-x} + 1}{\sin 2x} = \lim\limits_{x\to 0} \frac{e^x-1}{\sin 2x} - \lim\limits_{x\to 0} \frac{e^{-x}-1}{\sin 2x} =$

$= \lim\limits_{x\to 0} \frac{\frac{e^x-1}{x}}{\frac{\sin 2x}{2x}} \cdot \frac{1}{2} + \lim \frac{\frac{e^{-x}-1}{-x}}{\frac{\sin 2x}{2x}} \cdot \frac{1}{2} = \frac{1}{1} \cdot \frac{1}{2} + \frac{1}{1} \cdot \frac{1}{2} = \underline{\underline{1}}$

(př 23) $\lim\limits_{x\to 0} \frac{e^{3x}-1}{x} = \lim\limits_{x\to 0} \frac{e^{3x}-1}{3x} \cdot 3 = 1 \cdot 3 = \underline{\underline{3}}$

př 24
$$\lim_{x\to 0}\frac{\ln(x^2+1)}{8x^2} = \lim_{x\to 0}\frac{\ln(x^2+1)}{x^2}\cdot\frac{1}{8} = 1\cdot\frac{1}{8} = \underline{\underline{\frac{1}{8}}}$$

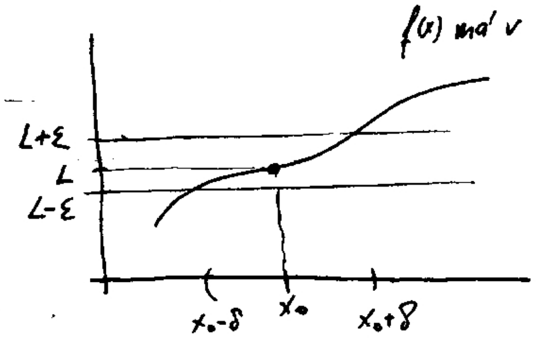
př 25
$$\lim_{x\to 0}\frac{\ln(7x+1)+e^{2x}-1}{x} = \lim_{x\to 0}\frac{\ln(7x+1)}{7x}\cdot 7 + \lim_{x\to 0}\frac{e^{2x}-1}{2x}\cdot 2 = 1\cdot 7 + 1\cdot 2 = \underline{\underline{9}}$$

př 26
$$\lim_{x\to\infty}\frac{\pi x+\sin x}{2x+\cos x}\,\frac{:x}{:x} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to\infty}\frac{\pi+\frac{\sin x}{x}}{2+\frac{\cos x}{x}} = \underline{\underline{\frac{\pi}{2}}}$$

př 27
$$\lim_{x\to 2}\frac{x^2}{x^2-3x+2} = \left\|\frac{4}{0}\right\| = \lim_{x\to 2}\frac{x^2}{(x-2)(x-1)} = \lim_{x\to 2}\frac{1}{x-2}\cdot\overbrace{\lim_{x\to 2}\frac{x^2}{x-1}}^{4}$$

$$\left.\begin{aligned}\lim_{x\to 2^+}\frac{1}{x-2} &= \left\|\frac{1}{+0}\right\| = \infty\\ \lim_{x\to 2^-}\frac{1}{x-2} &= \left\|\frac{1}{-0}\right\| = -\infty\end{aligned}\right\} \Rightarrow \text{limita } \nexists$$

<u>Definice limity</u>

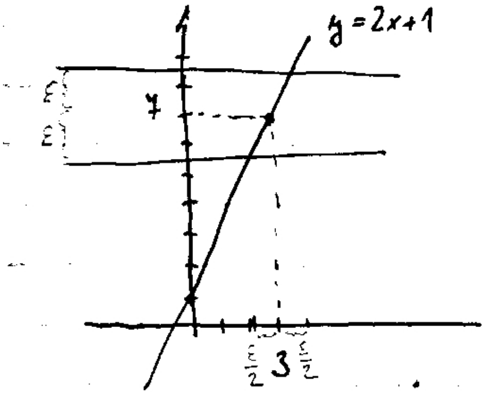


$f(x)$ má v x_0 limitu L, jestliže:
ke $\forall \varepsilon > 0$ $\exists \delta > 0$ takové, že $\forall x \in (x_0-\delta, x_0-\delta)$ platí:
čili $|x-x_0| < \delta$

$$|f(x) - L| < \varepsilon$$

př 28 Ukažte z definice, že $\lim_{x\to 3}(2x+1) = 7$.

máme libovolné $\varepsilon > 0$ a chceme najít $\delta > 0$ takové, aby $|f(x) - 7| < \varepsilon$



$$|2x+1-7| < \varepsilon$$
$$|2x-6| < \varepsilon$$
$$|x-3| < \frac{\varepsilon}{2}$$

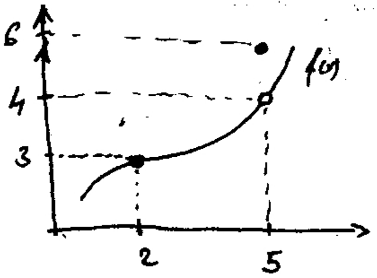
stačí tedy vzít $\delta = \frac{\varepsilon}{2}$

3. cvičení

• spojitost fce

$f(x)$ je v x_0 spojitá, jestliže: • je v x_0 definovaná

• limita v x_0 = fční hodnota v x_0 $\lim\limits_{x\to x_0} f(x) = f(x_0)$



bod $x_0 = 2$

• $f(x)$ je v $x_0 = 2$ def ✓

• $\lim\limits_{x\to 2} f(x) = 3$ = ✓

$f(2) = 3$

je v $x_0 = 2$ spojitá

bod $x_0 = 5$

• $f(x)$ je v $x_0 = 5$ def ✓

• $\lim\limits_{x\to 5} f(x) = 4$ ≠ ×

$f(5) = 6$

není v $x_0 = 5$ spojitá

(př 1) Určete, zda je fce $f(x)$ spojitá v bodech $x_0 = 2$ a $x_0 = 5$.

druhy nespojitosti

1, odstranitelná ∃ vl. limita $\lim\limits_{x\to x_0} f(x)$, ale $\neq f(x_0)$

2, nespojitost 1. druhu ∄ —"—, ale ∃ jednostranné 1. limity, ale ≠

3, 2. druhu aspoň 1 jednostranná limita je nevl. nebo ∄

(př 2) Najděte body nespojitosti a určete jejich druh.

a) $f(x) = \dfrac{\sin x}{|x|}$ ⇒ $x_0 = 0$ $\lim\limits_{x\to 0^+} \dfrac{\sin x}{x} = 1$, $\lim\limits_{x\to 0^-} \dfrac{\sin x}{-x} = -1$ ... 1. druh

b) $f(x) = \dfrac{x^3-1}{x^2-1} = \dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$

• $x = 1$ $\lim\limits_{x\to 1} \dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim\limits_{x\to 1} \dfrac{x^2+x+1}{x+1} = \dfrac{1^2+1+1}{1+1} = \dfrac{3}{2}$ ale $f(1)$ není def ...odstranitelná

• $x = -1$ $\lim\limits_{x\to -1^-} \dfrac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)} = \left\| \dfrac{1}{-0} \right\| = -\infty$

... 2. druh

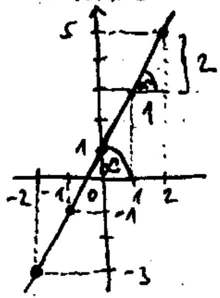
$\lim\limits_{x\to -1^+} \dfrac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)} = \left\| \dfrac{1}{+0} \right\| = +\infty$

• derivace funkce

směrnicový tvar přímky $y = kx + q$ k ... směrnice

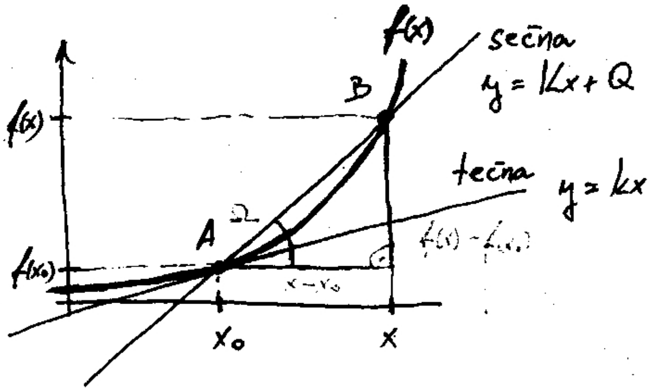
$y = 2x + 1$

x	-2	-1	0	1	2
y	-3	-1	1	3	5



$\mathrm{tg}\,\alpha = \frac{2}{1} = 2 = k$

$$\boxed{k = \mathrm{tg}\,\alpha}$$



směrnice sečny

$$K = \mathrm{tg}\,\Omega = \frac{f(x) - f(x_0)}{x - x_0}$$

směrnice tečny

$$k = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$ ... derivace $f(x)$ v x_0

$f'(x_0)$, $y'(x_0)$

pravidla pro počítání

$(c)' = 0$

$(x^n)' = n \cdot x^{n-1}$

$(e^x)' = e^x$

$(a^x)' = a^x \cdot \ln a$

$(\ln x)' = \frac{1}{x}$

$(\log_a x)' = \frac{1}{x \cdot \ln a}$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\mathrm{tg}\,x)' = \frac{1}{\cos^2 x}$

$(\mathrm{cotg}\,x)' = -\frac{1}{\sin^2 x}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\mathrm{arctg}\,x)' = \frac{1}{1+x^2}$

$(\mathrm{arccotg}\,x)' = -\frac{1}{1+x^2}$

$[c \cdot f(x)]' = c \cdot f'(x)$

$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

$(u \cdot v)' = u' \cdot v + u \cdot v'$

$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$

$f(g(x)) = f'(g(x)) \cdot g'(x)$

(př 3) Zderivujte

• $y = x^6 + 3x^5 + 2x^2 - 4x + 15$

$\underline{\underline{y' = 6x^5 + 15x^4 + 4x - 4}}$

• $y = \frac{1}{x^2} = x^{-2}$

$\underline{\underline{y = -2x^{-3} = -\frac{2}{x^3}}}$

• $y = \frac{1}{x} - \frac{5}{x^3} = x^{-1} - 5x^{-3}$

$y' = -x^{-2} + 15x^{-4} = \underline{\underline{-\frac{1}{x^2} + \frac{15}{x^4}}}$

$$\bullet\ y = 2x \cdot \sin x$$
$$\underline{\underline{y = 2\sin x + 2x\cos x}}$$

$$\bullet\ y = x^5 \cdot \ln x$$
$$y' = 5x^4 \cdot \ln x + x^5 \cdot \frac{1}{x} = 5x^4 \cdot \ln x + x^4 = \underline{\underline{x^4(5\ln x + 1)}}$$

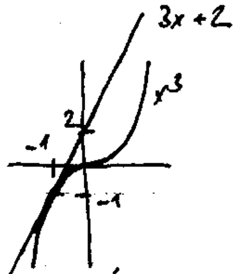
$$\bullet\ y = \frac{3x}{x^2+1}$$
$$y' = \frac{3(x^2+1) - 3x \cdot 2x}{(x^2+1)^2} = \frac{3x^2+3-6x^2}{(x^2+1)^2} = \underline{\underline{\frac{3-3x^2}{(x^2+1)^2}}}$$

$$\bullet\ y = \frac{\sin x + \cos x}{\cos x - \sin x}$$
$$y' = \frac{(\cos x - \sin x)(\cos x - \sin x) - (\sin x + \cos x)\cdot(-\sin x - \cos x)}{(\cos x - \sin x)^2} = \frac{\cos^2 x - 2\sin x\cos x + \sin^2 x + \sin^2 x + 2\sin x\cos x + \cos^2 x}{(\cos x - \sin x)^2} =$$
$$= \frac{2}{\cos^2 x - 2\cos x\sin x + \sin^2 x} = \underline{\underline{\frac{2}{1-\sin 2x}}}$$

$$\bullet\ y = \sin^2 x + \sin 3x$$
$$y' = \underline{\underline{2\sin x\cos x + \cos 3x \cdot 3}}$$

$$\bullet\ y = \sqrt{\ln(x^2+x+1)} = \left[\ln(x^2+x+1)\right]^{1/2}$$
$$y = \frac{1}{2}\left[\ln(x^2+x+1)\right]^{-1/2} \cdot \frac{1}{x^2+x+1} \cdot (2x+1) = \underline{\underline{\frac{2x+1}{2\sqrt{\ln(x^2+x+1)} \cdot (x^2+x+1)}}}$$

$$\bullet\ y = \ln\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} = \ln\frac{(x+1)^{1/2}-1}{(x+1)^{1/2}+1}$$
$$y' = \frac{1}{\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}} \cdot \frac{\frac{1}{2}(x+1)^{-1/2}\cdot\left[(x+1)^{1/2}+1\right] - \left[(x+1)^{1/2}-1\right]\cdot\frac{1}{2}(x+1)^{-1/2}}{\left[(x+1)^{1/2}+1\right]^2} =$$
$$= \frac{\cancel{\sqrt{x+1}+1}}{\sqrt{x+1}-1} \cdot \frac{\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{\sqrt{x+1}} - \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{\sqrt{x+1}}}{(\sqrt{x+1}+1)^{\cancel{2}}} = \frac{\frac{1}{\sqrt{x+1}}}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)} = \frac{\frac{1}{\sqrt{x+1}}}{x+1-1} =$$
$$= \frac{\frac{1}{\sqrt{x+1}}}{x} = \underline{\underline{\frac{1}{x\sqrt{x+1}}}}$$



př 4 Napište rovnici tečny fce v daném bodě

- $f(x) = x^3 \qquad A = [-1; ?]$, -1

$f'(x) = 3x^2$
$f'(-1) = 3\cdot(-1)^2 = 3$

$t: y = kx + q$
$y = 3x + q$

$A \in t: -1 = 3\cdot(-1) + q$
$q = 2$

$t: y = 3x + 2$

- $f(x) = \frac{1}{2}x^2 + x \qquad A = [4; ?]$, 12

$f'(x) = x + 1$
$f'(4) = 5$

$t: y = kx + q$
$y = 5x + q$
$A \in t: 12 = 5\cdot 4 + q$
$q = -8$

$t: y = 5x - 8$

- $f(x) = e^{2x} + \operatorname{tg} x^2 \qquad A = [0, ?]$, 1

$f'(x) = e^{2x}\cdot 2 + \frac{1}{\cos^2 x^2}\cdot 2x$
$f'(0) = 2$

$t: y = kx + q$
$y = 2x + q$
$A \in t: 1 = 2\cdot 0 + q$
$q = 1$

$t: y = 2x + 1$

př 5 Poloha hmotného bodu pohybujícího se po přímce je dána vztahem $x(t) = t^3 - \ln t$. Jaká je jeho rychlost a zrychlení v čase $t = 4\,s$?

$x(t) = t^3 - \ln t$
$v(t) = x'(t) = 3t^2 - \frac{1}{t}$
$v(4) = 3\cdot 16 - \frac{1}{4} = 47{,}75\ m/s$

$a(t) = v'(t) = 6t + \frac{1}{t^2}$
$a(4) = 6\cdot 4 + \frac{1}{16} = 24{,}0625\ m/s^2$

př 6 Zderivujte implicitně danou fci:

- $y^2 + \ln y + x^2 = 0$

$2yy' + \frac{1}{y}y' + 2x = 0$
$y'(2y + \frac{1}{y}) = -2x$
$y' = -\frac{2x}{2y + \frac{1}{y}}$
$y' = -\frac{2xy}{2y^2 + 1}$

- $4y = x^3 + \cos y$

$4y' = 3x^2 - \sin y \cdot y'$
$y'(4 + \sin y) = 3x^2$
$y' = \frac{3x^2}{4 + \sin y}$

- **L'Hospitalovo pravidlo**

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)} \quad \text{pro limity typu } \frac{0}{0},\ \frac{\infty}{\infty}$$

př 1 Vypočtěte limity

a) $\lim_{x\to 0}\frac{\sin x}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\cos x}{1} = \frac{1}{1} = \underline{1}$

b) $\lim_{x\to 0}\frac{e^x-1}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{e^x}{1} = \frac{1}{1} = \underline{1}$

c) $\lim_{x\to 0}\frac{\ln(x+1)}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\frac{1}{x+1}}{1} = \underline{1}$

d) $\lim_{x\to\infty}\frac{x^3+2x+1}{2x^3-4} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to\infty}\frac{3x^2+2x}{6x^2} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to\infty}\frac{6x}{12x} = \underline{\frac{1}{2}}$

e) $\lim_{x\to -2}\frac{\sqrt{x+3}-1}{x+2} = \left\|\frac{0}{0}\right\| = \lim_{x\to -2}\frac{\frac{1}{2}(x+3)^{-\frac{1}{2}}}{1} = \underline{\frac{1}{2}}$

limita typu $0\cdot\infty$ $\longrightarrow \frac{0}{0}, \frac{\infty}{\infty}$

a) $\lim_{x\to 0^+}(x\cdot\ln x) = \|0\cdot\infty\| = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = \underline{0}$

b) $\lim_{x\to\infty} x\cdot\sin\frac{\pi}{x} = \|\infty\cdot 0\| = \lim_{x\to\infty}\frac{\sin\frac{\pi}{x}}{\frac{1}{x}} = \left\|\frac{0}{0}\right\| = \lim_{x\to\infty}\frac{\cos\frac{\pi}{x}\cdot\left(-\frac{\pi}{\cancel{x^2}}\right)}{-\frac{1}{\cancel{x^2}}} = \lim_{x\to\infty}\pi\cdot\cos\frac{\pi}{x} = \underline{\pi}$

limity typu $0^0, \infty^0, 1^\infty$ $\nearrow 0\cdot\infty$

$$\lim_{x\to 0} f(x)^{g(x)} = e^{\ln \lim_{x\to x_0} f(x)^{g(x)}} = e^{\lim_{x\to x_0}\ln f(x)^{g(x)}} = e^{\lim_{x\to x_0}(g(x)\cdot\ln f(x))}$$

a) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \|1^\infty\| = e^{\ln\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x} = e^{\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x} = e^{\lim_{x\to\infty} x\cdot\ln\left(1+\frac{1}{x}\right)} = e^1 = \underline{e}$

$\lim_{x\to\infty} x\cdot\ln\left(1+\frac{1}{x}\right) = \|\infty\cdot 0\| = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \left\|\frac{0}{0}\right\| = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = \frac{1}{1} = 1$

b) $\lim_{x\to 0}(1-\cos x)^{\sin x} = \| 0^0 \| = e^{\lim_{x\to 0}\sin x\cdot\ln(1-\cos x)} = e^0 = \underline{\underline{1}}$

$$\lim_{x\to 0}\sin x\cdot\ln(1-\cos x) = \|0\cdot\infty\| = \lim_{x\to 0}\frac{\ln(1-\cos x)}{\frac{1}{\sin x}} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to 0}\frac{\frac{1}{1-\cos x}\cdot\sin x}{-\frac{\cos x}{\sin^2 x}} =$$

$$= \lim_{x\to 0}\frac{-\sin^3 x}{\underbrace{(1-\cos x)\cos x}_{\cos x-\cos^2 x}} = \left\|\frac{0}{0}\right\| = \lim_{x\to 0}\frac{-3\sin^2 x\cdot\cos x}{-\sin x+2\cos x\sin x} = \lim_{x\to 0}\frac{-3\sin x\cos x}{-1+2\cos x} = \frac{0}{-1+2} = \frac{0}{1} = 0$$

c) $\lim_{x\to 0^+}(\operatorname{cotg}x)^{\sin x} = \|\infty^0\| = e^{\lim_{x\to 0^+}\sin x\cdot\ln(\operatorname{cotg}x)} = e^0 = \underline{\underline{1}}$

$$\lim_{x\to 0^+}\sin x\cdot\ln(\operatorname{cotg}x) = \|0\cdot\infty\| = \lim_{x\to 0^+}\frac{\ln(\operatorname{cotg}x)}{\frac{1}{\sin x}} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to 0^+}\frac{\frac{1}{\operatorname{cotg}x}\cdot\left(-\frac{1}{\sin^2 x}\right)}{-\frac{\cos x}{\sin^2 x}} =$$

$$= \lim_{x\to 0^+}\frac{1}{\operatorname{cotg}x\cdot\cos x} = \lim_{x\to 0^+}\frac{1}{\frac{\cos x}{\sin x}\cdot\cos x} = \lim_{x\to 0^+}\frac{\sin x}{\cos^2 x} = \frac{0}{1} = 0$$

* <u>limita typu $\infty-\infty$</u> $\quad \lim_{x\to x_0}(f(x)-g(x)) = \lim_{x\to x_0}\left(\frac{1}{\frac{1}{f}}-\frac{1}{\frac{1}{g}}\right) = \lim_{x\to x_0}\frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{f\cdot g}} = \left\|\frac{0}{0}\right\|$

a) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \|\infty-\infty\| = \lim_{x\to 1}\frac{x-1-\ln x}{\ln x\cdot(x-1)} = \left\|\frac{0}{0}\right\| = \lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{1}{x}\cdot(x-1)+\ln x\cdot 1} =$

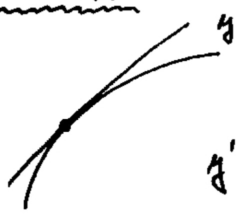
$$= \lim_{x\to 1}\frac{x-1}{x-1+x\cdot\ln x} = \left\|\frac{0}{0}\right\| = \lim_{x\to 1}\frac{1}{1+\ln x+\cancel{x}\frac{1}{\cancel{x}}} = \underline{\underline{\frac{1}{2}}}$$

b) $\lim_{x\to 0}\left(\operatorname{cotg}x-\frac{1}{x}\right) = \|\infty-\infty\| = \lim_{x\to 0}\left(\frac{\cos x}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin x} = \left\|\frac{0}{0}\right\| =$

$$= \lim_{x\to 0}\frac{\cos x-x\sin x-\cos x}{\sin x+x\cos x} = \lim_{x\to 0}-\frac{x\sin x}{\sin x+x\cos x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}-\frac{\sin x+x\cos x}{\cos x+\cos x-x\sin x} = \frac{0}{2} = \underline{\underline{0}}$$

- monotonie fce, lokální extrémy
  ↑ kde roste, klesá ↑ lok. max, lok. min.

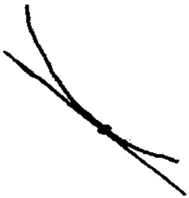
rostoucí fce



$y = \boxed{k}x + q$ kladná

$y' > 0$

klesající fce



$y' < 0$

$y = \boxed{k}x + q$ záporná

$y' = 0$

a)  lok. min

b)  lok. max

c)  ani lok. max ani lok. min

pozor!

 lok. min

$y' \nexists$

 lok. max

$y' \nexists$

rostoucí $y' > 0$

klesající $y' < 0$

lok. extrémy — $y' = 0$ ⊕ mění se znaménka y'

↘ $y' \nexists$

$y''(x_0) < 0$ ... lok. max

$y''(x_0) > 0$ ... lok. min

př 2. Určete, kde funkce klesá a kde roste, určete lok. extrémy, $D(f)$

a) $y = \frac{x}{2} - \operatorname{arctg} x$ $\quad D(f) = \mathbb{R}$

$$y' = \frac{1}{2} - \frac{1}{1+x^2} = \frac{1+x^2-2}{2(1+x^2)} = \frac{x^2-1}{2(1+x^2)} = \frac{(x+1)(x-1)}{\underbrace{2(1+x^2)}_{\text{vždy} > 0}}$$

↑ def. všude

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$x+1$	−	+	+
$x-1$	−	−	+
y'	+	−	+
	↗ max	↘ min	↗

b) $y = x \cdot e^{\frac{1}{x}}$ $\quad D(f) = \mathbb{R} \setminus \{0\}$

$$y' = e^{\frac{1}{x}} + x \cdot e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) = e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right) = e^{\frac{1}{x}} \, \frac{x-1}{x}$$

↑ není def. v 0

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$x-1$	−	−	+
x	−	+	+
y'	+	−	+
	↗ ~~max~~ !	↘ min	↗

- konvexnost, konkávnost fce, inflexní body

konvexní $y'' > 0$ $\cup$

konkávní $y'' < 0$ $\cap$

inflexní bod – $y'' = 0$ ⊕ mění se znaménko y'' 

– $y'' \nexists$

(př 3) Určete, kde je funkce konvexní a konkávní, určete inflexní body, $D(f)$.

a) $y = x^4 - 6x^2 - 2x + 5$ $\quad D(f) = \mathbb{R}$

$y' = 4x^3 - 12x - 2$

$y'' = 12x^2 - 12 = 12(x^2 - 1) = 12(x+1)(x-1)$

def. všude

		-1		1	
$x+1$	$-$		$+$		$+$
$x-1$	$-$		$-$		$+$
y''	$+$		$-$		$+$
	$\cup$	inf. bod	$\cap$	inf. bod	$\cup$

b) $y = \operatorname{arctg} \frac{x}{x+1}$ $\quad D(f) = \mathbb{R} \setminus \{-1\}$

$$y' = \frac{1}{1 + \left(\frac{x}{x+1}\right)^2} \cdot \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2 + x^2} \quad \text{vždy} > 0$$

$$= \frac{1}{2x^2 + 2x + 1}$$

y' není def v -1

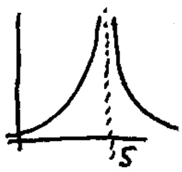
$$y'' = -\frac{4x+2}{(2x^2+2x+1)^2} \quad \leftarrow \text{vždy} > 0$$

		-1		$-\frac{1}{2}$	
$-(4x+2)$	$+$		$+$		$-$
y''	$+$		$+$		$-$
	$\cup$		$\cup$	inf. bod	$\cap$

- asymptoty

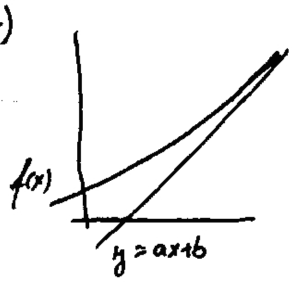
a) bez směrnice – má-li $f(x)$ v x_0 aspoň 1 jednostrannou limitu nevlastní

AB: $x = 5$



b) se směrnicí $y = ax + b$ $\rightarrow$ $a = \lim\limits_{x \to \pm\infty} \frac{f(x)}{x}$ $\quad b = \lim\limits_{x \to \pm\infty} (f(x) - ax)$

AS ↗



př.) Najděte asymptoty fce $y = \frac{x^3}{x^2-x-2} = \frac{x^3}{(x+1)(x-2)}$ $\quad D(f) = \mathbb{R} \setminus \{-1, 2\}$

a) BS $\boxed{x=-1}$ $\lim\limits_{x\to -1^+} \frac{x^3}{(x+1)(x-2)} = \left\| \frac{-1}{0} \right\| = \lim\limits_{x\to -1^+} \frac{x^3}{x-2} \cdot \lim\limits_{x\to -1^+} \frac{1}{x+1} = \left\| \frac{-1}{-3} \cdot \frac{1}{+0} \right\| = +\infty$

$\lim\limits_{x\to -1^-} \frac{x^3}{(x+1)(x-2)} = \left\| \frac{-1}{0} \right\| = \lim\limits_{x\to -1^-} \frac{x^3}{x-2} \cdot \lim\limits_{x\to -1^+} \frac{1}{x+1} = \left\| \frac{-1}{-3} \cdot \frac{1}{-0} \right\| = -\infty$

$\boxed{x=2}$ analogicky

$\Rightarrow$ asymptoty bez směrnice $\underline{\underline{x=-1}}$, $\underline{\underline{x=2}}$

b) SS $a = \lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \frac{x^2}{x^2-x-2} = \left\| \frac{\infty}{\infty} \right\| \overset{L.P.}{=} \lim\limits_{x\to\pm\infty} \frac{2x}{2x-1} = \left\| \frac{\infty}{\infty} \right\| \overset{L.P.}{=} \lim\limits_{x\to\pm\infty} \frac{2}{2} = \underline{\underline{1}}$

$b = \lim\limits_{x\to\infty} (f(x) - ax) = \lim\limits_{x\to\infty} \left(\frac{x^3}{x^2-x-2} - 1\cdot x \right) = \lim\limits_{x\to\infty} \frac{x^3 - x(x^2-x-2)}{x^2-x-2} = \lim\limits_{x\to\infty} \frac{x^2+2x}{x^2-x-2} \overset{L.P.}{=} \dots = \underline{\underline{1}}$

$\Rightarrow$ asymptota se směrnicí $\underline{\underline{y = x+1}}$

Průběh funkce

1. $D(f)$, lichá, sudá
2. Kladná, záporná
3. Rostoucí, klesající, extrémy
4. Konvexnost, konkávnost, inflexní body
5. Asymptoty

Vyšetřete průběh fce

př 5 $y = x^3 - 3x + 2 = (x-1)(x^2+x-2) = (x-1)^2(x+2)$

1) $D(f) = \mathbb{R}$

$f(-x) = (-x)^3 - 3(-x) + 2 = -x^3 + 3x + 2 \Rightarrow$ ani sudá ani lichá

2) Kladná, záporná

		-2		1	
$(x-1)^2$	$+$		$+$		$+$
$x+2$	$-$		$+$		$+$
y	$-$		$+$		$+$

3) Rostoucí, klesající, extrémy

$y' = 3x^2 - 3 = 3(x^2-1) = 3(x+1)(x-1)$

		-1		1	
$x+1$	$-$		$+$		$+$
$x-1$	$-$		$-$		$+$
y'	$+$		$-$		$+$
	$\nearrow$	max	$\searrow$	min	$\nearrow$

4) Konvexní, konkávní, inflexní body

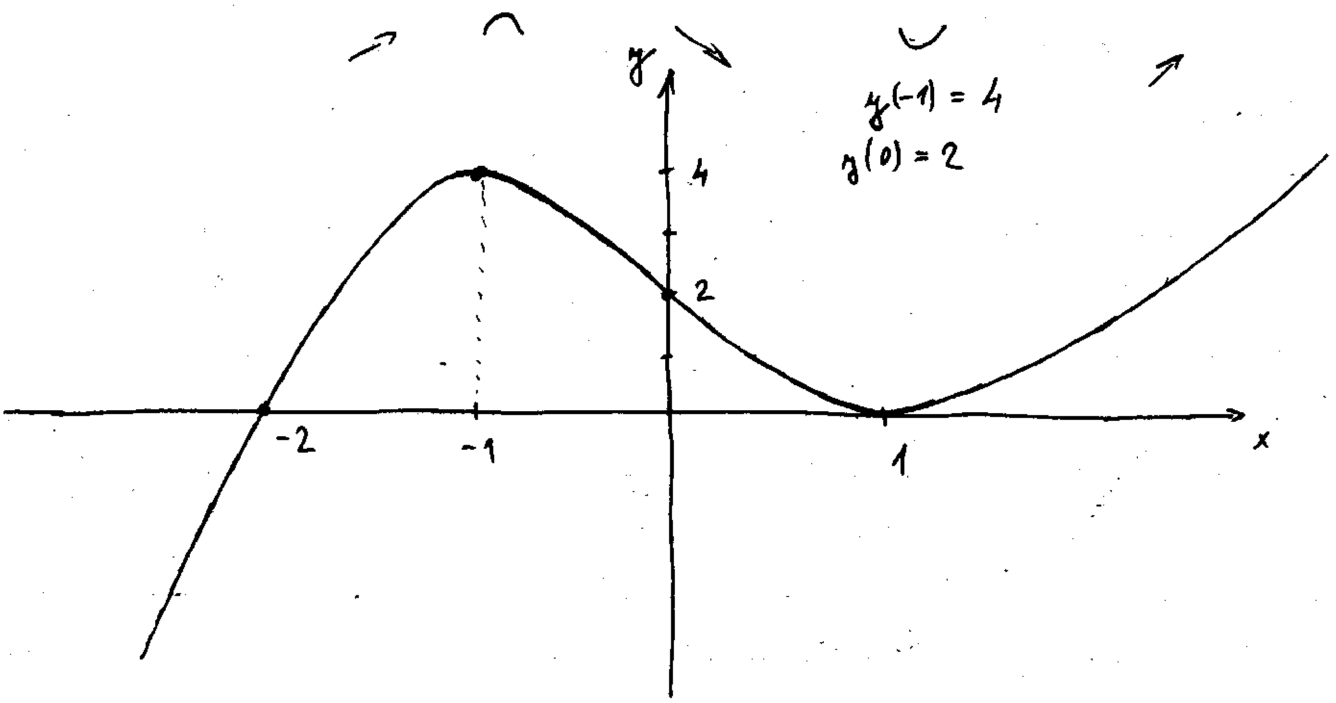
$y'' = 6x$

		0	
$6x$	$-$		$+$
y''	$-$		$+$
	$\cap$	inf. bod	$\cup$

5) Asymptoty

BS neex.

SS $a = \lim_{x \to \pm\infty} \frac{x^3 - 3x + 2}{x} = \left\| \frac{\infty}{\infty} \right\| = \lim_{x \to \pm\infty} (x^2 - 3) = \infty$ ... neex



př 6 $\quad y = \ln(x^2+1)$

1) $D(f) = \mathbb{R}$

$f(-x) = \ln((-x)^2+1) = \ln(x^2+1) \Rightarrow$ sudá

2) Kladná, záporná

$\ln(x^2+1) \geq 0$

$x^2+1 \geq 1$

$x^2 \geq 0$ – vždy

3) Rostoucí, klesající, extrémy

$y' = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1} \leftarrow$ vždy >0

		0	
$2x$	$-$		$+$
y'	$-$		$+$
	$\searrow$	min	$\nearrow$

4) Konvexní, konkávní, inflexní body

$y'' = \frac{2(x^2+1) - 2x\cdot 2x}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2} = \frac{2(1+x)(1-x)}{(x^2+1)^2}$

$(x^2+1)^2$ – vždy >0

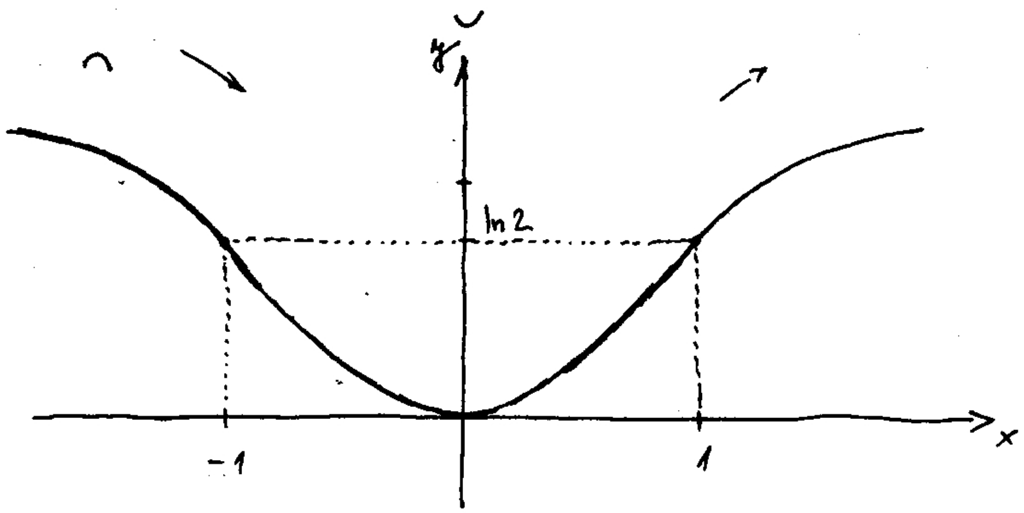
		-1		1	
$1+x$	$-$		$+$		$+$
$1-x$	$+$		$+$		$-$
y''	$-$ $\frown$	inf. bod	$+$ $\smile$	inf. bod	$-$ $\frown$

5) Asymptoty

BS neex.

SS $\quad a = \lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \frac{\ln(x^2+1)}{x} = \left\| \frac{\infty}{\infty} \right\| \overset{L.P.}{=} \lim\limits_{x\to\pm\infty} \frac{\frac{1}{x^2+1}\cdot 2x}{1} = \lim\limits_{x\to\pm\infty} \frac{2x}{x^2+1} \overset{\frac{\infty}{\infty}\, L.P.}{=} \lim\limits_{x\to\pm\infty} \frac{2}{2x} = 0$

$b = \lim\limits_{x\to\pm\infty} (f(x) - ax) = \lim\limits_{x\to\pm\infty} (\ln(x^2+1) - 0\cdot x) = \infty \Rightarrow$ neex



$y(-1) = \ln 2$

př 7 $\quad y=\dfrac{x^2-6x+13}{x-3}$ ← nerozložitelný, protože $D=(-6)^2-4\cdot 1\cdot 13 = 36-52 = -16<0$
a vždy >0

1) $D(f)=\mathbb{R}\setminus\{3\}$

$$f(-x)=\frac{(-x)^2-6(-x)+13}{-x-3}=\frac{x^2+6x+13}{-x-3}$$ ani sudá ani lichá

2) Kladná, záporná

		3	
$x-3$	$-$		$+$
y	$-$		$+$

3) Rostoucí, klesající, extrémy

$$y'=\frac{(2x-6)(x-3)-(x^2-6x+13)\cdot 1}{(x-3)^2}=\frac{2x^2-6x-6x+18-x^2+6x-13}{(x-3)^2}=\frac{x^2-6x+5}{(x-3)^2}=\frac{(x-1)(x-5)}{(x-3)^2}$$

		1		3		5	
$x-1$	$-$		$+$		$+$		$+$
$x-5$	$-$		$-$		$-$		$+$
$(x-3)^2$	$+$		$+$		$+$		$+$
y'	$+$		$-$		$-$		$+$
	$\nearrow$	max	$\searrow$		$\searrow$	min	$\nearrow$

4) Konvexní, konkávní, inflexní body

$$y''=\frac{(2x-6)(x-3)^{\cancel{2}}-(x^2-6x+5)\cancel{2(x-3)}}{(x-3)^{\cancel{4}3}}=\frac{2x^2-6x-6x+18-2x^2+12x-10}{(x-3)^3}=\frac{8}{(x-3)^3}$$

		3	
$(x-3)^3$	$-$		$+$
y''	$-$		$+$
	$\cap$		$\cup$

5) Asymptoty

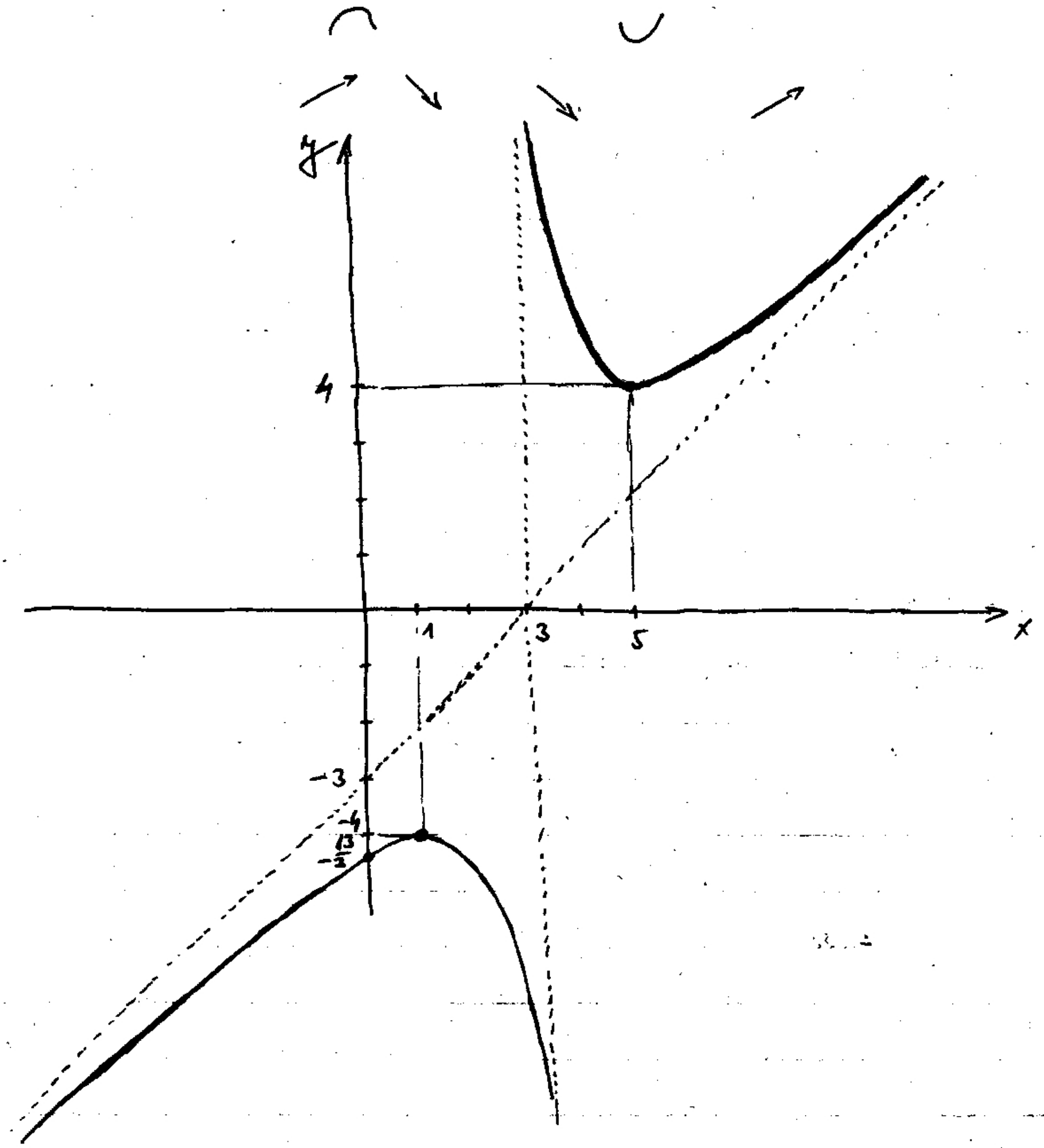
BS $\quad \lim\limits_{x\to 3^+}\dfrac{x^2-6x+13}{x-3}=\left\|\dfrac{4}{+0}\right\|=\infty$

$\lim\limits_{x\to 3^-}\dfrac{x^2-6x+13}{x-3}=\left\|\dfrac{4}{-0}\right\|=-\infty$ $\quad\Rightarrow$ asymptota bez směrnice $\boxed{x=3}$

SS $\quad a=\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\pm\infty}\dfrac{x^2-6x+13}{\underbrace{x(x-3)}_{x^2-3x}}=\left\|\dfrac{\infty}{\infty}\right\|\overset{L.P.}{=}\lim\limits_{x\to\pm\infty}\dfrac{2x-6}{2x-3}=\left\|\dfrac{\infty}{\infty}\right\|\overset{L.P.}{=}\lim\limits_{x\to\pm\infty}\dfrac{2}{2}=\underline{\underline{1}}$

$b=\lim\limits_{x\to\pm\infty}(f(x)-ax)=\lim\limits_{x\to\pm\infty}\left(\dfrac{x^2-6x+13}{x-3}-1\cdot x\right)=\lim\limits_{x\to\pm\infty}\dfrac{x^2-6x+13-x^2+3x}{x-3}=\lim\limits_{x\to\pm\infty}\dfrac{-3x+13}{x-3}=\left\|\dfrac{\infty}{\infty}\right\|\overset{L.P.}{=}$

$\overset{L.P.}{=}\lim\limits_{x\to\pm\infty}\dfrac{-3}{1}=\underline{\underline{-3}}$ $\quad\Rightarrow$ asymptota se směrnicí $\boxed{y=x-3}$



$y(1) = -4$

$y(0) = -\frac{13}{3} = -4\frac{1}{3}$

(př 8) Rozdělte číslo 10 na dvě čísla tak, aby jejich součin byl maximální!

$10 = a + b$

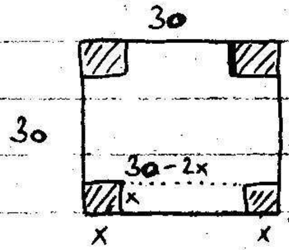
součin $\underline{y = a \cdot b} = a\cdot(10-a) = -a^2 + 10a$

$y' = 0$ $\qquad\qquad$ $y'' = -2$

$-2a + 10 = 0$ $\qquad\qquad$ $y''(5) = -2 < 0 \Rightarrow \max$ ✓

$\underline{a = 5} \Rightarrow \underline{b = 5}$

(př 9) Z rohů čtvercového kartonu o straně 30 cm odstřihněte čtverce o straně x tak, aby vznikla krabice měla co největší objem.



$V = (30-2x)^2 \cdot x$

$\underline{V = 900x - 120x^2 + 4x^3}$

$V' = 0$

$12x^2 - 240x + 900 = 0$

$x^2 - 20x + 75 = 0$

$x_{1,2} = \dfrac{20 \pm \sqrt{400-300}}{2} = \dfrac{20 \pm 10}{2} = \begin{cases} \cancel{15} \\ 5 \end{cases}$

$V'' = 24x - 240$

$V''(5) = 120 - 240 < 0 \Rightarrow \max$ ✓

$x \in (0, 15)$ $\qquad$ $\underline{\underline{x = 5\,cm}}$

Taylorův vzorec

V: f má v okolí x_0 derivace do řádu $n+1$ (vlastní), $n \in \mathbb{N}_0$. Pak pro $\forall x$ z okolí x_0 platí

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_n(x)\ \dots\ \text{Taylorův polynom } n\text{-tého stupně}} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)}(x-x_0)^{n+1}}_{R_n(x)\ \dots\ \text{zbytek}}$$

ξ – číslo mezi x_0, x

pro $x_0 = 0$ … Maclaurinův polynom

(př 10) Napište Taylorův polynom

a) 2. stupně pro fci $f(x) = 1 + 3x + 5x^2 - 2x^3$ v bodě $x_0 = -1$

$$P_2(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$f'(x) = 3 + 10x - 6x^2$ $\qquad$ $f''(x) = 10 - 12x$

$f(x_0) = 5$ $\qquad$ $f'(x_0) = -13$ $\qquad$ $f''(x_0) = 22$

$P_2(x) = 5 - 13(x+1) + \frac{22}{2!}(x+1)^2$

$\underline{\underline{P_2(x) = 5 - 13(x+1) + 11(x+1)^2}}$

b) 4. stupně pro fci $f(x) = \sin 2x$ v bodě $x_0 = 0$

$$P_4(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4$$

$f(x) = \sin 2x$	$f'(x) = 2\cos 2x$	$f''(x) = -4\sin 2x$	$f'''(x) = -8\cos 2x$	$f^{(4)}(x) = +16\sin 2x$
$f(x_0) = 0$	$f'(x_0) = 2$	$f''(x_0) = 0$	$f'''(x_0) = -8$	$f^{(4)}(x_0) = 0$

$$P_4(x) = 0 + \frac{2}{1!}(x-0) + \frac{0}{2!}(x-0)^2 + \frac{-8}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4$$

$$\underline{\underline{P_4(x) = 2x - \frac{4}{3}x^3}}$$

Integrály

integrace – opačná operace k derivaci

$$F(x) \underset{\int}{\overset{'}{\rightleftarrows}} f(x) \qquad \frac{x^3}{3} \underset{\int}{\overset{'}{\rightleftarrows}} x^2$$

vzorce

$$\int 0\,dx = c$$
$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$
$$\int \frac{1}{x}\,dx = \ln|x|$$
$$\vdots$$
$$\int e^x dx = e^x$$
$$\int a^x dx = \frac{a^x}{\ln a}$$

$$\int \sin x\,dx = -\cos x$$
$$\int \cos x\,dx = \sin x$$
$$\int \frac{1}{\sin^2 x}\,dx = -\operatorname{cotg} x$$
$$\int \frac{1}{\cos^2 x}\,dx = -\operatorname{tg} x$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x$$
$$\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$
$$\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$$
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$$

(př 11) Zintegrujte

a) $\int x^4 dx = \underline{\frac{x^5}{5} + C}$

b) $\int x^{\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \underline{\frac{2}{3} x^{\frac{3}{2}} + C}$

c) $\int x^3 \cdot \sqrt[3]{x}\,dx = \int x^3 \cdot x^{\frac{1}{3}} dx = \int x^{\frac{10}{3}} dx = \frac{x^{\frac{13}{3}}}{\frac{13}{3}} + c = \frac{3}{13} x^{\frac{13}{3}} + C = \underline{\underline{\frac{3}{13} x^4 \cdot \sqrt[3]{x} + C}}$

d) $\int \frac{\sqrt{x} - 2x}{x^2}\,dx = \int \left(\frac{x^{\frac{1}{2}}}{x^2} - \frac{2x}{x^2}\right) dx = \int \left(x^{-\frac{3}{2}} - 2\frac{1}{x}\right) dx = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - 2\ln|x| + C = \underline{\underline{-\frac{2}{\sqrt{x}} - 2\ln|x| + c}}$

e) $\int \frac{dx}{\sqrt{3-3x^2}} = \int \frac{1}{\sqrt{3}\sqrt{1-x^2}}\,dx = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{1-x^2}}\,dx = \underline{\underline{\frac{1}{\sqrt{3}} \arcsin x + c}}$

f) $\int \cos 2x\,dx = \underline{\frac{1}{2}\sin 2x + C}$

g) $\int \cos(2x+3)\,dx = \underline{\frac{1}{2}\sin(2x+3) + C}$

h) $\int \frac{2x}{x^2+3}\,dx = \ln|x^2+3| + C = \underline{\ln(x^2+3) + C}$ $>0 \;\forall x \in \mathbb{R}$

i) $\int \frac{4}{3x+8}\,dx = \int \frac{\frac{4}{3}3}{3x+8}\,dx = \frac{4}{3}\int \frac{3}{3x+8}\,dx = \underline{\frac{4}{3}\ln|3x+8| + C}$

• metoda per partes ... součin 2 fcí

$(u\cdot v)' = u'v + uv' \quad /\int$

$u\cdot v = \int u'v + \int uv' \quad \longrightarrow \boxed{\int u'v = uv - \int uv'}$

$\longrightarrow \boxed{\int uv' = uv - \int u'v}$

použití: $e^x \cdot \sin x$, $e^x \cos x$

$P(x)\cdot \sin$, $P(x)\cos x$

$P(x)\cdot e^x$, $P(x)e^x$

$P(x)\cdot \ln^n x$

i jinde :-)

př 12 Zintegrujte

a) $\int x\cdot \sin x\,dx = \left|\begin{matrix} u = x & v' = \sin x \\ u' = 1 & v = -\cos x\end{matrix}\right| = -x\cos x - \int -\cos x\,dx = \underline{-x\cos x + \sin x + C}$

b) $\int x^2 \cdot e^x\,dx = \left|\begin{matrix} u = x^2 & v' = e^x \\ u' = 2x & v = e^x\end{matrix}\right| = x^2 e^x - \int 2x e^x\,dx = \left|\begin{matrix} u = 2x & v' = e^x \\ u' = 2 & v = e^x\end{matrix}\right| =$

$= x^2e^x - \left(2xe^x - \int 2e^x\,dx\right) = \underline{x^2e^x - 2xe^x + 2e^x + C}$

c) $\int (3x^2+1)\cos x\,dx = \left|\begin{matrix} u = 3x^2+1 & v' = \cos x \\ u' = 6x & v = \sin x\end{matrix}\right| = (3x^2+1)\sin x - \int 6x\sin x\,dx = \left|\begin{matrix} u = 6x & v' = \sin x \\ u' = 6 & v = -\cos x\end{matrix}\right| =$

$= (3x^2+1)\sin x - \left(-6x\cos x - \int -6\cos x\,dx\right) = \underline{(3x^2+1)\sin x + 6x\cos x - 6\sin x + C}$

d) $\int x^2 \ln x\,dx = \left|\begin{matrix} u' = x^2 & v = \ln x \\ u = \frac{x^3}{3} & v' = \frac{1}{x}\end{matrix}\right| = \frac{x^3}{3}\ln x - \int \underbrace{\frac{x^3}{3}\cdot\frac{1}{x}}_{\frac{1}{3}x^2}\,dx = \frac{x^3}{3}\ln x - \frac{1}{3}\frac{x^3}{3} + C = \underline{\frac{x^3}{9}(3\ln x - 1) + C}$

e) $\int \ln x\,dx = \int 1\cdot \ln x\,dx = \left|\begin{matrix} u' = 1 & v = \ln x \\ u = x & v' = \frac{1}{x}\end{matrix}\right| = x\ln x - \int \underbrace{x\frac{1}{x}}_{1}\,dx = \underline{x\ln x - x + C}$

f) $\int e^x \sin x\,dx = \begin{vmatrix} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x \end{vmatrix} = -e^x\cos x - \int -e^x \cos x\,dx = -e^x\cos x + \int e^x \cos x\,dx =$

$= \begin{vmatrix} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x \end{vmatrix} = -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx$

$\int e^x \sin x\,dx = -e^x\cos x + e^x\sin x - \int e^x \sin x\,dx$

$2\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x \quad /:2$

$\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x)$

• substituční metoda

př 13) Zintegrujte

a) $\int 3x^2 e^{x^3}\,dx = \begin{vmatrix} t = x^3 \\ dt = 3x^2dx \end{vmatrix} = \int e^t\,dt = e^t + c = e^{x^3} + c$

b) $\int 2x\sqrt{x^2+5}\,dx = \begin{vmatrix} t = x^2+5 \\ dt = 2x\,dx \end{vmatrix} = \int \sqrt{t}\,dt = \int t^{1/2}\,dt = \frac{t^{3/2}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{t^3} + c = \frac{2}{3}\sqrt{(x^2+5)^3} + c$

c) $\int \frac{\sqrt{1+\ln x}}{x}\,dx = \int \sqrt{1+\ln x}\cdot\frac{1}{x}\,dx = \begin{vmatrix} t = 1+\ln x \\ dt = \frac{1}{x}dx \end{vmatrix} = \int \sqrt{t}\,dt = \int t^{1/2}\,dt = \frac{t^{3/2}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{(1+\ln x)^3} + c$

d) $\int \frac{4x}{\sqrt{1-x^4}}\,dx = \begin{vmatrix} t = x^2 \\ dt = 2x\,dx \quad /\cdot 2 \\ 2dt = 4x\,dx \end{vmatrix} = \int \frac{2}{\sqrt{1-t^2}}\,dt = 2\arcsin t + c = 2\arcsin x^2 + c$

e) $\int \underbrace{\cos^5 x}\sqrt{\sin x}\,dx = \begin{vmatrix} t = \sin x \\ dt = \cos x\,dx \end{vmatrix} = \int (1-t^2)^2\sqrt{t}\,dt = \int (1-2t^2+t^4)\cdot\sqrt{t}\,dt =$

$\cos^4 x \cdot \cos x$

$(1-\sin^2 x)^2\cdot\cos x$

$= \int (t^{1/2} - 2t^{5/2} + t^{9/2})\,dt = \frac{t^{3/2}}{\frac{3}{2}} - 2\frac{t^{7/2}}{\frac{7}{2}} + \frac{t^{11/2}}{\frac{11}{2}} + c = \sqrt{t^3}\left(\frac{2}{3} - \frac{4}{7}t^2 + \frac{2}{11}t^4\right) + c =$

$= \sqrt{\sin^3 x}\left(\frac{2}{3} - \frac{4}{7}\sin x + \frac{2}{11}\sin^2 x\right)$

7. cvičení

f) $\int x\sqrt{x-5}\,dx = \left| \begin{matrix} t = x-5 \to x = t+5 \\ dt = dx \end{matrix} \right| = \int (t+5)\sqrt{t}\,dt = \int \left(t^{3/2} + 5t^{1/2}\right) dt = \frac{t^{5/2}}{\frac{5}{2}} + 5\frac{t^{3/2}}{\frac{3}{2}} + c =$

$= t^{1/2}\left(\frac{2}{5}t^2 + \frac{10}{3}t\right) + c = \underline{\underline{\sqrt{x-5}\left(\frac{2}{5}(x-5)^2 + \frac{10}{3}(x-5)\right) + c}}$

g) $\int \sin\sqrt{x}\,dx = \left| \begin{matrix} t = \sqrt{x} \\ t^2 = x \\ 2t\,dt = dx \end{matrix} \right| = \int \sin t \cdot 2t\,dt = \int 2t \cdot \sin t\,dt = \left| \begin{matrix} u = 2t & v' = \sin t \\ u' = 2 & v = -\cos t \end{matrix} \right| =$

$-2t\cos t - \int -2\cos t = -2t\cos t + 2\sin t + c = \underline{\underline{2\left(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}\right) + c}}$

h) $\int \frac{\operatorname{arctg}\sqrt{x}}{\sqrt{x}(1+x)}\,dx = \left| \begin{matrix} t = \sqrt{x} \\ t^2 = x \\ 2t\,dt = dx \end{matrix} \right| = \int \frac{\operatorname{arctg} t}{t(1+t^2)} \cdot 2t\,dt = 2\int \frac{1}{1+t^2}\operatorname{arctg} t\,dt = \left| \begin{matrix} s = \operatorname{arctg} t \\ ds = \frac{1}{1+t^2}dt \end{matrix} \right| =$

$= 2\int s\,ds = 2\frac{s^2}{2} + c = s^2 + c = \operatorname{arctg}^2 t + c = \underline{\underline{\operatorname{arctg}^2\sqrt{x} + c}}$

- <u>integrace racionální lomené fce</u>

$\frac{P(x)}{Q(x)}$ neryzí $\quad$ st $P(x) \geq$ st $Q(x)$ $\quad \longrightarrow$ vydělíme $\Rightarrow$ polynom + ryzí

ryzí $\quad <$ $\quad \longrightarrow$ rozklad na parciální zlomky

$\frac{a}{(bx+c)^n}$, $\frac{ax+b}{(cx^2+dx+f)^n}$ nerozložitelný

(př 14) Zintegrujte

a) $\int \frac{3x+1}{x^2-3x+2}\,dx =$ $\qquad D = 9-8=1 \qquad x_{1,2} = \frac{+3\pm 1}{2} = \begin{cases} 2 \\ 1 \end{cases}$

$\frac{3x+1}{x^2-3x+2} = \frac{3x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \quad / \cdot (x-1)(x-2)$

$3x+1 = A(x-2) + B(x-1)$

$x: \; 3 = A + B$

$k: \; 1 = -2A - B$

$4 = -A \Rightarrow \underline{\underline{A = -4}} \Rightarrow B = 7$

$$= \int \left(\frac{-4}{x-1} + \frac{7}{x-2} \right) dx = -4 \ln|x-1| + 7 \ln|x-2| + c = -\ln(x-1)^4 + \ln|x-2|^7 + c =$$

$$= \underline{\underline{\ln \frac{|x-2|^7}{(x-1)^4} + c}}$$

b) $\displaystyle\int \frac{x^2+5x-4}{x^3+4x^2+4x}\,dx =$

$$\frac{x^2+5x-4}{x^3+4x^2+4x} = \frac{x^2+5x-4}{x(x^2+4x+4)} = \frac{x^2+5x-4}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \;/\cdot x(x+2)^2$$

$$x^2+5x-4 = A(x^2+4x+4) + B(x^2+2x) + Cx$$

$x^2:\ 1 = A + B$

$x:\ 5 = 4A + 2B + C$

$\underline{k:\ -4 = 4A}$ $\quad A = -1,\ B = 2,\ C = 5$

$$= \int \left(\frac{-1}{x} + \frac{2}{x+2} + \frac{5}{(x+2)^2} \right) dx = \left| \begin{matrix} t = x+2 \\ dt = dx \end{matrix} \right| = -\ln|x| + 2\ln|x+2| + 5 \int \underbrace{\frac{1}{t^2}}_{t^{-2}} dt =$$

$$= -\ln|x| + \ln(x+2)^2 + 5\,\frac{t^{-1}}{-1} + c = \underline{\underline{\ln \frac{(x+2)^2}{|x|} - \frac{5}{x+2} + c}}$$

c) $\displaystyle\int \frac{1}{x^2+2x+10}\,dx = \int \frac{1}{(x+1)^2+9}\,dx = \frac{1}{9} \int \frac{1}{\left(\frac{x+1}{3}\right)^2+1}\,dx = \left| \begin{matrix} t = \frac{x+1}{3} \\ dt = \frac{1}{3}dx \;/\cdot 3 \\ 3dt = dx \end{matrix} \right| = \frac{3}{9} \int \frac{1}{t^2+1}\,dt =$

← parc. zlomek

$D = 4 - 40 < 0$

$$= \frac{1}{3}\operatorname{arctg} t + c = \underline{\underline{\frac{1}{3}\operatorname{arctg}\frac{x+1}{3} + c}}$$

d) $\displaystyle\int \frac{2x+5}{x^2+4x+9}\,dx = \int \frac{2x+4+1}{x^2+4x+9}\,dx = \int \frac{2x+4}{x^2+4x+9}\,dx + \int \frac{1}{x^2+4x+9}\,dx =$

← parc. zlomek

$D = 16 - 36 < 0$

$$= \ln|x^2+4x+9| + \int \frac{1}{(x+2)^2+5}\,dx = \ln(x^2+4x+9) + \frac{1}{5} \int \frac{1}{\left(\frac{x+2}{\sqrt{5}}\right)^2+1}\,dx = \left| \begin{matrix} t = \frac{x+2}{\sqrt{5}} \\ dt = \frac{1}{\sqrt{5}}dx \;/\cdot\sqrt{5} \\ \sqrt{5}\,dt = dx \end{matrix} \right| =$$

$$= \ln(x^2+4x+9) + \frac{\sqrt{5}}{5} \int \frac{1}{t^2+1}\,dt = \ln(x^2+4x+9) + \frac{1}{\sqrt{5}}\operatorname{arctg} t + c = \underline{\underline{\ln(x^2+4x+9) + \frac{1}{\sqrt{5}}\operatorname{arctg}\frac{x+2}{\sqrt{5}} + c}}$$

parc. zlomek

$$e)\ \int \frac{3x+2}{x^2+4x+5}\,dx = \int \frac{\frac{3}{2}(2x+4)-4}{x^2+4x+5}\,dx = \frac{3}{2}\int \frac{2x+4}{x^2+4x+5} - 4\int \frac{1}{x^2+4x+5}\,dx =$$

$D = 16 - 4\cdot 5 < 0$

$$= \frac{3}{2}\ln|x^2+4x+5| - 4\int \frac{1}{(x+2)^2+1}\,dx = \left| \begin{matrix} t = x+2 \\ dt = dx \end{matrix} \right| = \frac{3}{2}\ln(x^2+4x+5) - 4\int \frac{1}{t^2+1}\,dt =$$

$$= \frac{3}{2}\ln(x^2+4x+5) - 4\,\mathrm{arctg}\,t + c = \underline{\underline{\frac{3}{2}\ln(x^2+4x+5) - 4\,\mathrm{arctg}\,(x+2) + c}}$$

$$f)\ \int \frac{x^2+3x+4}{x-1}\,dx = \int \left(x+4+\frac{8}{x-1}\right)dx = \underline{\underline{\frac{x^2}{2} + 4x + 8\ln|x-1| + c}}$$

$$\begin{array}{l} (x^2+3x+4) : (x-1) = x+4+\frac{8}{x-1} \\ \underline{-(x^2-x)} \\ \quad 4x+4 \\ \quad \underline{-(4x-4)} \\ \qquad 8 \end{array}$$

$$g)\ \int \frac{x^2+5x+4}{(x+1)^2}\,dx = \int\left(1+\frac{3x+3}{x^2+2x+1}\right)dx = x + \int\left(1+\frac{3(x+1)}{(x+1)^2}\right)dx = \int\left(1+\frac{3}{x+1}\right)dx =$$

$$\begin{array}{l} (x^2+5x+4) : (x^2+2x+1) = 1 + \frac{3x+3}{x^2+2x+1} \\ \underline{-(x^2+2x+1)} \\ \quad 3x+3 \end{array}$$

$$= \underline{\underline{x + 3\ln|x+1| + c}}$$

h) $$\int \frac{x^4-5x^3+9x^2-19x+8}{x^2-4x}\,dx = \int\left(x^2-x+5+\frac{x+8}{x^2-4x}\right)dx =$$

$$(x^4-5x^3+9x^2-19x+8):(x^2-4x) = x^2-x+5+\frac{x+8}{x^2-4x}$$

$$\begin{array}{l}
-(x^4-4x^3) \\
\hline
\quad -x^3+9x^2-19x+8 \\
\quad -(-x^3+4x^2) \\
\hline
\qquad 5x^2-19x+8 \\
\qquad -(5x^2-20x) \\
\hline
\qquad\quad x+8
\end{array}$$

$$\frac{x+8}{x^2-4x} = \frac{x+8}{x(x-4)} = \frac{A}{x}+\frac{B}{x-4} \quad /\cdot x(x-4)$$

$$x+8 = A(x-4)+Bx$$

$$x:\ 1 = A+B$$

$$k:\ 8 = -4A \qquad A=-2,\ B=3$$

$$= \frac{x^3}{3}-\frac{x^2}{2}+5x+\int\left(\frac{-2}{x}+\frac{3}{x-4}\right)dx = \underline{\underline{\frac{x^3}{3}-\frac{x^2}{2}+5x-2\ln|x|+3\ln|x-4|+c}}$$

$\swarrow$ nedělat (nestihne se :-)

i) $$\int \frac{4x^4-8x^3+26x^2-8x+5}{x^3-2x^2+5x}\,dx = \int\left(4x+\frac{6x^2-8x+5}{x^3-2x^2+5x}\right)dx = 2x^2+\int\frac{6x^2-8x+5}{x^3-2x^2+5x}\,dx =$$

$$(4x^4-8x^3+26x^2-8x+5):(x^3-2x^2+5x) = 4x + \frac{6x^2-8x+5}{x^3-2x^2+5x}$$
$$\underline{-(4x^4-8x^3+20x^2)}$$
$$6x^2-8x+5$$

$$\frac{6x^2-8x+5}{x^3-2x^2+5x} = \frac{6x^2-8x+5}{x(x^2-2x+5)} = \frac{A}{x}+\frac{Bx+C}{x^2-2x+5} \quad /\cdot x(x^2-2x+5)$$

$$6x^2-8x+5 = A(x^2-2x+5)+(Bx+C)x$$

x^2: $6 = A+B$

x: $-8 = -2A+C$

k: $5 = 5A$ $\quad A=1,\ B=5,\ C=-6$

$$= 2x^2+\int\left(\frac{1}{x}+\frac{5x-6}{x^2-2x+5}\right)dx = 2x^2+\ln|x|+\int\frac{\frac{5}{2}(2x-2)-1}{x^2-2x+5}\,dx =$$

$$= 2x^2+\ln|x|+\frac{5}{2}\int\frac{2x-2}{x^2-2x+5}\,dx-\int\frac{1}{x^2-2x+5}\,dx = 2x^2+\ln|x|+\frac{5}{2}\ln|x^2-2x+5|-\int\frac{1}{(x-1)^2+4}\,dx =$$

$$= 2x^2+\ln|x|+\frac{5}{2}\ln(x^2-2x+5)-\frac{1}{4}\int\frac{1}{\left(\frac{x-1}{2}\right)^2+1}\,dx = \left|\begin{array}{l} t=\frac{x-1}{2} \\ dt=\frac{1}{2}dx \ /\cdot 2 \\ 2dt = dx \end{array}\right| =$$

$$= \text{———}//\text{———} - \frac{2}{4}\int\frac{1}{t^2+1}\,dt =$$

$$= \text{———}//\text{———} - \frac{1}{2}\operatorname{arctg} t + c =$$

$$= \underline{\underline{2x^2+\ln|x|+\frac{5}{2}\ln(x^2-2x+5)-\frac{1}{2}\operatorname{arctg}\frac{x-1}{2}+c}}$$

j) $\int \frac{5\ln x}{x(\ln^3 x+\ln^2 x-2)}\,dx = \left| \begin{array}{l} t=\ln x \\ dt=\frac{1}{x}dx \end{array} \right| = \int \frac{5t}{t^3+t^2-2}\,dt =$

$$\frac{5t}{t^3+t^2-2} = \frac{5t}{(t-1)(t^2+2t+2)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+2t+2} \quad / \cdot (t-1)(t^2+2t+2)$$

$$\begin{array}{l} (t^3+t^2-2):(t-1) = t^2+2t+2 \\ \underline{-(t^3-t^2)} \\ \quad 2t^2-2 \\ \quad \underline{-(2t^2-2t)} \\ \qquad 2t-2 \\ \qquad \underline{-(2t-2)} \\ \qquad\quad 0 \end{array}$$

$$5t = A(t^2+2t+2) + (Bt+C)(t-1)$$

$$5t = A(t^2+2t+2) + B(t^2-t) + C(t-1)$$

$$t^2:\ 0 = A+B \qquad A=-B$$

$$t^1:\ 5 = 2A-B+C$$

$$t^0:\ 0 = 2A-C$$

$$5 = -3B+C$$

$$0 = -2B-C$$

$$5 = -5B \Rightarrow B=-1,\ A=1,\ C=2$$

$$= \int \left(\frac{1}{t-1} + \frac{-t+2}{t^2+2t+2} \right) dt = \ln|t-1| + \int \frac{-\frac{1}{2}(2t+2)+3}{t^2+2t+2}\,dt =$$

$$= \ln|t-1| - \frac{1}{2}\int \frac{2t+2}{t^2+2t+2}\,dt + 3\int \frac{1}{t^2+2t+2}\,dt = \ln|t-1| - \frac{1}{2}\ln(t^2+2t+2) + 3\int \frac{1}{(t+1)^2+1}\,dt =$$

$$= \left| \begin{array}{l} s=t+1 \\ ds=dt \end{array} \right| = \ln|t-1| - \frac{1}{2}\ln(t^2+2t+2) + 3\int \frac{1}{s^2+1}\,ds =$$

$$= \ln|t-1| - \frac{1}{2}\ln(t^2+2t+2) + 3\,\mathrm{arctg}\, s + c =$$

$$= \ln|t-1| - \frac{1}{2}\ln(t^2+2t+2) + 3\,\mathrm{arctg}\,(t+1) + c =$$

$$= \underline{\underline{\ln|\ln x-1| - \frac{1}{2}\ln(\ln^2 x+2\ln x+2) + 3\,\mathrm{arctg}\,(\ln x+1) + c}}$$

• integrace fcí obsahujících $\sin x$, $\cos x$

funkce	substituce
lichá vůči $\sin x$	$t = \cos x$
lichá vůči $\cos x$	$t = \sin x$

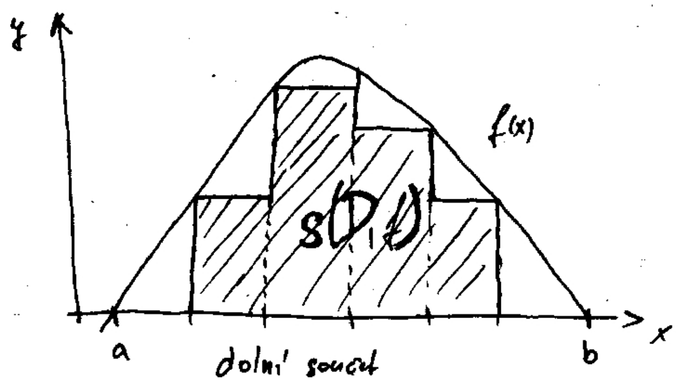
př 15 Zintegrujte

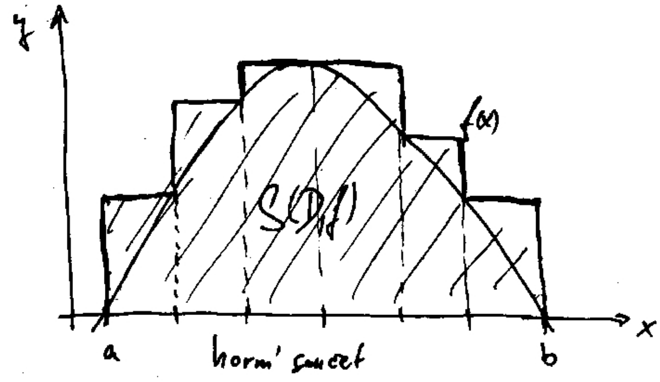
a) $\int \sin^3 x \, dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x \, dx \\ -dt = \sin x \, dx \end{array} \right| = \int \sin^2 x \sin x \, dx = \int (1-\cos^2 x) \sin x \, dx = \int (1-t^2)(-dt) =$

$= \int (t^2-1) \, dt = \frac{1}{3} t^3 - t + c = \underline{\underline{\frac{1}{3} \cos^3 x - \cos x + c}}$

b) $\int \cos x \sin^4 x \, dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \end{array} \right| = \int t^4 dt = \frac{1}{5} t^5 + c = \underline{\underline{\frac{1}{5} \sin^5 x + c}}$

Určitý integrál

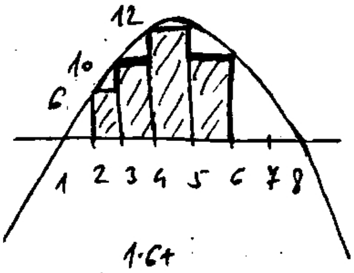


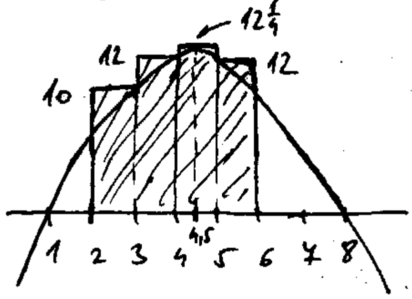


zjemňujeme dělení intervalu $\langle a, b \rangle$, až $S(D,f) = s(D,f)$

jinak : $S = \int_a^b f(x) \, dx = F(b) - F(a)$ ... Riemanův integrál př 16) Máme fci $f(x) = -x^2+9x-8$ na intervalu $\langle 2,6 \rangle$.

Určete dolní a horní součet pro dělící body $3;4;5$ a Riemannův integrál





$s(D,f) = 1\cdot 6 + 1\cdot 10 + 1\cdot 12 + 1\cdot 10 = 38$ $\qquad$ $S(D,f) = 1\cdot 10 + 1\cdot 12 + 1\cdot 12\tfrac{1}{4} + 1\cdot 12 = 46\tfrac{1}{4}$

$$\int_2^6 (-x^2+9x-8)\,dx = \left[-\frac{x^3}{3} + 9\frac{x^2}{2} - 8x\right]_2^6 = \left(-\frac{6^3}{3} + 9\frac{6^2}{2} - 8\cdot 6\right) - \left(-\frac{2^3}{3} + 9\frac{2^2}{2} - 8\cdot 2\right) =$$

$$= (-72+162-48) - \left(-\frac{8}{3}+18-16\right) = 42 + \frac{2}{3} = \underline{\underline{42\tfrac{2}{3}}}$$

př 17) Vypočítejte

a) $$\int_1^4 \frac{x+3}{x}\,dx = \int_1^4 \left(1+\frac{3}{x}\right)dx = \Big[x + 3\ln|x|\Big]_1^4 = (4+3\ln 4) - (1+3\overset{0}{\ln 1}) = \underline{\underline{3+3\ln 4}}$$

b) $$\int_0^{\frac{\pi}{2}} x\sin x\,dx = \left|\begin{matrix} u=x & v'=\sin x \\ u'=1 & v=-\cos x \end{matrix}\right| = \Big[-x\cos x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x\,dx =$$

$$= \Big[-x\cos x\Big]_0^{\frac{\pi}{2}} + \Big[\sin x\Big]_0^{\frac{\pi}{2}} = \Big[-x\cos x + \sin x\Big]_0^{\frac{\pi}{2}} = \left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (-0\cdot\cos 0 + \sin 0) = \underline{\underline{1}}$$

c) $$\int_{-2}^0 x^2\sqrt{1-x^3}\,dx = \left|\begin{matrix} t=1-x^3 & x=-2 \to t=9 \\ dt=-3x^2dx & x=0 \to t=1 \\ -\frac{1}{3}dt = x^2dx & \end{matrix}\right| = -\frac{1}{3}\int_9^1 \sqrt{t}\,dt = -\frac{1}{3}\int_9^1 t^{1/2}\,dt = -\frac{1}{3}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_9^1 =$$

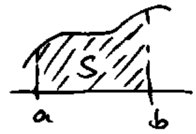
$$= -\frac{2}{9}\Big[t^{3/2}\Big]_9^1 = -\frac{2}{9}\Big[t\cdot\sqrt{t}\Big]_9^1 = -\frac{2}{9}(1-27) = \underline{\underline{\frac{52}{9}}}$$

d) $$\int_{-1}^1 x\,dx = \left[\frac{x^2}{2}\right]_{-1}^1 = \frac{1}{2} - \frac{1}{2} = \underline{\underline{0}}$$

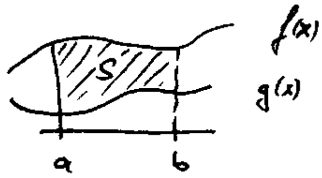
Aplikace určitého integrálu

$\boxed{\text{obsah plochy}}$

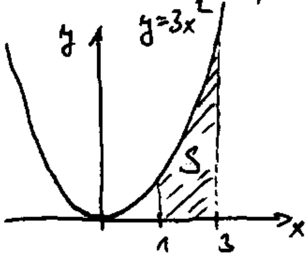
$$S = \int_a^b f(x)\,dx \qquad f(x) \text{ ... nezáporná fce!!!}$$



$$S = \int_a^b (f(x) - g(x))\,dx \qquad f(x) \geqq g(x) \text{ na } \langle a, b \rangle$$



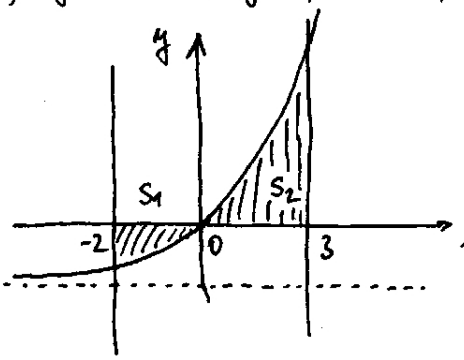
(př 18) Určete obsah plochy pod grafem funkce $y = 3x^2$ pro x od 1 do 3.



$$S = \int_1^3 3x^2\,dx = \left[3\frac{x^3}{3}\right]_1^3 = \left[x^3\right]_1^3 = 27 - 1 = \underline{\underline{26}}$$

(př 19) Určete obsah plochy ohraničené křivkami:

a) $y = e^x - 1,\ y = 0,\ x = -2,\ x = 3$



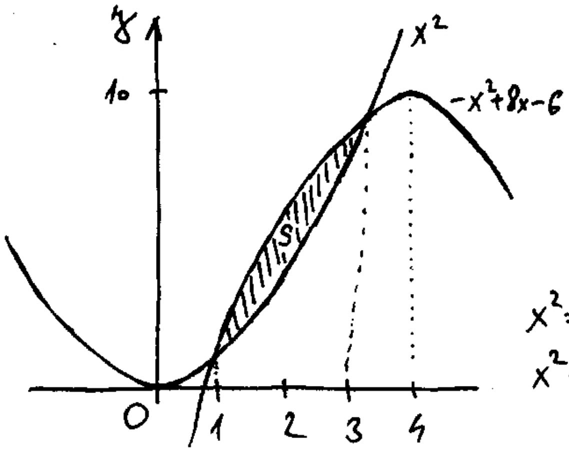
$$S_1 = \int_{-2}^{0} [0 - (e^x - 1)]\,dx = \int_{-2}^{0} (1 - e^x)\,dx = \left[x - e^x\right]_{-2}^{0} = (0-1) - (-2 - e^{-2}) =$$
$$= 1 + \frac{1}{e^2}$$

$$S_2 = \int_0^3 [(e^x - 1) - 0]\,dx = \left[e^x - x\right]_0^3 = (e^3 - 3) - (1 - 0) = e^3 - 4$$

$$S = S_1 + S_2 = 1 + \frac{1}{e^2} + e^3 - 4 = \underline{\underline{e^3 + \frac{1}{e^2} - 3}}$$

b) $y = x^2,\ y = -x^2 + 8x - 6$

$\hookrightarrow y = -x^2 + 8x - 6 = -[x^2 - 8x + 6] = -[(x-4)^2 - 10] = -(x-4)^2 + 10$



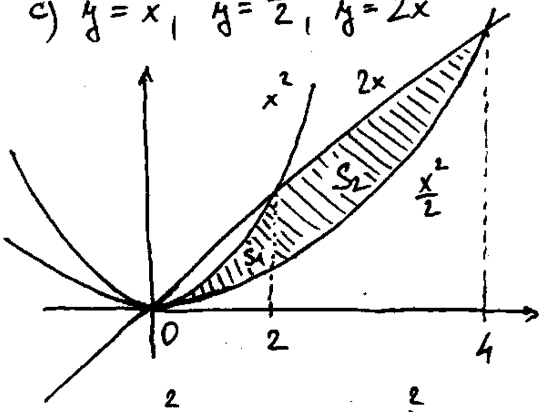
$$S = \int_1^3 [(-x^2 + 8x - 6) - x^2]\,dx = -2\int_1^3 (x^2 - 4x + 3)\,dx =$$
$$= -2\left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 = -2\left[(9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3\right)\right] =$$
$$= -2\left[0 - \frac{4}{3}\right] = \underline{\underline{\frac{8}{3}}}$$

$x^2 = -x^2 + 8x - 6$

$x^2 - 4x + 3 = 0$

$x_1 = 1,\ x_2 = 3$

c) $y = x^2, \ y = \frac{x^2}{2}, \ y = 2x$



$x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0 \vee x = 2$

$\frac{x^2}{2} = 2x$

$x^2 = 4x$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0 \vee x = 4$

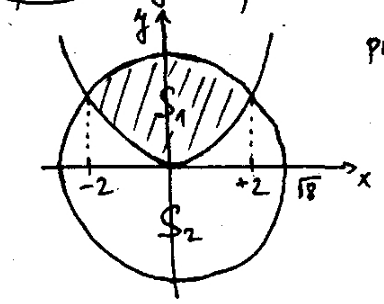
$$S_1 = \int_0^2 \left(x^2 - \frac{x^2}{2}\right) dx = \int_0^2 \frac{x^2}{2} dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{8}{6} - 0 = \underline{\frac{4}{3}}$$

$$S_2 = \int_2^4 \left(2x - \frac{x^2}{2}\right) dx = \left[x^2 - \frac{x^3}{6}\right]_2^4 = \left(16 - \frac{64}{6}\right) - \left(4 - \frac{8}{6}\right) = \frac{48 - 32 - 12 + 4}{3} = \underline{\frac{8}{3}}$$

$$S = S_1 + S_2 = \frac{4}{3} + \frac{8}{3} = \frac{12}{3} = \underline{4}$$

9. cvičení

(př 2a) V jakém poměru dělí křivka $y=\frac{1}{2}x^2$ plochu kruhu $x^2+y^2=8$?



průsečíky $x^2=2y \to 2y+y^2=8$

$y^2+2y-8=0$

$y_1=2,\; y_2=-4$

$\frac{1}{2}x^2=2 \qquad \frac{1}{2}x^2=-4$

$x=\pm 2 \qquad \Downarrow$ n.ř.

$$S_1=\int_{-2}^{2}\left(\sqrt{8-x^2}-\frac{1}{2}x^2\right)dx=\int_{-2}^{2}\sqrt{8-x^2}\,dx-\int_{-2}^{2}\frac{1}{2}x^2\,dx$$

$$\cos^2 t=\frac{1+\cos 2t}{2}$$

$$\bullet\ \int_{-2}^{2}\sqrt{8-x^2}\,dx=\left|\begin{matrix} x=\sqrt{8}\sin t=2\sqrt{2}\sin t & x=-2 \to t=-\frac{\pi}{4}\\ dx=\sqrt{8}\cos t\,dt & x=2 \to t=\frac{\pi}{4}\end{matrix}\right|=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sqrt{8-8\sin^2 t}\,\sqrt{8}\cos t\,dt=$$

$$=8\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\underbrace{\sqrt{1-\sin^2 t}}_{|\cos t|>0 \text{ na } \langle -\frac{\pi}{4},\frac{\pi}{4}\rangle}\cos t\,dt=8\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos^2 t\,dt=4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1+\cos 2t)\,dt=4\left[t+\frac{1}{2}\sin 2t\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=$$

$$=4\left[\left(\frac{\pi}{4}+\frac{1}{2}\right)-\left(-\frac{\pi}{4}-\frac{1}{2}\right)\right]=4+2\pi$$

$$\bullet\ \int_{-2}^{2}\frac{1}{2}x^2\,dx=\frac{1}{6}\left[x^3\right]_{-2}^{2}=\frac{1}{6}(8-(-8))=\frac{8}{3}$$

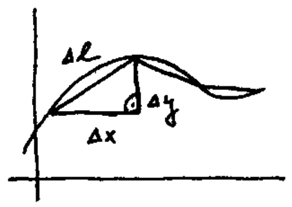
celkem $S_1=(4+2\pi)-\frac{8}{3}=\frac{4}{3}+2\pi$

$S_2=8\pi-\left(\frac{4}{3}+2\pi\right)=6\pi-\frac{4}{3}$

$$\frac{S_1}{S_2}=\frac{2\pi+\frac{4}{3}}{6\pi-\frac{4}{3}}=\frac{6\pi+4}{18\pi-4}=\frac{3\pi+2}{9\pi-2}$$

$$\frac{S_2}{S_1}=\frac{9\pi-2}{3\pi+2}\doteq 2{,}23$$ ... tolikrát je plocha S_2 větší než plocha S_1

délka křivky



$$l:\ \sum \Delta l=\sum \sqrt{(\Delta x)^2+(\Delta y)^2}$$

$\downarrow$ zjemňujeme dělení

$$l=\int_a^b\sqrt{(dx)^2+(dy)^2}=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \Rightarrow l=\int_a^b\sqrt{1+f'^2}\,dx$$

$$=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \Rightarrow l=\int_a^b\sqrt{x'^2+y'^2}\,dt$$ ... pro parametrické vyjádření křivky

př 21 Určete délku křivky $y = \ln\cos x$ v intervalu $\langle 0, \frac{\pi}{6} \rangle$. $\quad y' = \frac{1}{\cos x}\cdot(-\sin x)$

$$l = \int_a^b \sqrt{1+y'^2}\,dx = \int_0^{\frac{\pi}{6}} \sqrt{1+\frac{\sin^2 x}{\cos^2 x}}\,dx = \int_0^{\frac{\pi}{6}} \sqrt{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}\,dx = \int_0^{\frac{\pi}{6}} \sqrt{\frac{1}{\cos^2 x}}\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{\cos x}\,dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x\,dx \\ x = \frac{\pi}{6} \rightarrow t = \frac{1}{2} \\ x = 0 \rightarrow t = 0 \end{array} \right| =$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos x}{\cos^2 x}\,dx = \int_0^{\frac{\pi}{6}} \frac{\cos x}{1-\sin^2 x}\,dx = \int_0^{\frac{1}{2}} \frac{1}{1-t^2}\,dt =$$

$$\frac{1}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t} \quad /\cdot(1-t)(1+t)$$

$$1 = A(1+t) + B(1-t)$$

$$t:\ 0 = A - B$$

$$k:\ 1 = A + B$$

$$A = \frac{1}{2},\ B = \frac{1}{2}$$

$$= \int_0^{\frac{1}{2}} \left(\frac{\frac{1}{2}}{1-t} + \frac{\frac{1}{2}}{1+t}\right)dt = \int_0^{\frac{1}{2}} \left[\left(-\frac{1}{2}\,\frac{-1}{1-t}\right) + \frac{1}{2}\,\frac{1}{1+t}\right]dt = \left[-\frac{1}{2}\ln|1-t| + \frac{1}{2}\ln|1+t|\right]_0^{\frac{1}{2}} =$$

$$= \left[\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|\right]_0^{\frac{1}{2}} = \frac{1}{2}\ln\frac{\frac{3}{2}}{\frac{1}{2}} - \frac{1}{2}\underbrace{\ln 1}_{0} = \underline{\underline{\frac{1}{2}\ln 3}}$$

př 22 Určete délku kružnice o poloměru r.

$$x = r\cos t \qquad t \in \langle 0, 2\pi \rangle$$

$$y = r\sin t$$

$$x' = -r\sin t$$

$$y' = r\cos t$$

$$l = \int_a^b \sqrt{x'^2 + y'^2}\,dt = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt = \int_0^{2\pi} r\,dt = r\,[t]_0^{2\pi} = r(2\pi - 0) = \underline{\underline{2\pi r}}$$

(pří 23) Určete délku křivky

$x = \ln t \qquad t \in \langle \sqrt[4]{3}, \sqrt[4]{15} \rangle$

$y = \frac{1}{2} t^2$

$x' = \frac{1}{t}$

$y' = t$

$$\ell = \int_a^b \sqrt{x'^2 + y'^2}\, dt = \int_{\sqrt[4]{3}}^{\sqrt[4]{15}} \sqrt{\frac{1}{t^2} + t^2}\, dt = \int_{\sqrt[4]{3}}^{\sqrt[4]{15}} \frac{1}{t} \sqrt{1+t^4}\, dt = \left| \begin{array}{ll} s = \sqrt{1+t^4} & t = \sqrt[4]{3} \to s = 2 \\ s^2 = 1 + t^4 \Rightarrow t^4 = s^2 - 1 & t = \sqrt[4]{15} \to s = 4 \\ 2s\,ds = 4t^3 dt & \\ \cancel{2}s\,ds = \cancel{4}^{2} t^4 \frac{1}{t} dt & \\ s\,ds = 2(s^2-1)\frac{1}{t} dt & \\ \frac{s}{2(s^2-1)} ds = \frac{1}{t} dt & \end{array} \right| =$$

$$= \int_2^4 s \cdot \frac{s}{2(s^2-1)} ds = \frac{1}{2} \int_2^4 \frac{s^2}{s^2-1} ds = \frac{1}{2} \int_2^4 \left(1 + \frac{1}{s^2-1}\right) ds =$$

$$s^2 : (s^2-1) = 1 + \frac{1}{s^2-1}$$
$$\frac{-(s^2-1)}{1}$$

$$\frac{1}{s^2-1} = \frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} \quad / \cdot (s-1)(s+1)$$

$$1 = A(s+1) + B(s-1)$$

$$s: \ 0 = A + B$$

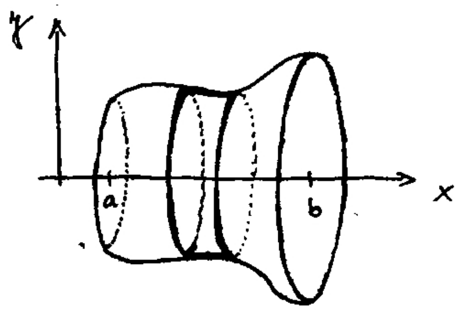
$$k: \ 1 = A - B$$

$$A = \frac{1}{2}, \ B = -\frac{1}{2}$$

$$= \frac{1}{2} \int_2^4 \left(1 + \frac{\frac{1}{2}}{s-1} - \frac{\frac{1}{2}}{s+1}\right) ds = \frac{1}{2}\left[s + \frac{1}{2}\ln|s-1| - \frac{1}{2}\ln|s+1|\right]_2^4 = \left[\frac{1}{2}s + \frac{1}{4}\ln\left|\frac{s-1}{s+1}\right|\right]_2^4 =$$

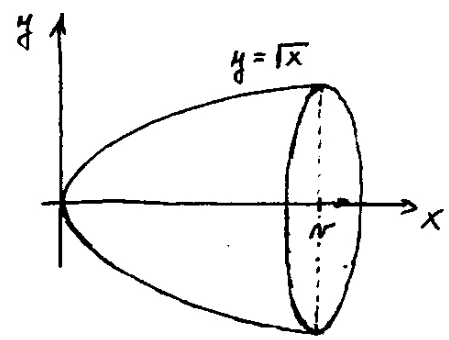
$$= \left(2 + \frac{1}{4}\ln\frac{3}{5}\right) - \left(1 + \frac{1}{4}\ln\frac{1}{3}\right) = 1 + \frac{1}{4}\ln\frac{\frac{3}{5}}{\frac{1}{3}} = \underline{\underline{1 + \frac{1}{4}\ln\frac{9}{5}}}$$

objem rotačního tělesa



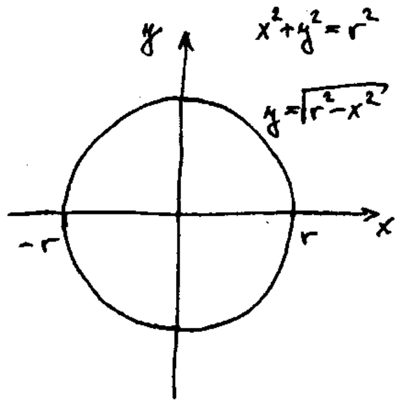
$$V = \pi \int_a^b f^2(x)\,dx$$

př 24 Určete objem rotačního paraboloidu o výšce v.



$$V = \pi \int_a^b f^2(x)\,dx = \pi \int_0^v (\sqrt{x})^2\,dx = \pi \int_0^v x\,dx = \pi \left[\frac{x^2}{2}\right]_0^v = \pi\left(\frac{v^2}{2} - 0\right) = \underline{\underline{\frac{1}{2}\pi v^2}}$$

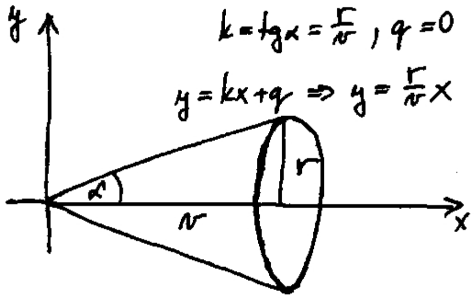
př 25 Určete objem koule o poloměru r.



$$V = \pi \int_a^b f^2(x)\,dx = \pi \int_{-r}^{r} \left(\sqrt{r^2-x^2}\right)^2 dx = \pi \int_{-r}^{r} (r^2-x^2)\,dx = \pi \left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} =$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right] = \pi\left(\frac{2}{3}r^3 + \frac{2}{3}r^3\right) = \underline{\underline{\frac{4}{3}\pi r^3}}$$

př 26 Určete objem kužele o poloměru podstavy r a výšce v.

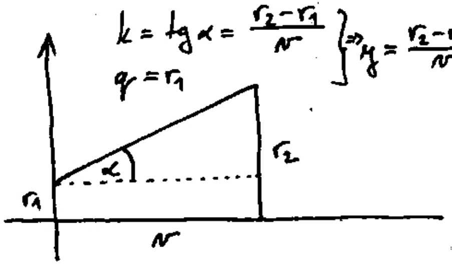


$$V = \pi \int_a^b f^2(x)\,dx = \pi \int_0^v \left(\frac{r}{v}x\right)^2 dx = \frac{\pi r^2}{v^2} \int_0^v x^2\,dx =$$

$$= \frac{\pi r^2}{v^2}\left[\frac{x^3}{3}\right]_0^v = \frac{\pi r^2}{v^2}\left(\frac{v^3}{3} - 0\right) = \underline{\underline{\frac{1}{3}\pi r^2 \cdot v}}$$

př 27 Určete objem komolého kužele o poloměrech podstav r_1, r_2 a výšce v.

$$\left.\begin{array}{l} k = \operatorname{tg}\alpha = \dfrac{r_2 - r_1}{v} \\ q = r_1 \end{array}\right\} \Rightarrow y = \frac{r_2 - r_1}{v} x + r_1$$



$$V = \pi \int_a^b f^2(x)\,dx = \pi \int_0^v \left(\frac{r_2 - r_1}{v} x + r_1 \right)^2 dx = \pi \int_0^v \left[\frac{(r_2 - r_1)^2}{v^2} x^2 + \frac{2r_1(r_2 - r_1)}{v} x + r_1^2 \right] dx =$$

$$= \pi \left[\frac{(r_2 - r_1)^2}{v^2} \frac{x^3}{3} + \frac{2r_1(r_2 - r_1)}{v} \frac{x^2}{2} + r_1^2 x \right]_0^v = \pi \left(\frac{(r_2 - r_1)^2}{v^2} \frac{v^3}{3} + \frac{r_1(r_2 - r_1)}{v} \cdot v^2 + r_1^2 v \right) =$$

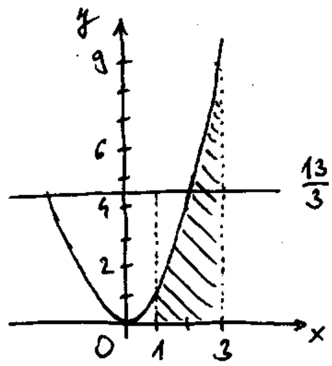
$$= \frac{\pi v}{3} \left(r_2^2 - 2r_2 r_1 + r_1^2 + 3r_1 r_2 - 3r_1^2 + 3r_1^2 \right) = \underline{\underline{\frac{1}{3} \pi v \left(r_1^2 + r_1 r_2 + r_2^2 \right)}}$$

<u>průměr funkce</u>

$$av_{[a,b]} f(x) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

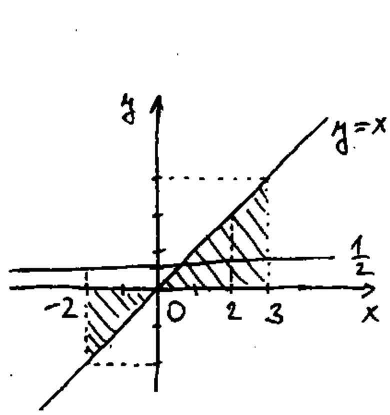
př 28 Vypočtěte průměr funkce $f(x) = x^2$ na intervalu $\langle 1, 3 \rangle$.

$$av_{[1,3]} f(x) = \frac{1}{3-1} \int_1^3 x^2\,dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_1^3 = \frac{1}{2} \left(9 - \frac{1}{3} \right) = \frac{1}{2} \cdot \frac{26}{3} = \underline{\underline{\frac{13}{3}}}$$



př 29 Vypočtěte průměr funkce $f(x) = x$ na intervalu $\langle -2, 3 \rangle$.

$$av_{[-2,3]} f(x) = \frac{1}{3-(-2)} \int_{-2}^3 x\,dx = \frac{1}{5} \left[\frac{x^2}{2} \right]_{-2}^3 = \frac{1}{5} \left(\frac{9}{2} - \frac{4}{2} \right) = \underline{\underline{\frac{1}{2}}}$$



10. cvičení

nevlastní integrály

1. druhu

$$\int_{-\infty}^{a} f(x)dx \quad \int_{a}^{\infty} f(x)dx \quad \int_{-\infty}^{\infty} f(x)dx \quad ; \text{ např: } \int_{a}^{\infty} f(x)dx = \lim_{b\to\infty} \int_{a}^{b} f(x)dx$$

př. 30 Vypočítejte obsah plochy pod grafem funkce

a) $f(x) = \frac{1}{x^2}$ v intervalu $(-\infty,-1\rangle$

$$S = \int_{-\infty}^{-1} \frac{1}{x^2}dx = \int_{-\infty}^{-1} x^{-2}dx = -\left[\frac{1}{x}\right]_{-\infty}^{-1} = -\left(\frac{-1}{1} - 0\right) = \underline{1}$$ ... integrál konverguje

b) $f(x) = \frac{1}{x}$ v intervalu $\langle 1,\infty)$

$$S = \int_{1}^{\infty} \frac{1}{x}\,dx = [\ln x]_1^{\infty} = \infty - 0 = \infty$$ ... integrál diverguje

c) $f(x) = \frac{1}{1+x^2}$ v intervalu $(-\infty,\infty)$

$$S = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = [\operatorname{arctg} x]_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \underline{\pi}$$ ... integrál konverguje

př. 31 Vypočtěte $\int_0^{\infty} \cos x\,dx = [\sin x]_0^{\infty} = \underbrace{\lim_{x\to\infty} \sin x}_{\nexists} - 0$ ... integrál $\nexists$, osciluje

2. druhu

$f(x)$ je na $\langle a,b\rangle$ neohraničená (jde do $+\infty$ nebo $-\infty$)

př. 32 Vypočtěte obsah plochy pod grafem funkce

a) $y = \frac{1}{x^2}$ na intervalu $\langle -1,0\rangle$

$$S = \int_{-1}^{0} \frac{1}{x^2}\,dx = -\left[\frac{1}{x}\right]_{-1}^{0} = -\left(\lim_{x\to 0^-} \frac{1}{x} - (-1)\right) = -(-\infty + 1) = \infty$$ ... diverguje

b) $y = \frac{1}{x}$ v intervalu $\langle 0,1\rangle$

$$S = \int_{0}^{1} \frac{1}{x}dx = [\ln x]_0^1 = 0 - \lim_{x\to 0^+} \ln x = 0 - (-\infty) = \infty$$ ... diverguje

Nekonečné řady

nekonečná řada ... součet forem $a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n$

a_n ... n-tý člen

např. $1+2+3+\dots+n+\dots = \sum_{n=1}^{\infty} n$

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\dots = \sum_{n=1}^{\infty} \frac{1}{n}$

Aritmetická řada $a_{n+1} = a_n + d$, diference ... d, $d \in \mathbb{R}$

např. $1+4+7+10+13+\dots$

součet prvních n členů $s_n = \frac{n}{2}(a_1 + a_n)$

Geometrická řada $a_{n+1} = a_n \cdot q$, $q \in \mathbb{R}$ ... kvocient

např. $1+2+4+8+16+\dots = \sum_{n=0}^{\infty} aq^n$

součet prvních n členů: (1) $s_n = a + a q + a q^2 + \dots + a q^{n-1} \quad | \cdot q$

(2) $q \cdot s_n = \quad a q + a q^2 + \quad + a q^{n-1} + a q^n$

(1)-(2) $s_n(1-q) = a - a q^n$

$$s_n = a \frac{1-q^n}{1-q}$$

součet nekonečné řady ... $s = \lim_{n \to \infty} s_n$ ← součet prvních n členů

píšeme $\sum_{n=1}^{\infty} a_n = s$, $\sum_{n=1}^{\infty} a_n = $ ∞ ... diverguje; číslo ... konverguje

pokud $\lim_{n\to\infty} s_n$ neexistuje, pak osciluje

součet nekonečné geometrické řady pro $q \in (-1,1)$ $\quad s = \lim_{n\to\infty} a \frac{1-q^n}{1-q} = \frac{a}{1-q}$

$$\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q}$$

př 1 Určete součet řad

a) $\sum_{n=0}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} 1\cdot\left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$ ... K

b) $\sum_{n=0}^{\infty} \frac{3\cdot(-1)^n}{4^n} = \sum_{n=0}^{\infty} 3\cdot\left(-\frac{1}{4}\right)^n = \frac{3}{1-(-\frac{1}{4})} = \frac{12}{5}$ ... K

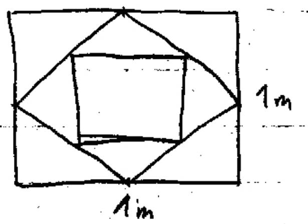
c) $\sum_{n=0}^{\infty} 5^n = 1+5+25+125+\dots = \infty$ ... D

d) $\sum_{n=0}^{\infty} \frac{3^n-2}{6^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty} \frac{2}{6^n} = \frac{1}{1-\frac{1}{2}} - \frac{2}{1-\frac{1}{6}} = 2 - \frac{12}{5} = -\frac{2}{5}$

aby jakákoli nek. řada konvergovala, musí platit $\lim_{n\to\infty} a_n = 0$ ... nutná podmínka konvergence

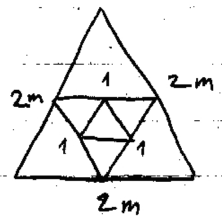
př 2 Určete ~~obsahů~~ součet obsahů čtverců na obrázku.

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots = \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} 1\cdot\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2\,m^2$



př 3 Určete max délku drátu potřebného ke zhotovení obrazce.

$6+3+\frac{3}{2}+\frac{3}{4}+\dots = 6+\sum_{n=0}^{\infty} \frac{3}{2^n} = 6+\frac{3}{1-\frac{1}{2}} = 6+6 = 12\,m$



nebo $= \sum_{n=0}^{\infty} 6\cdot\left(\frac{1}{2}\right)^n = \frac{6}{1-\frac{1}{2}} = 12\,m$

př 4 Určete součet řad

a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \quad / \cdot n(n+1)$

$1 = A(n+1) + Bn$

$n:\ 0 = A+B$

$k:\ 1 = A \Rightarrow B = -1$

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) =$$

$$= 1 - \frac{1}{n+1}$$

$$s = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

b) $$\sum_{n=1}^{\infty} \frac{1}{n^3+3n^2+2n} = \sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}\right)$$

$$\frac{1}{n^3+3n^2+2n} = \frac{1}{n(n^2+3n+2)} = \frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} \quad / \cdot n(n+1)(n+2)$$

$$1 = A(n^2+3n+2) + B(n^2+2n) + C(n^2+n)$$

n^2: $0 = A + B + C$

n: $0 = 3A + 2B + C$

k: $1 = 2A \quad \rightarrow A = \frac{1}{2}$

$-\frac{1}{2} = B + C$

$-\frac{3}{2} = 2B + C$

$-1 = B \quad C = \frac{1}{2}$

$$s_n = \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{4} + \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{5} + \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{6} + \frac{1}{14}\right) + \dots$$

$$\dots + \left(\frac{1}{2(n-2)} - \frac{1}{n-1} + \frac{1}{2n}\right) + \left(\frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}\right) + \left(\frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}\right) = \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$s = \lim_{n\to\infty}\left(\frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{n+1} + \frac{1}{2(n+2)}\right) = \frac{1}{4} + 0 - 0 + 0 = \frac{1}{4} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^3+3n^2+2n} = \frac{1}{4}$$

c) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(1) $s_n = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \dots + \frac{n}{2^n} \quad / \cdot \frac{1}{2}$

(2) $\frac{1}{2} s_n = \quad \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \dots + \frac{n-1}{2^n} + \frac{n}{2^{n+1}}$

(1)-(2) $\frac{1}{2} s_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} - \frac{n}{2^{n+1}} \quad / \cdot 2$

$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}} - \frac{n}{2^n}$

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \Big(\overbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}}}^{\text{geometrická řada}} - \frac{n}{2^n}\Big) = \overbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots}^{\text{nekonečná geom. řada}} - \lim_{n\to\infty} \frac{n}{2^n} = \left\| \frac{\infty}{\infty} \right\| \overset{L.P.}{=}$$

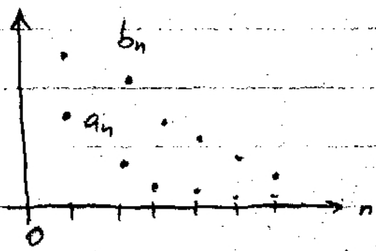
$$= \frac{1}{1-\frac{1}{2}} - \lim_{n\to\infty} \frac{1}{2^n \cdot \ln 2} = 2 - 0 = 2 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{n}{2^n} = 2$$

A) Řady s nezápornými členy

- konvergují nebo divergují k $+\infty$

1) Srovnávací kritérium

$\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ posloupnosti nezáporných čísel, $a_n \leq b_n$ pro $n \in \{N, N+1, \dots\}$



$$\sum_{n=0}^{\infty} b_n \ K \Rightarrow \sum_{n=0}^{\infty} a_n \ K$$

$$\sum_{n=0}^{\infty} a_n \ D \Rightarrow \sum_{n=1}^{\infty} b_n \ D$$

(př. 5) Rozhodněte o konvergenci či divergenci řad:

a) $\sum_{n=1}^{\infty} \frac{1}{n \cdot 4^n}$ — řada $\sum_{n=1}^{\infty} \frac{1}{4^n}$ K $\oplus$ $\frac{1}{n 4^n} \leq \frac{1}{4^n}$ pro $\forall n \in N \Rightarrow$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n \cdot 4^n} \ K$$

b) $\sum_{n=1}^{\infty} \frac{n^2+5}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n} + \frac{5}{n^3}$ — řada $\sum_{n=1}^{\infty} \frac{1}{n}$ D $\oplus$ $\frac{1}{n} + \frac{5}{n^3} \geq \frac{1}{n}$ $\forall n \in N \Rightarrow$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2+5}{n^3} \ D$$

2) <u>Integrální kritérium</u>

$$\sum_{n=0}^{\infty} a_n \ K \iff \int_N^{\infty} f(x)\,dx \ K \qquad N \text{ ... nějaké z } \langle 0,\infty)$$

$$D \qquad\qquad = \infty$$

(př 6) <u>Rozhodněte o konvergenci či divergenci řad</u> a) $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\int_{1}^{\infty} \frac{1}{x}\,dx = \left[\ln x\right]_1^{\infty} = \infty - 1 = \infty \Rightarrow \text{řada } \sum_{n=1}^{\infty} \frac{1}{n} \ D$$

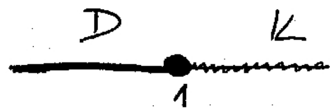
(1) zvolíme

b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = -\left[\frac{1}{x}\right]_1^{\infty} = -(0-1) = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \ K$$

obecně $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ ... konverguje pro $\alpha > 1$

divergují pro $\alpha \leqq 1$



3) <u>Podílové kritérium</u>

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q \qquad \begin{array}{l} q < 1 \ \dots\ K \\ q > 1,\ q = \infty \ \dots\ D \\ q = 1 \ \dots\ \text{nelze rozhodnout} \end{array}$$

(př 7) Rozhodněte o konvergenci či divergenci řad

a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)\cdot n!\, n^n}{(n+1)^{n+1}\cdot n!} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} =$$

$$= \lim_{n\to\infty} \frac{1}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{e}} = \frac{1}{e} < 1 \Rightarrow \text{řada } \underline{\text{konverguje}}$$

b) $\sum_{n=1}^{\infty} \frac{n!}{n^2}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^2}}{\frac{n!}{n^2}} = \lim_{n\to\infty} \frac{(n+1)\cdot n!\cdot n^2}{(n+1)^2\, n!} = \lim_{n\to\infty} \frac{n^2}{n+1} = \left\| \frac{\infty}{\infty} \right\| \overset{LP}{=} \lim_{n\to\infty} \frac{2n}{1} = \infty \Rightarrow$$

$\Rightarrow$ řada <u>diverguje</u>

4) Odmocninové kritérium

$$\lim_{n\to\infty} \sqrt[n]{a_n} = q \qquad \begin{array}{l} q < 1 \; \ldots\ldots \; K \\ q > 1 \; (= \infty) \; \ldots\ldots \; D \\ q = 1 \; \ldots\ldots \; \text{nelze rozhodnout} \end{array}$$

(př 8) Rozhodněte o konvergenci či divergenci řad

a) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1 \Rightarrow \text{řada konverguje}$$

b) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)^{n^2}\right]^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n^2\cdot\frac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1 \Rightarrow$$

$\Rightarrow$ řada diverguje

B) Alternující řady

jsou tvaru

$$\sum_{n=0}^{\infty} (-1)^n a_n, \qquad a_n \geq 0$$

← členy pravidelně střídají znaménka

Leibnitzovo kritérium

Je-li $\{a_n\}$ nerostoucí posloupnost kladných čísel ⊕ platí $\lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{n=0}^{\infty} (-1)^n a_n$ K

(př 9) Ukažte, že následující řady konvergují

a) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

$$\left.\begin{array}{l} \{\frac{1}{n}\} \text{ nerostoucí } \checkmark \\ \frac{1}{n} > 0 \quad \checkmark \\ \lim_{n\to\infty} \frac{1}{n} = 0 \quad \checkmark \end{array}\right\} \Rightarrow K$$

b) $\sum_{n=1}^{\infty} (-1) \frac{1}{n^2}$

$$\left.\begin{array}{l} \{\frac{1}{n^2}\} \text{ nerostoucí } \checkmark \\ \frac{1}{n^2} > 0 \quad \checkmark \\ \lim_{n\to\infty} \frac{1}{n^2} = 0 \quad \checkmark \end{array}\right\} \Rightarrow K$$

absolutní / relativní konvergence

Def.: $\sum a_n$ konverguje absolutně, jestliže $\sum |a_n|$ K

relativně D

př $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ ...... K
$\sum_{n=1}^{\infty} \frac{1}{n}$ ...... D
$\Rightarrow \sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ konverguje relativně

$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ ...... K
$\sum_{n=1}^{\infty} \frac{1}{n^2}$ ...... K
$\Rightarrow \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ konverguje absolutně

př 10 Rozhodněte, zda řada konverguje absolutně, relativně nebo nekonverguje.

a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+3}$

$\left\{\frac{1}{2n+3}\right\}$ nerostoucí ✓
$\frac{1}{2n+3} > 0 \quad \forall n \in \mathbb{N}$ ✓
$\lim_{n\to\infty} \frac{1}{2n+3} = 0$ ✓
$\Rightarrow \sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+3}$ K

- $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ $\quad \int_1^{\infty} \frac{1}{2x+3}dx = \left| \begin{array}{ll} t=2x+3 & x=1 \to t=5 \\ dt = 2dx & x=\infty \to t=\infty \\ \frac{1}{2}dt = dx & \end{array} \right| = \frac{1}{2}\int_5^{\infty} \frac{1}{t}\, dt = \frac{1}{2}\left[\ln|t|\right]_5^{\infty} =$

$= \frac{1}{2}(\infty - \ln 5) = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{2n+3}$ D

$\Rightarrow$

$\Rightarrow$ řada $\sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+3}$ K relativně

b) $\sum_{n=2}^{\infty} \frac{1}{n\cdot \ln^2 n}\cdot(-1)^{n+1}$

$$\left.\begin{array}{l} \left\{\frac{1}{n\cdot\ln^2 n}\right\} \text{ nerostoucí } \checkmark \\ \frac{1}{n\cdot\ln^2 n} > 0 \quad \forall n \geq 2 \ \checkmark \\ \lim\limits_{n\to\infty} \frac{1}{n\cdot\ln^2 n} = \left\| \frac{1}{\infty} \right\| = 0 \ \checkmark \end{array}\right\} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n\cdot\ln^2 n}\cdot(-1)^{n+1} \ K$$

• $\sum_{n=2}^{\infty} \frac{1}{n\cdot\ln^2 n}$ $\quad \int_2^{\infty} \frac{1}{x\cdot\ln^2 x}\,dx = \left| \begin{array}{ll} t=\ln x & x=2 \Rightarrow t=\ln 2 \\ dt=\frac{1}{x}dx & x=\infty \Rightarrow t=\infty \end{array}\right| = \int_{\ln 2}^{\infty} \frac{1}{t^2}\,dt = -\left[\frac{1}{t}\right]_{\ln 2}^{\infty} =$

$$= -\left(0 - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n\cdot\ln^2 n} \ K$$

$\Rightarrow$

$\Rightarrow$ řada $\sum_{n=2}^{\infty} \frac{1}{n\cdot\ln^2 n}\cdot(-1)^{n+1}$ <u>K absolutně</u>

c) $\sum_{n=1}^{\infty} \frac{\sin\frac{\pi}{n}}{n^2}$ $\qquad \frac{\pi}{n} = \pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \dots \Rightarrow \frac{\pi}{n} \in \langle 0, \pi \rangle \Rightarrow \sin\frac{\pi}{n} \in \langle 0,1 \rangle$

$$\frac{\overset{\langle 0,1\rangle}{\sin\frac{\pi}{n}}}{n^2} \leq \frac{1}{n^2} \oplus \sum_{n=1}^{\infty} \frac{1}{n^2} \ K \Rightarrow \sum_{n=1}^{\infty} \frac{\sin\frac{\pi}{n}}{n^2} \ K$$

• $\sum_{n=1}^{\infty} \left|\frac{\sin\frac{\pi}{n}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{\sin\frac{\pi}{n}}{n^2} \ldots K$

$\Rightarrow$ řada $\sum_{n=1}^{\infty} \frac{\sin\frac{\pi}{n}}{n^2}$ <u>K absolutně</u>

d) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

platí Věta: $\boxed{\text{pokud } \sum_{n=1}^{\infty} |a_n| \ K \Rightarrow \sum_{n=1}^{\infty} a_n \ K}$

k tomu, aby $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ konvergovala, stačí, aby konvergovala $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| = \sum \frac{|\sin n|}{n^2}$$

$$\frac{\overset{\langle 0,1\rangle}{|\sin n|}}{n^2} \leq \frac{1}{n^2} \oplus \sum_{n=1}^{\infty} \frac{1}{n^2} \ K \Rightarrow \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \ K \Rightarrow \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \ K$$

a protože konverguje řada $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ i řada $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$ $\Rightarrow$ řada $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ <u>K absolutně</u>

12. cvičení

Mocninné řady

sčítáme čísla → číselné řady

sčítame funkce → řady funkcí (jedněmi z nich jsou mocninné řady)

tvaru: $\sum_{n=0}^{\infty} a_n x^n$ ... mocninná řada se středem $x_0 = 0$, $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ ... se středem v x_0

$a_1, a_2, \dots, a_n, \dots$ – koeficienty

např.: $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots + \dots = \sum_{n=0}^{\infty} (n+1) x^n$

př 11 Určete součet řad

a) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \underline{\underline{\frac{1}{1-x}}}$ pro $x \in (-1,1)$

↑ geometrická řada s kvocientem x

b) $\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots = \frac{1}{1-(-x)} = \underline{\underline{\frac{1}{1+x}}}$ $x \in (-1,1)$

poloměr konvergence $r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$ ... pokud existuje

div. | konverguje abs. | div

$-r$ | 0 (střed) | r

nutno rozhodnout v $\underline{\underline{\pm r}}$

př 12 Určete poloměr konvergence, obor konvergence

a) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{2^n}{n^2}}{\frac{2^{n+1}}{(n+1)^2}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \underline{\underline{\frac{1}{2}}}$$ ... poloměr konvergence

- $x = \frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{2^n}{n^2} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ ... K
- $x = -\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{2^n}{n^2} \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ ... K (dle Leibnitzova kritéria)

celkem – obor konvergence $\underline{\underline{\langle -\frac{1}{2}, \frac{1}{2} \rangle}}$

b) $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)\cdot 3^n}$

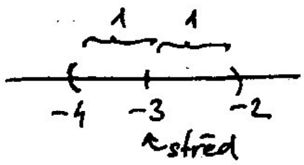
$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)3^n}}{\frac{1}{(n+2)3^{n+1}}} \right| = \lim_{n\to\infty} \frac{(n+2)\cdot 3}{(n+1)} \overset{\frac{\infty}{\infty}}{=} 3$$

- $x = 3$ $\quad \sum_{n=0}^{\infty} \frac{3^n}{(n+1)3^n} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} \quad \dots D$
- $x = -3$ $\quad \sum_{n=0}^{\infty} \frac{(-3)^n}{(n+1)3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \quad \dots K$

obor konvergence $\underline{\underline{\langle -3, 3)}}$

c) $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2}$

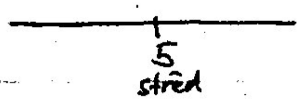
$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}} \right| = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{2n+2}{2n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{2}{2} = 1$$



- $x = -4$ $\quad \sum_{n=1}^{\infty} \frac{(-4+3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \dots K$
- $x = -2$ $\quad \sum_{n=1}^{\infty} \frac{(-2+3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \dots K$ $\qquad$ obor konvergence $\underline{\underline{\langle -4, -2 \rangle}}$

d) $\sum_{n=1}^{\infty} n^n (x-5)^n$

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^{n+1}} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} \cdot \lim_{n\to\infty} \frac{1}{n+1} = \left\| \frac{1}{e} \cdot \frac{1}{\infty} \right\| = 0$$



$x = 5$ $\quad \sum_{n=1}^{\infty} n^n (5-5)^n = \sum_{n=1}^{\infty} n^n \cdot 0 = \sum_{n=1}^{\infty} 0 = 0+0+0+\dots = 0 \quad \dots K$

obor konvergence $\{5\}$

e) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim_{n\to\infty} \frac{(n+1)\cdot n!}{n!} = \lim_{n\to\infty} (n+1) = \infty$$

obor konvergence $\underline{\underline{(-\infty,\infty)}}$

mejme mocninnou radu $\sum_{n=0}^{\infty} a_n x^n$, $r>0$ nebo $r=\infty$

- součet $s(x) = \sum_{n=0}^{\infty} a_n x^n$ je spojitý na $(-r,r)$; pokud $\sum a_n x^n$ K v r ... $s(r) = \lim_{x\to r^-} s(x)$
  $-r$ $s(-r) = \lim_{x\to r^+} s(x)$

- derivace $\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} (a_n x^n)'$
- integrál $\int \sum_{n=0}^{\infty} a_n x^n \, dx = \sum_{n=0}^{\infty} \int a_n x^n \, dx$

nemění se poloměr konvergence

př 13 Určete obor konvergence a součet řady $\sum_{n=1}^{\infty} n x^{n-1}$.
Pomocí získaného výpočtu určete součet řady .... $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$$

$x=-1$ $\quad \sum_{n=1}^{\infty} n(-1)^{n-1} = 1-2+3-4+5-6+\ldots$ nekonverguje

$x=1$ $\quad \sum_{n=1}^{\infty} n\, 1^{n-1} = \sum_{n=1}^{\infty} n$ ...... D

$\rightarrow (-1,1)$

$$\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} (x^n)' = \left(\sum_{n=1}^{\infty} x^n\right)' = (x + x^2 + x^3 + \ldots)' = \left(\frac{x}{1-x}\right)' = \frac{1\cdot(1-x) - x(-1)}{(1-x)^2} = \underline{\underline{\frac{1}{(1-x)^2}}}$$

geom. řada s kvocientem x

pro $(-1,1)$

pro $x = \frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{(1-\frac{1}{2})^2} = \underline{\underline{4}}$$

př 14 Určete obor konvergence a součet řady $\sum_{n=1}^{\infty} 2n\,x^{2n-1}$.
Pomocí získaného výpočtu určete součet řady $\sum_{n=1}^{\infty} \frac{n}{2^{2n-2}}$

kde řada $\sum_{n=1}^{\infty} 2n\,x^{2n-1}$ konverguje? $\to$ podílové kritérium

(vzorec $\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ nelze použít, protože máme x^{2n-1})

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)\,x^{2(n+1)-1}}{2n\,x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{2n+1}{2n}\cdot\frac{x^{2n+1}}{x^{2n-1}}\right| = |1\cdot x^2| = |x^2| = x^2$$

řada K tam, kde $x^2 < 1 \Rightarrow (-1,1)$

$x = -1$ $\quad \sum_{n=1}^{\infty} 2n\underbrace{(-1)^{2n-1}}_{=-1} = \sum_{n=1}^{\infty}(-2n)$ .... D

celkem $\underline{(-1,1)}$

$x = 1$ $\quad \sum_{n=1}^{\infty} 2n\,(1)^{2n-1} = \sum_{n=1}^{\infty} 2n$ .... D

$$\sum_{n=1}^{\infty} 2n\,x^{2n-1} = \sum_{n=1}^{\infty}(x^{2n})' = \left(\sum_{n=1}^{\infty} x^{2n}\right)' = (x^2+x^4+x^6+\dots)' = \left(\frac{x^2}{1-x^2}\right)' = \frac{2x(1-x^2)-x^2\cdot(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

$$\sum_{n=1}^{\infty}\frac{n}{2^{2n-2}} = \sum_{n=1}^{\infty} 2n\left(\frac{1}{2}\right)^{2n-1} = \frac{2\cdot\frac{1}{2}}{\left(1-\left(\frac{1}{2}\right)^2\right)^2} = \frac{1}{\left(\frac{3}{4}\right)^2} = \underline{\underline{\frac{16}{9}}}$$

př 15 Určete obor konvergence a součet řady $\sum_{n=1}^{\infty}\frac{x^n}{n}$
Pomocí získaného výpočtu určete součet řady $\sum_{n=1}^{\infty}\frac{1}{n\cdot 3^n}$

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{n+1}{n} = 1$$

$x = -1$ $\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ .... K

$x = 1$ $\quad \sum_{n=1}^{\infty}\frac{1}{n}$ .... D $\qquad$ celkem $\langle -1,1)$

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\int x^{n-1}dx = \int\sum_{n=1}^{\infty}x^{n-1}dx = \int(1+x+x^2+\dots)\,dx = \int\frac{1}{1-x}dx = -\ln|1-x|+C$$

↑ geom. řada s kvocientem x, $x\in(-1,1)$

pro $x=0$ je součet řady 0 : $0=-\ln|1-0|+c \Rightarrow c=0$

původní řada však K ještě v -1 : $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ $\quad S(-1)=\lim_{x\to -1^+}(-\ln|1-x|)=-\ln 2$

celkově tedy platí : $\sum_{n=1}^{\infty}\frac{x^n}{n}=-\ln|1-x| \quad$ pro $x\in\langle -1,1)$

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot 3^n}=\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)^n}{n}=-\ln\left|1-\frac{1}{3}\right|=-\ln\frac{2}{3}=\ln\left(\frac{2}{3}\right)^{-1}=\underline{\underline{\ln\frac{3}{2}}}$$

(př 16) Určete obor konvergence a součet řady $x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots \quad \left(=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\right)$

opět použijeme podílového kritéria (viz (př 14))

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)+1}}{2(n+1)+1}}{\frac{x^{2n+1}}{2n+1}}\right|=\lim_{n\to\infty}\left|\frac{2n+1}{2n+3}\cdot\frac{x^{2n+3}}{x^{2n+1}}\right|=\quad|1\cdot x^2|=|x^2|=x^2$$

řada konverguje tam, kde $x^2<1 \Rightarrow (-1,1)$

$$x=-1 \qquad \sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}\cdot\underbrace{(-1)^{2n}}_{=1}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}\ \ldots\ K$$

$$x=1 \qquad \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\ \ldots\ K \qquad \text{celkově } \langle -1,1\rangle$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\int x^{2n}\,dx=\int\sum_{n=0}^{\infty}(-1)^n x^{2x}\,dx=\int(1-x^2+x^4-x^6+\ldots)\,dx=$$

geom. řada s kvocientem $\underline{-x^2}$

$$=\int\frac{1}{1+x^2}\,dx=\operatorname{arctg}x+c \qquad \ldots \text{pro } (-1,1)$$

součet řady pro $x=0$ je 0 $\Rightarrow 0=\operatorname{arctg}0+c \Rightarrow c=0$

pro $x=-1 \qquad \sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{2n+1}}{2n+1}=\lim_{x\to -1^+}(\operatorname{arctg}x)=-\frac{\pi}{4}$

pro $x=1 \qquad \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}=\lim_{x\to 1^-}(\operatorname{arctg}x)=\frac{\pi}{4}$

celkem: $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=\operatorname{arctg}x \qquad$ pro $x\in\langle -1,1\rangle$

Taylorova řada

– je mocninná řada

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{\text{Taylorův polynom}} + \ldots$$

Taylorova řada

pro $x_0 = 0$ ... Maclaurinova řada (resp. polynom)

př. 17 Rozviňte funkce do mocninných řad

a) $f(x) = \sin x$, $x_0 = 0$

$f(x) = \sin x$ $\quad f'(x) = \cos x$ $\quad f''(x) = -\sin x$ $\quad f'''(x) = -\cos x$ $\quad f^{(4)}(x) = \sin x$

$f(x_0) = 0$ $\quad f'(x_0) = 1$ $\quad f''(x_0) = 0$ $\quad f'''(x_0) = -1$ $\quad f^{(4)}(x_0) = 0$ $\quad \ldots$

$$\sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \ldots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

obor konvergence?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)!}{(2n+3)!}\cdot\frac{x^{2n+3}}{x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{(2n+3)(2n+2)}\cdot x^2\right| = 0 < 1$$

vždy ✓

$\Rightarrow \underline{(-\infty, \infty)}$

b) $f(x) = e^x \quad , \quad x_0 = 0$

$$f(x) = f'(x) = f''(x) = \dots = f^{(n)}(x) = \dots = e^x$$

$$f(x_0) = f'(x_0) = f''(x_0) = \dots = f^{(n)}(x_0) = \dots = 1$$

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

obor konvergence ?

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} \frac{(n+1)n!}{n!} = \lim_{n\to\infty} (n+1) = \infty \Rightarrow \underline{(-\infty, \infty)}$$

př 18 Vypočtete

$$\int \frac{\sin x}{x} dx \qquad \Big\| \quad \frac{\sin x}{x} = \frac{1}{x} \cdot \sin x = \frac{1}{x} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

$$\int \frac{\sin x}{x} dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)}$$

př 19 Ze znalosti Taylorova rozvoje funkce $f(x) = e^x$ odvoďte Taylorův rozvoj pro funkci $f(x) = e^{-x^2}$

víme: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Longrightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \underline{\underline{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}}}$

místo x dosadíme $-x^2$

13. cvičení

Diferenciální rovnice

- obsahují y' (popř. y'', y''', ...)

(A) rovnice se separovatelnými proměnnými – typu $y' = f(x) \cdot g(y)$

(př 1) Řešte diferenciální rovnici

$$y' = \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1 \quad / \cdot 2$$

$$y^2 = x^2 + c \quad \text{... obecné řešení}$$

(př 2) Řešte diferenciální rovnici

$$y' = (2-y)\,\mathrm{tg}\,x$$

$$\frac{dy}{dx} = (2-y)\,\mathrm{tg}\,x \quad / \cdot dx$$

$$dy = (2-y)\,\mathrm{tg}\,x \quad / : (2-y) \quad \rightarrow \quad 2-y \neq 0$$

$$\boxed{y \neq 2}$$

$$\int \frac{1}{2-y}\,dy = \int \mathrm{tg}\,x\,dx$$

$$-\int \frac{-1}{2-y}\,dy = -\int \frac{-\sin x}{\cos x}\,dx$$

$$-\ln|2-y| = -\ln|\cos x| + c_1 \quad / \cdot (-1)$$

$$\ln|2-y| = \ln|\cos x| + c_2$$

$$e^{\ln|2-y|} = e^{\ln|\cos x| + c_2}$$

$$|2-y| = e^{\ln|\cos x|} \cdot e^{c_2}$$

$$|2-y| = |\cos x| \cdot c_3 \quad , \quad c_3 \geqq 0$$

$$2-y = c_4 \cdot \cos x \qquad c_4 \in \mathbb{R}$$

$$-y = c_4 \cdot \cos x - 2 \quad / \cdot (-1)$$

obecné řešení .... $y = 2 + c \cdot \cos x$

není náhodou $y = 2$ řešením původní rovnice?

$$(2)' = (2-2) \cdot \mathrm{tg}\,x$$

$$0 = 0 \ \checkmark \qquad \Rightarrow \text{ano je}$$

a je obsaženo v obecném řešení?

→ ano je, pro $c = 0$

vše je v pořádku ✓

(př3) Řešte diferenciální rovnici

$$\underline{1+y^2 - xy.(1+x^2)\,y' = 0}$$

$$xy(1+x^2)\,y' = 1+y^2$$

$$xy(1+x^2)\,\frac{dy}{dx} = 1+y^2 \quad /\cdot dx$$

$$xy(1+x^2)\,dy = (1+y^2)\,dx \quad /: x(1+x^2)$$

$$y\,dy = \frac{1+y^2}{x(1+x^2)}\,dx \quad /:(1+y^2)$$

$$\int \frac{y}{1+y^2}\,dy = \int \frac{1}{x(1+x^2)}\,dx$$

$$\frac{1}{2}\int \frac{2y}{1+y^2}\,dy = \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$$

$$\frac{1}{2}\ln|1+y^2| = \ln|x| - \frac{1}{2}\ln|1+x^2| + c_1 \quad /\cdot 2$$

$$\ln|1+y^2| = \ln|x|^2 - \ln|1+x^2| + c_2$$

$$\ln|1+y^2| = \ln\frac{x^2}{1+x^2} + c_2$$

$$e^{\ln(1+y^2)} = e^{\ln\frac{x^2}{1+x^2} + c_2}$$

$$1+y^2 = \frac{x^2}{1+x^2}\cdot c$$

$$\underline{\underline{(1+x^2)(1+y^2) = c\cdot x^2}}$$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \quad /\cdot x(1+x^2)$$

$$1 = A(1+x^2) + (Bx+C)\,x$$

$$1 = A(1+x^2) + Bx^2 + Cx$$

x^2: $0 = A+B$

x: $0 = C$

k: $1 = A$, $B = -1$

(B) homogenní – typu $y' = f\left(\frac{y}{x}\right)$ substituce: $u = \frac{y}{x} \;/\cdot x$

$ux = y \;/\,'$

$u'x + u = y'$

(př1) Řešte diferenciální rovnici

$$\underline{xy' = y \ln \frac{y}{x}}$$

$$y' = \frac{y}{x} \cdot \ln \frac{y}{x}$$

subst. $u = \frac{y}{x} \;/\cdot x$

$u \cdot x = y \;/\,'$

$u'x + u = y'$

$$u'x + u = u \cdot \ln u$$

$$u'x = u \cdot \ln u - u$$

$$\frac{du}{dx} \cdot x = u(\ln u - 1) \;/\cdot dx$$

$$du\, x = u(\ln u - 1)dx \;/ : x$$

$$du = u(\ln u - 1)\frac{dx}{x} \;/ : u(\ln u - 1)$$

$\boxed{u \neq 0} \Rightarrow \frac{y}{x} \neq 0 \Rightarrow \underline{y \neq 0}$

$\ln u - 1 \neq 0$

$\ln u \neq 1$

$\boxed{u \neq e} \Rightarrow \frac{y}{x} = e \Rightarrow \underline{y \neq ex}$

$$\int \frac{1}{u(\ln u - 1)}\,du = \int \frac{1}{x}\,dx$$

integrál $\int \frac{1}{u(\ln u - 1)}\,du$ vypočítáme zvlášť:

$$\int \frac{1}{u(\ln u - 1)}\,du = \left| \begin{array}{l} t = \ln u - 1 \\ dt = \frac{1}{u}\,du \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| = \ln|\ln u - 1|$$

zpět k rovnici:

$$\ln|\ln u - 1| = \ln|x| + c_1$$

$$e^{\ln|\ln u - 1|} = e^{\ln|x| + c_1}$$

$$|\ln u - 1| = |x| \cdot c_2 \quad , \; c_2 \geq 0$$

$$\ln u - 1 = x \cdot c \quad , \; c \in \mathbb{R}$$

$$\ln u = cx + 1$$

$$e^{\ln u} = e^{cx+1}$$

$$u = e^{cx+1}$$

$$\frac{y}{x} = e^{cx+1}$$

$$\underline{y = x \cdot e^{cx+1}}$$

↑ obecné řešení

• není náhodou $y = 0$ řešením původní rovnice?

$$x \cdot (0)' = 0 \cdot \ln \frac{0}{x}$$

$$0 = 0 \cdot \ln 0 \text{ – není definováno} \Rightarrow$$

$\Rightarrow y = 0$ vůbec neuvažujeme

• není náhodou $y = ex$ řešením původní rovnice?

$$x \cdot (ex)' = ex \cdot \ln \frac{ex}{x}$$

$$x \cdot e = ex \cdot \ln e$$

$$x \cdot e = ex \;\checkmark$$

a je obsaženo v obecném řešení pro nějakou konstantu c? → ANO, pro $c = 0$ ⇒

⇒ vše je v pořádku, nic nedopisujeme

př 5) Řešte diferenciální rovnici

$$y' = \frac{6y+4x}{y+6x}$$

$$y' = \frac{6\frac{y}{x}+4}{\frac{y}{x}+6}$$

$u = \frac{y}{x}$

$u \cdot x = y$

$u'x + u = y'$

$$u'x + u = \frac{6u+4}{u+6}$$

$$u'x = \frac{6u+4}{u+6} - u$$

$$u'x = \frac{6u+4-u(u+6)}{u+6}$$

$$u'x = \frac{-u^2+4}{u+6}$$

$$\frac{du}{dx}x = \frac{-u^2+4}{u+6} \quad /\cdot dx \cdot \frac{u+6}{-u^2+4}$$

$$\int \frac{u+6}{-u^2+4}\,du = \int \frac{1}{x}\,dx$$

$$\frac{u+6}{-u^2+4} = \frac{A}{2-u} + \frac{B}{2+u} \quad /\cdot(2-u)(2+u)$$

$u+6 = A(2+u) + B(2-u)$

$u: \; 1 = A - B \quad /\cdot 2$

$k: \; 6 = 2A + 2B$

$8 = 4A$

$A = 2, \; B = 1$

$$\int \left(\frac{2}{2-u} + \frac{1}{2+u}\right) = \ln|x| + c_1$$

$$-2\ln|2-u| + \ln|2+u| = \ln|x| + c_1$$

$$\ln\left|\frac{2+u}{(2-u)^2}\right| = \ln|x| + c_1$$

$$e^{\ln\frac{|2+u|}{(2-u)^2}} = e^{\ln|x|+c_1}$$

$$\frac{|2+u|}{(2-u)^2} = |x| \cdot c_2, \quad c_2 \geq 0$$

$$\frac{2+u}{(2-u)^2} = c \cdot x, \quad c \in \mathbb{R}$$

$$\frac{2+\frac{y}{x}}{(2-\frac{y}{x})^2} = c \cdot x \quad /\cdot x^2$$

$$\frac{2x^2+yx}{(2x-y)^2} = cx \quad /:x$$

$$\frac{2x+y}{(2x-y)^2} = c$$

$$\underline{\underline{2x+y = c\cdot(2x-y)^2, \; y = 2x}}$$

- $u \neq -6 \rightarrow \frac{y}{x} \neq -6 \rightarrow \underline{y \neq -6x}$

funkcí $y=6x$ se nezabýváme, protože ji nesmíme dosadit do původní rovnice (ve jmenovateli by byla 0)

- $u \neq 2 \rightarrow \frac{y}{x} \neq 2 \rightarrow y \neq 2x$

není náhodou $y = 2x$ řešením původní rovnice?

$(2x)' = \frac{6\cdot 2x+4x}{2x+6x}$

$2 = \frac{16x}{8x}$

$2 = 2 \Rightarrow$ ano, je; není však obsaženo v obecném řešení $\Rightarrow$ musíme jej dopsat

- $u \neq -2 \rightarrow \frac{y}{x} \neq -2 \rightarrow y \neq -2x$

není náhodou $y = -2x$ řešením původní rovnice?

$(-2x)' = \frac{6\cdot(-2x)+4x}{-2x+6x}$

$-2 = \frac{-8x}{4x}$

$-2 = -2$ ano, je; a je obsaženo v obecném řešení pro $c = 0$

ⓒ <u>lineární</u> – tvaru $y' = a(x)\,y + b(x)$ ... metoda variace konstanty

(př 6) Řešte diferenciální rovnici

$$\underline{y' + 2xy = x e^{-x^2}}$$

nejprve vyřešíme příslušnou homogenní rovnici (čili $b(x) = 0$)

$$y' + 2xy = 0$$
$$\frac{dy}{dx} = -2xy$$
$$\int \frac{dy}{y} = \int -2x\,dx$$
$$\ln|y| = -x^2 + c_1$$
$$e^{\ln|y|} = e^{-x^2 + c_1}$$
$$|y| = e^{-x^2} \cdot c_2\,, \quad c_2 \geq 0$$
$$y = c \cdot e^{-x^2} + \underline{c(x)\,e^{-x^2}}\,, \quad c \in \mathbb{R}$$

dosadíme do původní rovnice

$$\left(c(x)\,e^{-x^2}\right)' + 2x\,c(x)\,e^{-x^2} = x e^{-x^2}$$
$$c'(x)\,e^{-x^2} + \underbrace{c(x) \cdot e^{-x^2} \cdot (-2x) + 2x\,c(x)\,e^{-x^2}}_{=0} = x e^{-x^2}$$
$$c'(x)\,e^{-x^2} = x e^{-x^2}$$
$$c'(x) = x$$
$$c(x) = \int x\,dx$$
$$c(x) = \frac{x^2}{2}$$

celkem: $y = c \cdot e^{-x^2} + \frac{x^2}{2} e^{-x^2} \Rightarrow \underline{\underline{y = \left(\frac{x^2}{2} + c\right) e^{-x^2}}}$

(př 7) Řešte lineární rovnici

$$\underline{y' = -3y + x}$$

homogenní:

$$y' + 3y = 0$$
$$\frac{dy}{dx} = -3y$$
$$\int \frac{dy}{y} = \int -3\,dx$$
$$\ln|y| = -3x + c_1$$
$$e^{\ln|y|} = e^{-3x + c_1}$$
$$|y| = e^{-3x} \cdot c_2\,, \quad c_2 \geq 0$$
$$y = c e^{-3x} + c(x)\,e^{-3x}\,, \quad c \in \mathbb{R}$$

$$\left(c(x)\,e^{-3x}\right)' = -3c(x)\,e^{-3x} + x$$
$$c'(x)\,e^{-3x} + \underline{c(x)\,e^{-3x} \cdot (-3)} = \underline{-3c(x)\,e^{-3x}} + x$$
$$c'(x) \cdot e^{-3x} = x$$
$$c'(x) = x e^{3x}$$
$$c(x) = \int x e^{3x}\,dx = \left| \begin{matrix} u = x & v' = e^{3x} \\ u' = 1 & v = \frac{1}{3} e^{3x} \end{matrix} \right| =$$
$$= \frac{1}{3} x e^{3x} - \int \frac{1}{3} e^{3x}\,dx = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x}$$

celkem: $y = c \cdot e^{-3x} + \left(\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x}\right) e^{-3x}$

$$\underline{\underline{y = c\,e^{-3x} + \frac{1}{3}x - \frac{1}{9}}}$$