MB102 Matematika II
domácí úkoly

DÚ 1

1. Najděte polynom, který prochází body [-1,2], [0,1], [1,0], [2,5].

$f(x) = ax^3 + bx^2 + cx + d$
$2 = -a + b - c + d$
$1 = d$
$0 = a + b + c + d$
$\underline{5 = 8a + 4b + 2c + d}$ dosazením d = 1 (viz 2. rovnice) do zbývajících rovnic dostáváme:

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 8 & 4 & 2 & 4 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 12 & -6 & 12 \end{array}\right)$$

b = 0
c = -2
a = 1
d = 1

$\underline{f(x) = x^3 - 2x + 1}$

nebo (Lagrangeův interpolační polynom):

$$f(x) = 2\cdot\frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} + 1\cdot\frac{(x-(-1))(x-1)(x-2)}{(0-(-1))(0-1)(0-2)} + 0\cdot\ldots + 5\cdot\frac{(x-(-1))(x-0)(x-1)}{(2-(-1))(2-0)(2-1)} =$$

$$= -\tfrac{2}{6}x\left(x^2-3x+2\right) + \tfrac{1}{2}\left(x^2-1\right)(x-2) + \tfrac{5}{6}x\left(x^2-1\right) = -\tfrac{1}{3}\left(x^3-3x^2+2x\right) + \tfrac{1}{2}\left(x^3-2x^2-x+2\right) + \tfrac{5}{6}\left(x^3-x\right)$$

$$f(x) = \underline{\underline{x^3-2x+1}}$$

2. Najděte polynom, pro který platí: P(2) = 5, P(4) = 21, P’(-2) = -7.

$P(x) = ax^2 + bx + c$ $P'(x) = 2ax + b$
$5 = 4a + 2b + c$ $-7 = -4a + b$
$21 = 16a + 4b + c$

$$\left(\begin{array}{ccc|c} 4 & 2 & 1 & 5 \\ 16 & 4 & 1 & 21 \\ -4 & 1 & 0 & -7 \end{array}\right) \approx \left(\begin{array}{ccc|c} 4 & 2 & 1 & 5 \\ 0 & -4 & -3 & 1 \\ 0 & 3 & 1 & -2 \end{array}\right) \approx \left(\begin{array}{ccc|c} 4 & 2 & 1 & 5 \\ 0 & -4 & -3 & 1 \\ 0 & 0 & -5 & -5 \end{array}\right)$$

c = 1
b = -1
a = 3/2

$\underline{\underline{f(x) = 3/2x^2 - x + 1}}$

3. Rozložte na parciální zlomky

a) $$-\frac{\mathbf{x^2+4x+3}}{\mathbf{(x-1)(x^2+3)}}$$

$$\underline{-\frac{x^2+4x+3}{(x-1)(x^2+3)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+3}} \quad / \cdot (x-1)\left(x^2+3\right)$$

$$-x^2-4x-3 = A\left(x^2+3\right) + (Bx+C)(x-1)$$

$$-x^2-4x-3 = Ax^2+3A+Bx^2-Bx+Cx-C$$

$x^2:\ -1 = A + B$
$x:\ -4 = -B + C$
$\underline{k:\ -3 = 3A - C}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 3 & 0 & -1 & -3 \\ 0 & -1 & 1 & -4 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -3 & -1 & 0 \\ 0 & -1 & 1 & -4 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & -4 & 12 \end{array}\right)$$

c = -3
b = 1
a = -2

$$\underline{\underline{\frac{-x^2-4x-3}{(x-1)(x^2+3)} = \frac{-2}{x-1} + \frac{x-3}{x^2+3}}}$$

b) $\mathbf{\dfrac{-2x^2+10x+13}{x^3+4x^2-3x-18}}$

Nejprve je potřeba rozložit jmenovatele. Na první pohled nelze nic vytknout. Jedná se o polynom 3. stupně – první kořen je potřeba prostě uhádnout :-).
Je to číslo 2 (protože po dosazení čísla dva vyjde 0). Z toho plyne, že jmenovatel je dělitelný výrazem (x-2). Provedeme písemné dělení:

$$(x^3+4x^2-3x-18):(x-2)=x^2+6x+9$$
$$\underline{-(x^3-2x^2)}$$
$$6x^2-3x-18$$
$$\underline{-(6x^2-12x)}$$
$$9x-18$$
$$\underline{-(9x-18)}$$
$$0$$

$$\frac{-2x^2+10x+13}{x^3+4x^2-3x-18}=\frac{-2x^2+10x+13}{(x-2)(x^2+6x+9)}=\underline{\frac{-2x^2+10x+13}{(x-2)(x+3)^2}=\frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{(x+3)^2}}\quad /\cdot(x-2)(x+3)^2$$

$$-2x^2+10x+13=A(x+3)^2+B(x-2)(x+3)+C(x-2)$$
$$-2x^2+10x+13=A(x^2+6x+9)+B(x^2+x-6)+C(x-2)$$
$$x^2:\ -2=A+B$$
$$x:\quad 10=6A+B+C$$
$$\underline{k:\quad 13=9A-6B-2C}$$

$$\left(\begin{array}{ccc|c}1&1&0&-2\\6&1&1&10\\9&-6&-2&13\end{array}\right)\approx\left(\begin{array}{ccc|c}1&1&0&-2\\0&-5&1&22\\0&-15&-2&31\end{array}\right)\approx\left(\begin{array}{ccc|c}1&1&0&-2\\0&-5&1&22\\0&0&-5&-35\end{array}\right)$$

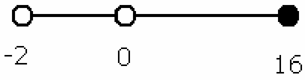
c = 7
b = -3
a = 1

$$\underline{\underline{\frac{-2x^2+10x+13}{x^3+4x^2-3x-18}=\frac{1}{x-2}-\frac{3}{x+3}+\frac{7}{(x+3)^2}}}$$

DÚ 2

1. Najděte infimum a supremum množin

a) A = (-2,16] – {0}



$\underline{\underline{\inf A = -2}}$
$\underline{\underline{\sup A = 16}}$

b) $\mathbf{B = \left\{\frac{n+1}{n}, n \in N\right\}}$

$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots, \frac{1001}{1000}, \dots$

$\underline{\underline{\inf B = 1}}$
$\underline{\underline{\sup B = 2}}$

2. Vypočtete limity

a) $$\lim_{x\to 2}\frac{x+4}{(x-2)^6} = \left\|\frac{6}{+0}\text{ pro obě jednostranné limity}\right\| = \underline{\underline{\infty}}$$

b) $$\lim_{x\to 2}\frac{3x-1}{x(x-2)^5} = \left\|\frac{5}{0}\right\| = \lim_{x\to 2}\frac{3x-1}{x}\cdot\lim_{x\to 2}\frac{1}{(x-2)^5} = \frac{5}{2}\cdot\lim_{x\to 2}\frac{1}{(x-2)^5} =$$

$$= \left\|\frac{5}{2}\cdot\frac{1}{+0} = +\infty \text{ pro limitu zprava, } \frac{5}{2}\cdot\frac{1}{-0} = -\infty \text{ pro limitu zleva}\right\| \Rightarrow \underline{\underline{\text{limita neexistuje}}}$$

c) $$\lim_{x\to 4}\frac{x^2-6x+8}{x^2-5x+4} = \left\|\frac{0}{0}\right\| = \lim_{x\to 4}\frac{(x-4)(x-2)}{(x-4)(x-1)} = \lim_{x\to 4}\frac{x-2}{x-1} = \frac{4-2}{4-1} = \underline{\underline{\frac{2}{3}}}$$

d) $$\lim_{x\to\infty}\frac{x^2-6x+8}{x^2-5x+4} = \left\|\frac{\infty}{\infty}\text{ vydelime nejvyssi mocninou vyskytujici se ve jmenovateli(cili dole :-)}\right\| =$$

$$= \lim_{x\to\infty}\frac{1-\frac{6}{x}+\frac{8}{x^2}}{1-\frac{5}{x}+\frac{4}{x^2}} = \frac{1-0+0}{1-0+0} = \underline{\underline{1}}$$

e) $$\lim_{x\to 1}\frac{x^3-x^2-x+1}{x^3+x-2} \overset{\frac{0}{0}}{=} \lim_{x\to 1}\frac{x^2(x-1)-(x-1)}{(x-1)(x^2+x+2)} = \lim_{x\to 1}\frac{(x-1)(x^2-1)}{(x-1)(x^2+x+2)} = \lim_{x\to 1}\frac{x^2-1}{x^2+x+2} = \frac{1-1}{1+1+2} = \frac{0}{4} = \underline{\underline{0}}$$

f) $$\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos 2x} = \left\|\frac{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}{0} = \frac{0}{0}\right\| = \lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos 2x}\cdot\frac{\cos x+\sin x}{\cos x+\sin x} = \lim_{x\to\frac{\pi}{4}}\frac{\cos^2 x-\sin^2 x}{\cos 2x(\cos x+\sin x)} =$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\cos 2x}{\cos 2x(\cos x+\sin x)} = \lim_{x\to\frac{\pi}{4}}\frac{1}{\cos x+\sin x} = \frac{1}{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{2}} = \underline{\underline{\frac{\sqrt{2}}{2}}}$$

g) $$\lim_{x\to\infty}\frac{\sqrt{x^2+3x}}{\sqrt[3]{2x^3-2x}} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to\infty}\frac{\sqrt{1+\frac{3}{x}}}{\sqrt[3]{2-\frac{2}{x^2}}} = \frac{\sqrt{1+0}}{\sqrt[3]{2-0}} = \underline{\underline{\frac{1}{\sqrt[3]{2}}}}$$

h) $$\lim_{x\to\infty}(\sqrt{x^2+x+1}-\sqrt{x^2-4x+1}) = \|\infty-\infty\| = \lim_{x\to\infty}\frac{\sqrt{x^2+x+1}-\sqrt{x^2-4x+1}}{1}\cdot\frac{\sqrt{x^2+x+1}+\sqrt{x^2-4x+1}}{\sqrt{x^2+x+1}+\sqrt{x^2-4x+1}} =$$

$$= \lim_{x\to\infty}\frac{x^2+x+1-(x^2-4x+1)}{\sqrt{x^2+x+1}+\sqrt{x^2-4x+1}} = \lim_{x\to\infty}\frac{5x}{\sqrt{x^2+x+1}+\sqrt{x^2-4x+1}} = \lim_{x\to\infty}\frac{5}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}} = \frac{5}{1+1} = \underline{\underline{\frac{5}{2}}}$$

i) $$\lim_{x\to\infty}(\sqrt{x^2+7}+\sqrt{x^2-1}) = \|\infty+\infty\| = \underline{\underline{\infty}}$$

j) $$\lim_{x\to 0}\frac{\sin 6x-\ln(9x+1)+e^{2x}-1}{3x} = = \lim_{x\to 0}\left(\frac{\sin 6x}{3x}-\frac{\ln(9x+1)}{3x}+\frac{e^{2x}-1}{3x}\right) =$$

$$= \lim_{x\to 0}\left(\frac{\sin 6x}{6x}\cdot 2-\frac{\ln(9x+1)}{9x}\cdot 3+\frac{e^{2x}-1}{2x}\cdot\frac{2}{3}\right) = 1\cdot 2-1\cdot 3+1\cdot\frac{2}{3} = \underline{\underline{-\frac{1}{3}}}$$

DÚ 3

1. Najděte body nespojitosti a určete jejich druh.

a) $\mathbf{f(x) = \dfrac{x^3 - x^2}{x - 1}} = \dfrac{x^2(x-1)}{x-1}$

$$\lim_{x\to 1} \frac{x^2(x-1)}{x-1} = \lim_{x\to 1} x^2 = 1$$

f(x) není v bodě x_o = 1 definována a existuje vlastní limita, která se nerovná funkční hodnotě => $\underline{\underline{x_o = 1 \ldots \text{nespojitost odstranitelná}}}$

b) $\mathbf{f(x) = \dfrac{\ln(x+1)}{|x|}}$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln(x+1)}{x} = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\ln(x+1)}{-x} = -1$$

f(x) není v bodě x_o = 0 definována a neexistuje vlastní limita. Existují pouze obě vlastní jednostranné limity, které se však nerovnají => $\underline{\underline{x_o = 0 \ldots \text{nespojitost 1. druhu}}}$

c) $\mathbf{f(x) = e^{\frac{1}{x-3}}}$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} e^{\frac{1}{x-3}} = \left\| e^{\frac{1}{+0}} = e^{\infty} \right\| = \infty$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} e^{\frac{1}{x-3}} = \left\| e^{\frac{1}{-0}} = e^{-\infty} \right\| = 0$$

f(x) není v bodě x_o = 3 definována a neexistuje vlastní limita. Jedna z jednostranných limit je dokonce nevlastní => $\underline{\underline{x_o = 3 \ldots \text{nespojitost 2. druhu}}}$

2. Nalezněte derivaci funkce

a) $\mathbf{y = x^8 + \dfrac{3}{x^5} - \ln x} = x^8 + 3x^{-5} - \ln x$

$$y' = 8x^7 - 15x^{-6} - \frac{1}{x} = \underline{\underline{8x^7 - \frac{15}{x^6} - \frac{1}{x}}}$$

b) $\mathbf{y = (x^4 - 2x) \cdot \sin x}$

$$\underline{\underline{y' = (4x^3 - 2)\sin x + (x^4 - 2x)\cos x}}$$

c) $\mathbf{y = \dfrac{\sin 2x}{x^3}}$

$$y' = \frac{2\cos 2x \cdot x^3 - \sin 2x \cdot 3x^2}{(x^3)^2} = \underline{\underline{\frac{2x^3 \cos 2x - 3x^2 \sin 2x}{x^6}}}$$

d) $\mathbf{y = \sqrt[3]{\dfrac{1+x^3}{1-x^3}}} = \left(\dfrac{1+x^3}{1-x^3}\right)^{\frac{1}{3}}$

$$y' = \frac{1}{3}\left(\frac{1+x^3}{1-x^3}\right)^{-\frac{2}{3}} \cdot \frac{3x^2(1-x^3) - (1+x^3)(-3x^2)}{(1-x^3)^2} = \frac{1}{3}\,\frac{(1-x^3)^{\frac{2}{3}}}{(1+x^3)^{\frac{2}{3}}}\,\frac{3x^2 - 3x^5 + 3x^2 + 3x^5}{(1-x^3)^2} =$$

$$= \frac{1}{3}\frac{1}{(1+x^3)^{\frac{2}{3}}}\frac{6x^2}{(1-x^3)^{\frac{4}{3}}} = \frac{2x^2}{(1+x^3)^{\frac{2}{3}}(1-x^3)^{\frac{4}{3}}} = \underline{\underline{\frac{2x^2}{\sqrt[3]{(1+x^3)^2(1-x^3)^4}}}}$$

e) $\mathbf{y = \frac{1}{4}\ln\frac{x^2-1}{x^2+1}}$

$$y' = \frac{1}{4}\cdot\frac{1}{\frac{x^2-1}{x^2+1}}\cdot\frac{2x(x^2+1)-(x^2-1)\cdot 2x}{(x^2+1)^2} = \frac{1}{4}\cdot\frac{x^2+1}{x^2-1}\cdot\frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = \frac{1}{4}\cdot\frac{4x}{(x^2-1)(x^2+1)} = \underline{\underline{\frac{x}{x^4-1}}}$$

f) $\mathbf{y = \sqrt{\sin\sqrt{x}}} = \left(\sin(x)^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$$y' = \frac{1}{2}\left(\sin\sqrt{x}\right)^{-\frac{1}{2}}\cdot\cos\sqrt{x}\cdot\frac{1}{2}x^{-\frac{1}{2}} = \frac{\cos\sqrt{x}}{4\sqrt{\sin\sqrt{x}}\cdot\sqrt{x}} = \underline{\underline{\frac{\cos\sqrt{x}}{4\sqrt{x\cdot\sin\sqrt{x}}}}}$$

g) $\mathbf{y = x\cdot tg(\ln(x))}$

$$y' = 1\cdot tg(\ln(x)) + x\cdot\frac{1}{\cos^2(\ln(x))}\cdot\frac{1}{x} = \underline{\underline{tg(\ln(x)) + \frac{1}{\cos^2(\ln(x))}}}$$

h) $\mathbf{y^4 + \cot g y + \sin^3 5x = 0}$

$$4y^3\cdot y' - \frac{1}{\sin^2 y}\cdot y' + 3\sin^2 5x\cdot\cos 5x\cdot 5 = 0$$

$$y'\cdot\left(4y^3 - \frac{1}{\sin^2 y}\right) = -15\sin^2 5x\cdot\cos 5x$$

$$y' = -\frac{15\sin^2 5x\cdot\cos 5x}{4y^3 - \frac{1}{\sin^2 y}} = \underline{\underline{-\frac{15\sin^2 5x\cdot\cos 5x\cdot\sin^2 y}{4y^3\sin^2 y - 1}}}$$

3) Napište rovnici tečny funkce v daném bodě

a) $\mathbf{f(x) = tgx + x^3 + 2,}$ $\quad\mathbf{A = [0,?]}$ … $A = [0,2]$

$$f'(x) = \frac{1}{\cos^2 x} + 3x^2$$

$$k = f'(0) = \frac{1}{1^2} + 3\cdot 0^2 = 1$$

$t: y = kx + q$

$y = x + q$

$2 = 0 + q$

$q = 2$

$\underline{\underline{t: y = x + 2}}$

b) $\mathbf{f(x) = \sin x,}$ $\quad\mathbf{A = [\frac{\pi}{6}, ?]}$ … $A = [\frac{\pi}{6}, \frac{1}{2}]$

$f'(x) = \cos x$

$k = f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$t: y = kx + q$

$y = \frac{\sqrt{3}}{2}x + q$

$\frac{1}{2} = \frac{\sqrt{3}}{2}\cdot\frac{\pi}{6} + q$

$q = \frac{1}{2} - \frac{\pi\sqrt{3}}{12}$

$\underline{\underline{t: y = \frac{\sqrt{3}}{2}x + \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{12}\right)}}$

DÚ 4

1. Vypočtete limity

a) $$\lim_{x\to0^+}\frac{\ln x}{\ln(\sin x)}=\left\|\frac{\infty}{\infty}\right\|\overset{\text{L.P.}}{=}\lim_{x\to0^+}\frac{\frac{1}{x}}{\frac{1}{\sin x}\cdot\cos x}=\lim_{x\to0^+}\frac{\sin x}{x\cos x}=\left\|\frac{0}{0}\right\|\overset{\text{L.P.}}{=}\lim_{x\to0^+}\frac{\cos x}{\cos x-x\sin x}=\frac{1}{1-0\cdot0}=\underline{\underline{1}}$$

b) $$\lim_{x\to\infty}\frac{x^2-7x}{x^3-15x+6}=\left\|\frac{\infty}{\infty}\right\|\overset{\text{L.P.}}{=}\lim_{x\to\infty}\frac{2x-7}{3x^2-15}=\left\|\frac{\infty}{\infty}\right\|\overset{\text{L.P.}}{=}\lim_{x\to\infty}\frac{2}{6x}=\left\|\frac{2}{\infty}\right\|=\underline{\underline{0}}$$

c) $$\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\left\|\infty-\infty\right\|=\lim_{x\to0}\frac{e^x-1-x}{x\left(e^x-1\right)}=\left\|\frac{0}{0}\right\|\overset{\text{L.P.}}{=}\lim_{x\to0}\frac{e^x-1}{e^x-1+xe^x}=\left\|\frac{0}{0}\right\|\overset{\text{L.P.}}{=}\lim_{x\to0}\frac{e^x}{e^x+e^x+xe^x}=\underline{\underline{\frac{1}{2}}}$$

d) $$\lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\mathrm{tg}x=\left\|0\cdot\infty\right\|=\lim_{x\to\frac{\pi}{2}}\frac{\frac{\pi}{2}-x}{\frac{1}{\mathrm{tg}x}}=\left\|\frac{0}{0}\right\|\overset{\text{L.P.}}{=}\lim_{x\to\frac{\pi}{2}}\frac{-1}{-\frac{1}{\mathrm{tg}^2x}\cdot\frac{1}{\cos^2x}}=\lim_{x\to\frac{\pi}{2}}\frac{1}{\frac{\cos^2x}{\sin^2x}\cdot\frac{1}{\cos^2x}}=\lim_{x\to\frac{\pi}{2}}\sin^2x=\underline{\underline{1}}$$

e) $$\lim_{n\to\infty}\sqrt[n]{n}=\lim_{n\to\infty}n^{\frac{1}{n}}=\left\|\infty^0\right\|=e^{\ln\lim\limits_{n\to\infty}n^{\frac{1}{n}}}=e^{\lim\limits_{n\to\infty}\ln(n^{\frac{1}{n}})}=e^{\lim\limits_{n\to\infty}\frac{1}{n}\ln n}$$

$$\lim_{n\to\infty}\frac{1}{n}\ln n=\left\|0\cdot\infty\right\|=\lim_{n\to\infty}\frac{\ln n}{n}=\left\|\frac{\infty}{\infty}\right\|\overset{\text{L.P.}}{=}\lim_{n\to\infty}\frac{\frac{1}{n}}{1}=\lim_{n\to\infty}\frac{1}{n}=0\quad\Rightarrow\quad\lim_{n\to\infty}\sqrt[n]{n}=e^0=\underline{\underline{1}}$$

f) $$\lim_{x\to0}(\cos x)^{\mathrm{cotg}^2x}=\left\|1^\infty\right\|=e^{\ln\lim\limits_{x\to0}(\cos x)^{\mathrm{cotg}^2x}}=e^{\lim\limits_{x\to0}\ln(\cos x)^{\mathrm{cotg}^2x}}=e^{\lim\limits_{x\to0}\mathrm{cotg}^2x\cdot\ln(\cos x)}$$

$$\lim_{x\to0}\mathrm{cotg}^2x\cdot\ln(\cos x)=\left\|\infty\cdot0\right\|=\lim_{x\to0}\frac{\ln(\cos x)}{\frac{1}{\mathrm{cotg}^2x}}=\left\|\frac{0}{0}\right\|\overset{\text{L.P.}}{=}\lim_{x\to0}\frac{\frac{1}{\cos x}\cdot(-\sin x)}{-2\cdot\frac{1}{\mathrm{cotg}^3x}\cdot\left(-\frac{1}{\sin^2x}\right)}=$$

$$=\lim_{x\to0}\frac{-\sin x\cdot\sin^2x\cdot\mathrm{cotg}^3x}{2\cos x}=\lim_{x\to0}\frac{-\sin^3x\cdot\frac{\cos^3x}{\sin^3x}}{2\cos x}=\lim_{x\to0}\frac{-\cos^2x}{2}=-\frac{1}{2}\quad\Rightarrow\quad\lim_{x\to0}(\cos x)^{\mathrm{cotg}^2x}=e^{-\frac{1}{2}}=\underline{\underline{\frac{1}{\sqrt{e}}}}$$

g) $$\lim_{x\to0}(1-\cos x)^{\sin x}=\left\|0^0\right\|=e^{\ln\lim\limits_{x\to0}(1-\cos x)^{\sin x}}=e^{\lim\limits_{x\to0}\ln(1-\cos x)^{\sin x}}=e^{\lim\limits_{x\to0}\sin x\cdot\ln(1-\cos x)}$$

$$\lim_{x\to0}\sin x\cdot\ln(1-\cos x)=\left\|0\cdot\infty\right\|=\lim_{x\to0}\frac{\ln(1-\cos x)}{\frac{1}{\sin x}}=\left\|\frac{\infty}{\infty}\right\|\overset{\text{L.P.}}{=}\lim_{x\to0}\frac{\frac{1}{1-\cos x}\cdot\sin x}{-\frac{1}{\sin^2x}\cdot\cos x}=\lim_{x\to0}\frac{-\sin^3x}{(1-\cos x)\cos x}=\left\|\frac{0}{0}\right\|=$$

$$=\lim_{x\to0}\frac{\sin^3x}{-\cos x+\cos^2x}\overset{\text{L.P.}}{=}\lim_{x\to0}\frac{3\sin^2x\cdot\cos x}{\sin x-2\cos x\cdot\sin x}=\lim_{x\to0}\frac{3\sin^2x\cdot\cos x}{\sin x\cdot(1-2\cos x)}=\lim_{x\to0}\frac{3\sin x\cdot\cos x}{1-2\cos x}=\frac{0}{-1}=0$$

$$\Rightarrow\quad\lim_{x\to0}(1-\cos x)^{\sin x}=e^0=\underline{\underline{1}}$$

2. Určete, kde funkce klesá a roste, určete její lokální extrémy a definiční obor.

a) $y=e^{x^3-12x}$ D(f) = **R**

$$y'=e^{x^3-12x}\cdot\left(3x^2-12\right)=3e^{x^3-3x}\cdot\left(x^2-4\right)=$$
$$=3e^{x^3-3x}\cdot(x-2)(x+2)$$

x − 2	−		−		+
x + 2	−		+		+
		-2		2	
y′	+		−		+
	↗	max	↘	min	↗

b) $y=3x^4+4x^3-36x^2-7$ D(f) = **R**

$$y'=12x^3+12x^2-72x=12x\left(x^2+x-6\right)=$$
$$=12x(x-2)(x+3)$$

x	−		−		+		+
x − 2	−		−		−		+
x + 3	−		+		+		+
		-3		0		2	
y′	−		+		−		+
	↘	min	↗	max	↘	min	↗

DÚ 5

Vyšetřete průběh funkcí:

A) $\mathbf{y = 2x \cdot e^{x}}$

1) $D(f) = \mathbf{R}$

$f(-x) = -2x \cdot e^{-x} = -\dfrac{2x}{e^{x}} \Rightarrow$ ani sudá ani lichá

2) Kladná, záporná

e^{x} ... vždy >0

2x	−	0	+
y	−		+

3) Rostoucí, klesající, extrémy

$y' = 2e^{x} + 2xe^{x} = 2e^{x}(x+1)$

x + 1	−	-1	+
	−		+
y′	↘	min	↗

4) Konvexní, konkávní, inflexní body

$y'' = 2e^{x}(x+1) + 2e^{x} = 2e^{x}(x+2)$

x + 2	−	-2	+
	−		+
y″	∩	inf	∪

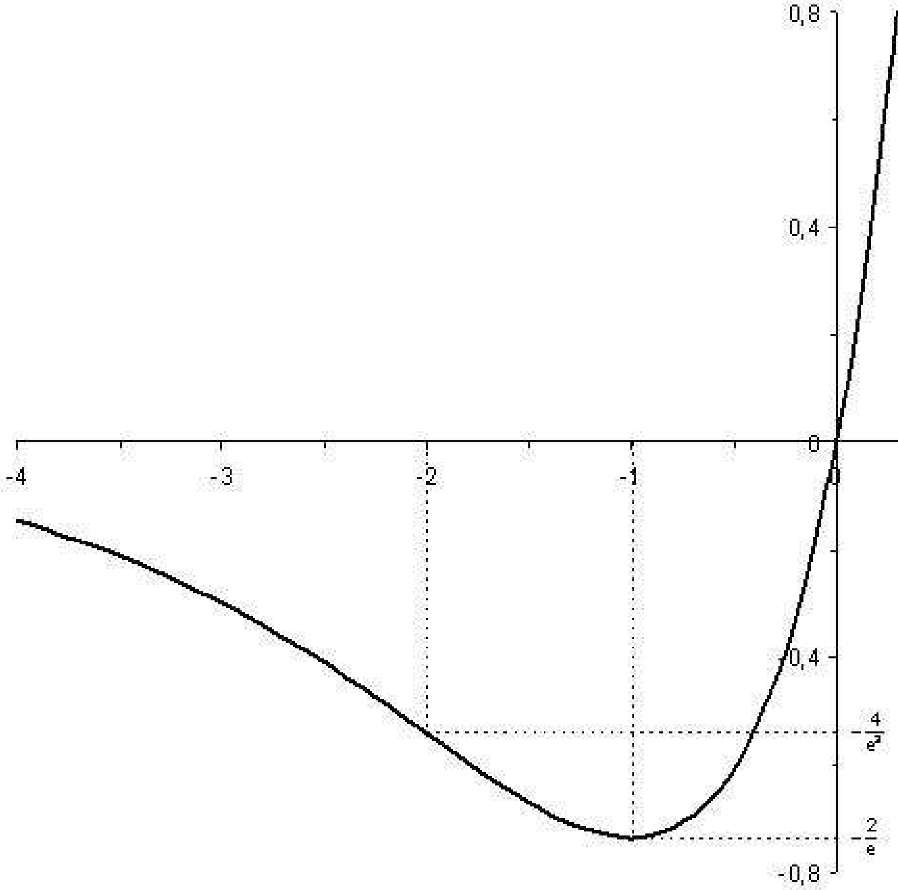
5) Asymptoty

BS ... neexistují

SS $a = \lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \dfrac{2xe^{x}}{x} = \lim\limits_{x\to\infty} 2e^{x} = \left\| 2e^{\infty} \right\| = \infty \Rightarrow$ asymptota bez směrnice pro ∞ neexistuje

$$\left.\begin{aligned} a &= \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{2xe^{x}}{x} = \lim_{x\to-\infty} 2e^{x} = \left\| 2e^{-\infty} = 2 \cdot 0 \right\| = 0 \\ b &= \lim_{x\to-\infty} \left(f(x) - a \cdot x \right) = \lim_{x\to-\infty} 2xe^{x} = \left\| \infty \cdot 0 \right\| = \lim_{x\to-\infty} \frac{2x}{e^{-x}} = \left\| \frac{\infty}{\infty} \right\| \overset{\text{L.P.}}{=} \lim_{x\to-\infty} \frac{2}{-e^{-x}} = \lim_{x\to-\infty} (-2e^{x}) = \left\| -2 \cdot 0 \right\| = 0 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow$ asymptota se směrnicí pro $-\infty$ je y = 0 (tedy osa x)



B) $\mathbf{y = x \cdot e^{\frac{1}{x}}}$

1) $D(f) = \mathbf{R} - \{0\}$

$f(-x) = -x \cdot e^{\frac{1}{-x}} \Rightarrow$ ani sudá ani lichá

2) Kladná, záporná $e^{\frac{1}{x}}$ ... vždy >0

x	–		+
		0	
y	–		+

3) Rostoucí, klesající, extrémy

$$y' = e^{\frac{1}{x}} + xe^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) = e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right) = e^{\frac{1}{x}}\,\frac{x-1}{x}$$

x - 1	–		–		+
x	–		+		+
		0		1	
y′	+		–		+
	↗		↘	min	↗

4) Konvexní, konkávní, inflexní body

$$y'' = e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)\left(1 - \frac{1}{x}\right) + e^{\frac{1}{x}}\,\frac{1}{x^2} = e^{\frac{1}{x}}\,\frac{1}{x^3}$$

x^3	–		+
		0	
	–		+
y″	∩		∪

5) Asymptoty

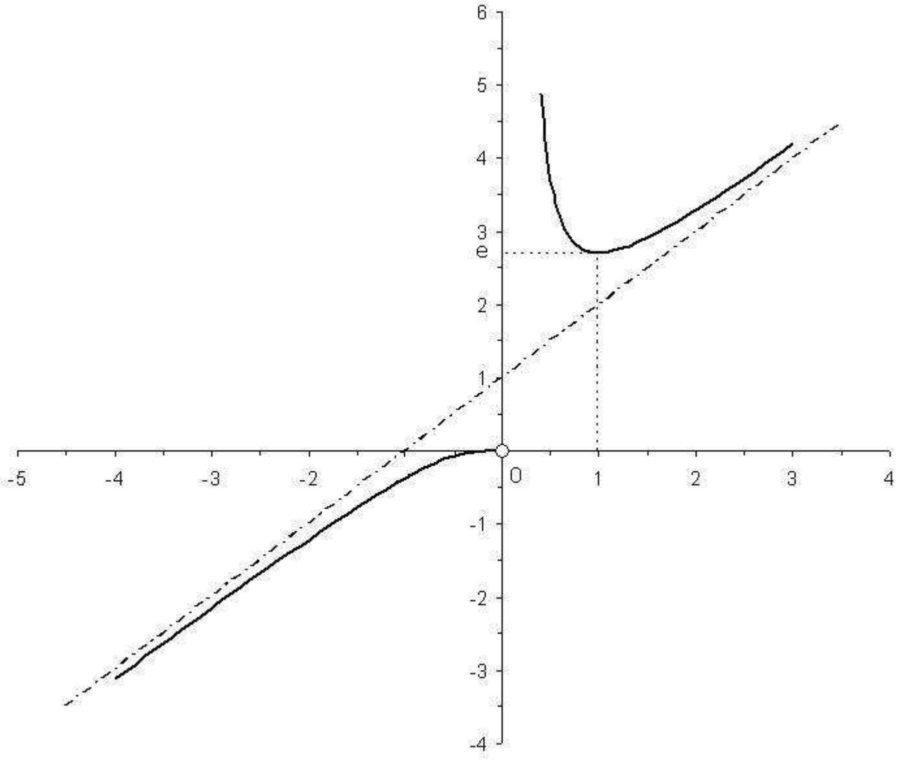
BS $\lim\limits_{x\to 0^+} xe^{\frac{1}{x}} = \left\|0 \cdot e^{+\infty} = 0 \cdot \infty\right\| = \lim\limits_{x\to 0^+} \dfrac{e^{\frac{1}{x}}}{\frac{1}{x}} = \left\|\dfrac{\infty}{\infty}\right\| \overset{\text{L.P.}}{=} \lim\limits_{x\to 0^+} \dfrac{e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} e^{\frac{1}{x}} = \infty \Rightarrow x = 0$ asymptota BS

$$\lim_{x\to 0^-} xe^{\frac{1}{x}} = \left\|0 \cdot e^{-\infty} = 0 \cdot 0\right\| = 0$$

SS $a = \lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \dfrac{xe^{\frac{1}{x}}}{x} = \lim\limits_{x\to\pm\infty} e^{\frac{1}{x}} = e^0 = 1$

$b = \lim\limits_{x\to\pm\infty}\left(f(x) - a \cdot x\right) = \lim\limits_{x\to\pm\infty}\left(xe^{\frac{1}{x}} - x\right) \overset{\infty-\infty}{=} \lim\limits_{x\to\pm\infty} x\left(e^{\frac{1}{x}} - 1\right) \overset{\infty\cdot 0}{=} \lim\limits_{x\to\pm\infty} \dfrac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} \overset{\frac{0}{0}\,\text{L.P.}}{=} \lim\limits_{x\to\pm\infty} \dfrac{e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim\limits_{x\to\pm\infty} e^{\frac{1}{x}} = e^0 = 1$ $\Bigg\} \Rightarrow$

$\Rightarrow$ asymptota se směrnicí pro ±∞ je y = x + 1



C) $y = \operatorname{arctg}\frac{x}{2-x}$

1) D(f) = **R** - $\{2\}$

$$f(-x) = \operatorname{arctg}\frac{-x}{2+x} \Rightarrow \text{ ani sudá ani lichá}$$

2) Kladná, záporná

y = arctgx má takové znaménko jako x

x	−		+		+
2 − x	+		+		−
		● 0		○ 2	
y′	−		+		−

3) Rostoucí, klesající, extrémy

$$y' = \frac{1}{1+\left(\frac{x}{2-x}\right)^2} \cdot \frac{2-x-x(-1)}{(2-x)^2} = \frac{2}{(2-x)^2+x^2} =$$

$$= \frac{2}{2x^2-4x+4} = \frac{1}{x^2-2x+2}$$

Čitatel i jmenovatel zlomku vždy kladný ⇒ fce poroste v každém bodě definičního oboru

4) Konvexní, konkávní, inflexní body

$$y'' = -(x^2-2x+2)^{-2} \cdot (2x-2) = \frac{2(1-x)}{(x^2-2x+2)^2}$$

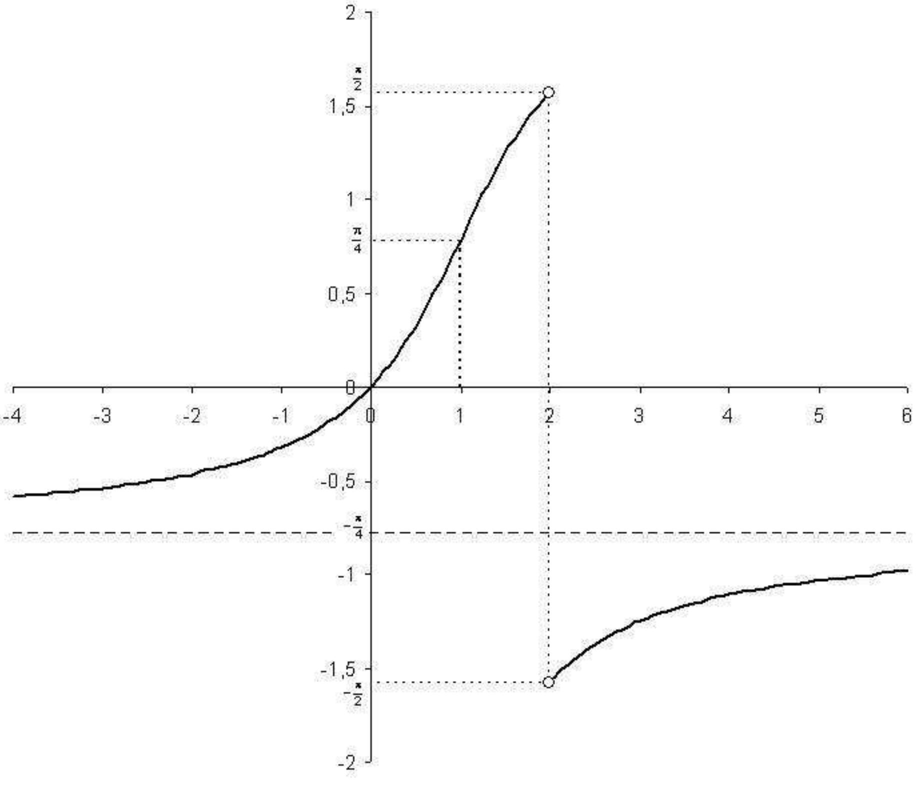
1 − x	+		−
		● 1	
y″	+ ∪	inf	− ∩

5) Asymptoty

$$\left.\begin{aligned} \text{BS}\quad &\lim_{x\to 2^+} \operatorname{arctg}\frac{x}{2-x} = \left\|\operatorname{arctg}\frac{2}{-0} = \operatorname{arctg}(-\infty)\right\| = -\frac{\pi}{2} \\ &\lim_{x\to 2^-} \operatorname{arctg}\frac{x}{2-x} = \left\|\operatorname{arctg}\frac{2}{+0} = \operatorname{arctg}(\infty)\right\| = \frac{\pi}{2} \end{aligned}\right\} \Rightarrow \text{asymptoty bez směrnice neexistují}$$

$$\left.\begin{aligned} \text{SS}\quad a &= \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\operatorname{arctg}\frac{x}{2-x}}{x} = \left\|\frac{\operatorname{arctg}(-1)}{\infty} = \frac{-\frac{\pi}{4}}{\infty}\right\| = 0 \\ b &= \lim_{x\to\pm\infty} (f(x) - a\cdot x) = \lim_{x\to\pm\infty} \operatorname{arctg}\frac{x}{2-x} = \operatorname{arctg}(-1) = -\frac{\pi}{4} \end{aligned}\right\} \Rightarrow \text{ asymptota SS pro } \pm\infty \text{ je } y = -\frac{\pi}{4}$$

pozn.: při výpočtu **a** i **b** se využije: $\lim_{x\to\pm\infty} \frac{x}{2-x} = \left\|\frac{\infty}{\infty}\right\| \overset{\text{L.P.}}{=} \lim_{x\to\pm\infty} \frac{1}{-1} = -1$



DÚ 6

1. Číslo 100 rozdělte na dvě čísla tak, aby součet jejich druhých mocnin byl minimální.

$100 = a + b$

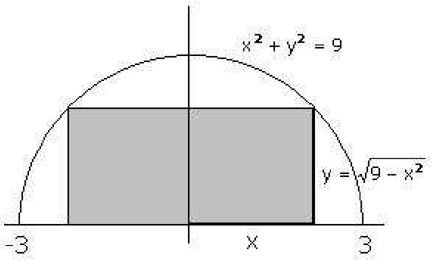
$y = a^2 + b^2 = a^2 + (100 - a)^2 = 2a^2 - 200a + 10000$

$y' = 0$

$4a - 200 = 0 \qquad y'' = 4$

$\underline{\underline{a = 50}} \Rightarrow \underline{\underline{b = 50}} \qquad y''(50) = 4 > 0 \Rightarrow \text{minimum}$

2. Do půlkružnice o poloměru 3 cm vepište obdélník o co největším obsahu.



$S = 2 \cdot x \cdot y$

$y = 2x\sqrt{9 - x^2}$

$y' = 0$

$$2 \cdot \sqrt{9 - x^2} + 2x \cdot \tfrac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x) = 0$$

$$2\sqrt{9 - x^2} - \frac{2x^2}{\sqrt{9 - x^2}} = 0$$

$$\frac{2(9 - x^2) - 2x^2}{\sqrt{9 - x^2}} = 0$$

$$\frac{18 - 4x^2}{\sqrt{9 - x^2}} = 0$$

$$18 - 4x^2 = 0$$

$$2x^2 = 9$$

$$\underline{\underline{x = \frac{3}{\sqrt{2}}}}$$

$$y'' = \frac{-8x \cdot \sqrt{9 - x^2} - (18 - 4x^2) \cdot \tfrac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x)}{9 - x^2} = \frac{-8x(9 - x^2) + 18x - 4x^3}{(9 - x^2)^{\frac{3}{2}}} = \frac{x(4x^2 - 54)}{(9 - x^2)^{\frac{3}{2}}}$$

$$y''(\frac{3}{\sqrt{2}}) = \frac{\frac{3}{\sqrt{2}}\left(4\left(\frac{3}{\sqrt{2}}\right)^2 - 54\right)}{(9 - \left(\frac{3}{\sqrt{2}}\right)^2)^{\frac{3}{2}}} = \frac{\frac{3}{\sqrt{2}}(-36)}{\left(\frac{9}{2}\right)^{\frac{3}{2}}} = \frac{\frac{3}{\sqrt{2}}(-36)}{\frac{9}{2} \cdot \frac{3}{\sqrt{2}}} = -\frac{36}{\frac{9}{2}} = -8 \; < \; 0 \Rightarrow \text{maximum}$$

3. Zintegrujte

$$\textbf{a)} \int \frac{\sqrt{x} - 5x + 1}{3x} dx = \frac{1}{3}\int \left(\frac{x^{\frac{1}{2}}}{x} - \frac{5x}{x} + \frac{1}{x}\right) dx = \frac{1}{3}\int \left(x^{-\frac{1}{2}} - 5 + \frac{1}{x}\right) dx = \frac{1}{3}\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 5x + \ln|x|\right) + c =$$

$$= \underline{\underline{\frac{1}{3}\left(2\sqrt{x} - 5x + \ln|x|\right) + c}}$$

$$\textbf{b)} \int \frac{-6x^2 - 24}{x^3 + 12x + 7} dx = -2\int \frac{3x^2 + 12}{x^3 + 12x + 7} dx = |\text{derivace spodku} = \text{vrch}| = \underline{\underline{-2\ln|x^3 + 12x + 7| + c}}$$

$$\textbf{c)} \int 5x \sin x dx = \begin{vmatrix} u = 5x & u' = 5 \\ v' = \sin x & v = -\cos x \end{vmatrix} = -5x\cos x - \int -5\cos x dx = \underline{\underline{-5x\cos x + 5\sin x + c}}$$

$$\textbf{d)} \int (x^2 - 3)e^x dx = \begin{vmatrix} u = x^2 - 3 & u' = 2x \\ v' = e^x & v = e^x \end{vmatrix} = (x^2 - 3)e^x - \int 2xe^x dx = \begin{vmatrix} u = 2x & u' = 2 \\ v' = e^x & v = e^x \end{vmatrix} =$$

$$= (x^2 - 3)e^x - \left(2xe^x - \int 2e^x\right) = (x^2 - 3)e^x - 2xe^x + 2e^x + c = \underline{\underline{e^x(x^2 - 2x - 1) + c}}$$

e) $\int \mathbf{x^4 \ln x dx} = \left|\begin{array}{ll} u' = x^4 & u = \frac{x^5}{5} \\ v = \ln x & v' = \frac{1}{x} \end{array}\right| = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\frac{1}{x}dx = \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4 dx = \underline{\underline{\frac{x^5}{5}\ln x - \frac{x^5}{25} + c}}$

f) $\int \mathbf{e^x \cos x dx} = \left|\begin{array}{ll} u = e^x & u' = e^x \\ v' = \cos x & v = \sin x \end{array}\right| = e^x \sin x - \int e^x \sin x dx = \left|\begin{array}{ll} u = e^x & u' = e^x \\ v' = \sin x & v = -\cos x \end{array}\right| =$

$$= e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x dx\right) = e^x \sin x + e^x \cos x - \int e^x \cos x dx \Rightarrow$$

$$\Rightarrow \int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x dx = \underline{\underline{\frac{e^x}{2}(\sin x + \cos x) + c}}$$

g) $\int \mathbf{6x^2 e^{x^3} dx} = \left|\begin{array}{l} t = x^3 \\ dt = 3x^2 dx \\ 2dt = 6x^2 dx \end{array}\right| = \int e^t 2dt = 2e^t + c = \underline{\underline{2e^{x^3} + c}}$

h) $\int \mathbf{\frac{3x^2}{\sqrt{1 - x^6}} dx} = \left|\begin{array}{l} t = x^3 \\ dt = 3x^2 dx \end{array}\right| = \int \frac{1}{\sqrt{1 - t^2}} dt = \arcsin t + c = \underline{\underline{\arcsin x^3 + c}}$

i) $\int \mathbf{\frac{(1 + \ln x)^5}{x} dx} = \int \frac{1}{x}(1 + \ln x)^5 dx = \left|\begin{array}{l} t = 1 + \ln x \\ dt = \frac{1}{x} dx \end{array}\right| = \int t^5 dt = \frac{t^6}{6} + c = \underline{\underline{\frac{1}{6}(1 + \ln x)^6 + c}}$

j) $\int \mathbf{\cos x \sqrt{\sin x} dx} = \left|\begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array}\right| = \int \sqrt{t} dt = \int t^{\frac{1}{2}} dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \underline{\underline{\frac{2}{3}\sin^{\frac{3}{2}} x + c}}$

DÚ 7

Zintegrujte:

1) $\displaystyle \int \frac{\mathbf{2x^2+11x+32}}{\mathbf{x^3+3x^2+6x-10}}\,\mathbf{dx} =$

$$\frac{2x^2+11x+32}{x^3+3x^2+6x-10}=\underline{\frac{2x^2+11x+32}{(x-1)(x^2+4x+10)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4x+10}}\qquad /\cdot(x-1)(x^2+4x+10)$$

$2x^2+11x+32=A(x^2+4x+10)+(Bx+C)(x-1)$

$2x^2+11x+32=A(x^2+4x+10)+B(x^2-x)+C(x-1)$

$x^2:\quad 2=A+B$

$x:\quad 11=4A-B+C$

$\underline{k:\quad 32=10A-C}$

$A=3, B=-1, C=-2$

$$=\int\left(\frac{3}{x-1}+\frac{-x-2}{x^2+4x+10}\right)dx=3\ln|x-1|-\int\frac{x+2}{x^2+4x+10}dx=3\ln|x-1|-\frac{1}{2}\int\frac{2x+4}{x^2+4x+10}dx=$$

$$=\underline{\underline{3\ln|x-1|-\frac{1}{2}\ln(x^2+4x+10)+c}}$$

2) $\displaystyle \int \frac{\mathbf{3x^3+2x^2-68x-8}}{\mathbf{(x+2)^2(x^2-6x+12)}}\,\mathbf{dx} =$

$$\underline{\frac{3x^3+2x^2-68x-8}{(x+2)^2(x^2-6x+12)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{Cx+D}{x^2-6x+12}}\qquad /\cdot(x+2)^2(x^2-6x+12)$$

$3x^3+2x^2-68x-8=A(x+2)(x^2-6x+12)+B(x^2-6x+12)+(Cx+D)(x+2)^2$

$3x^3+2x^2-68x-8=A(x^3-4x^2+24)+B(x^2-6x+12)+C(x^3+4x^2+4x)+D(x^2+4x+4)$

$x^3:\quad 3=A+C$

$x^2:\quad 2=-4A+B+4C+D$

$x:\quad -68=-6B+4C+4D$

$\underline{k:\quad -8=24A+12B+4D}$

$A=0, B=4, C=3, D=-14$

$$=\int\left(\frac{4}{(x+2)^2}+\frac{3x-14}{x^2-6x+12}\right)dx=\left|\begin{array}{l}t=x+2\\ dt=dx\end{array}\right|=4\int\frac{1}{t^2}dt+\int\frac{\frac{3}{2}(2x-6)-5}{x^2-6x+12}dx=$$

$$=4\int t^{-2}dt+\frac{3}{2}\int\frac{2x-6}{x^2-6x+12}dx-\int\frac{5}{x^2-6x+12}dx=4\frac{t^{-1}}{-1}+\frac{3}{2}\ln(x^2-6x+12)-5\int\frac{1}{(x-3)^2+3}dx=$$

$$=-\frac{4}{t}+\frac{3}{2}\ln(x^2-6x+12)-\frac{5}{3}\int\frac{1}{\left(\frac{x-3}{\sqrt{3}}\right)^2+1}dx=\left|\begin{array}{l}s=\dfrac{x-3}{\sqrt{3}}\\ ds=\dfrac{1}{\sqrt{3}}dx\to\sqrt{3}ds=dx\end{array}\right|=$$

$$=-\frac{4}{x+2}+\frac{3}{2}\ln(x^2-6x+12)-\frac{5}{3}\sqrt{3}\int\frac{1}{s^2+1}ds=-\frac{4}{x+2}+\frac{3}{2}\ln(x^2-6x+12)-\frac{5}{\sqrt{3}}\mathrm{arctg}s+c=$$

$$=\underline{\underline{-\frac{4}{x+2}+\frac{3}{2}\ln(x^2-6x+12)-\frac{5}{\sqrt{3}}\mathrm{arctg}\frac{x-3}{\sqrt{3}}+c}}$$

3) $\displaystyle \int \mathbf{\frac{3x^3+x^2+4x-6}{x^3-2x^2-3x}dx} = |\text{vydelime}| = \int\left(3+\frac{7x^2+13x-6}{x^3-2x^2-3x}\right)dx =$

$$\frac{7x^2+13x-6}{x^3-2x^2-3x} = \frac{7x^2+13x-6}{x(x^2-2x-3)} = \underline{\frac{7x^2+13x-6}{x(x+1)(x-3)} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}} \qquad /\cdot\, x(x+1)(x-3)$$

$7x^2+13x-6 = A(x+1)(x-3)+Bx(x-3)+Cx(x+1)$

$7x^2+13x-6 = A(x^2-2x-3)+B(x^2-3x)+C(x^2+x)$

$x^2: \quad 7 = A+B+C$

$x: \quad 13 = -2A-3B+C$

$\underline{k: \quad -6 = -3A}$

$A = 2, B = -3, C = 8$

$$= \int\left(3+\frac{2}{x}+\frac{-3}{x+1}+\frac{8}{x-3}\right)dx = \underline{\underline{3x+2\ln|x|-3\ln|x+1|+8\ln|x-3|+c}}$$

4) $\displaystyle \int \mathbf{\frac{x^4-x^3-7x^2+15x+4}{x^2-4x+5}dx} = |\text{vydelime}| = \int\left(x^2+3x+\frac{4}{x^2-4x+5}\right)dx =$

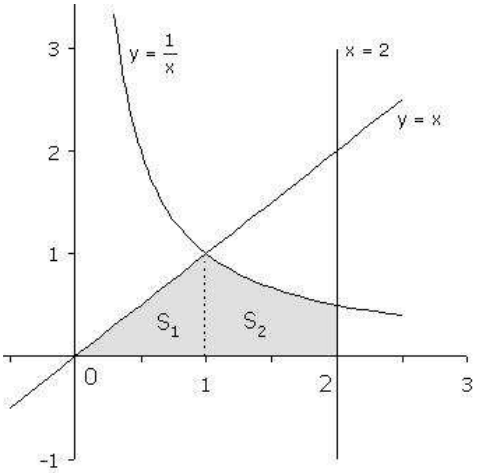
$$= \frac{x^3}{3}+3\frac{x^2}{2}+4\int\frac{1}{(x-2)^2+1}dx = \begin{vmatrix} t = x-2 \\ dt = dx \end{vmatrix} = \frac{x^3}{3}+\frac{3x^2}{2}+4\int\frac{1}{t^2+1}dt = \frac{x^3}{3}+\frac{3x^2}{2}+4\text{arctg}t+c =$$

$$\underline{\underline{= \frac{x^3}{3}+\frac{3x^2}{2}+4\text{arctg}(x-2)+c}}$$

DÚ 8

1. Vypočtete obsah plochy ohraničené křivkami

a) $\mathbf{y = x\,,\ y = \frac{1}{x}\,,\ y = 0\,,\ x = 2}$



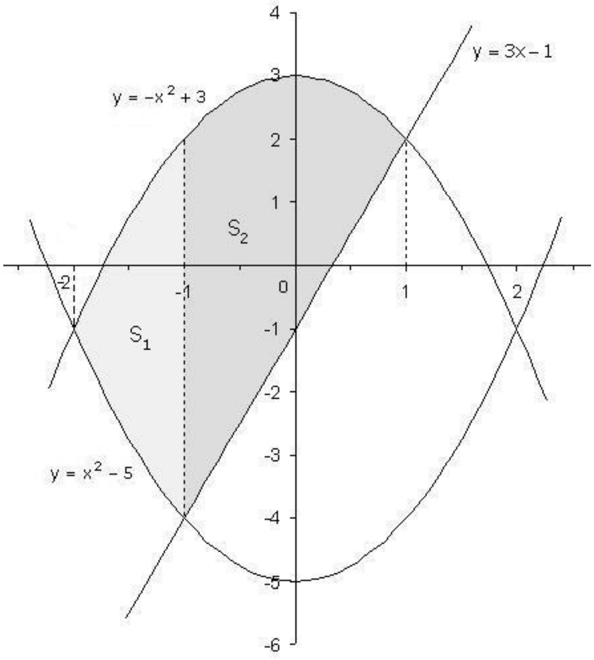
$$x = \frac{1}{x} \quad / \cdot x$$
$$x^2 - 1 = 0$$
$$x = -1 \quad \vee \quad x = 1$$

$$S_1 = \int_0^1 (x - 0)dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$S_2 = \int_1^2 (\frac{1}{x} - 0)dx = \left[\ln|x|\right]_1^2 = \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2$$

$$S = S_1 + S_2 = \underline{\underline{\tfrac{1}{2} + \ln 2}}$$

b) $\mathbf{y = x^2 - 5\,,\ y = -x^2 + 3\,,\ y = 3x - 1}$ **(tu část, která obsahuje počátek souřadnic)**



$$\begin{array}{lll} x^2 - 5 = -x^2 + 3 & x^2 - 5 = 3x - 1 & -x^2 + 3 = 3x - 1 \\ 2x^2 = 8 & x^2 - 3x - 4 = 0 & x^2 + 3x - 4 = 0 \\ x^2 = 4 & \underline{x = -1} \quad \vee \quad x = 4 & \underline{x = 1} \quad \vee \quad x = -4 \\ x = \pm 2 & & \end{array}$$

$$S_1 = \int_{-2}^{-1} [(-x^2 + 3) - (x^2 - 5)]dx = \int_{-2}^{-1} (-2x^2 + 8)dx =$$
$$= \left[-2\frac{x^3}{3} + 8x\right]_{-2}^{-1} = \left(-2\frac{-1}{3} - 8\right) - \left(-2\frac{-8}{3} - 16\right) =$$
$$= \frac{2}{3} - 8 - \frac{16}{3} + 16 = \frac{2 - 24 - 16 + 48}{3} = \frac{10}{3}$$

$$.S_2 = \int_{-1}^{1} [(-x^2 + 3) - (3x - 1)]dx = \int_{-1}^{1} (-x^2 - 3x + 4)dx =$$
$$= \left[-\frac{x^3}{3} - 3\frac{x^2}{2} + 4x\right]_{-1}^{1} = \left(-\frac{1}{3} - \frac{3}{2} + 4\right) - \left(-\frac{-1}{3} - \frac{3}{2} - 4\right) =$$
$$= -\frac{1}{3} - \frac{3}{2} + 4 - \frac{1}{3} + \frac{3}{2} + 4 = -\frac{1}{3} - \frac{1}{3} + 8 = \frac{22}{3}$$

$$S = S_1 + S_2 = \frac{10}{3} + \frac{22}{3} = \frac{32}{3} = \underline{\underline{10\tfrac{2}{3}}}$$

2. Zintegrujte

a) $$\int \frac{\mathbf{1}}{\mathbf{\cos x}}\mathbf{dx} = \left| \begin{array}{ll} \text{licha vuci cosx} \Rightarrow & t = \sin x dx \\ & dt = \cos x dx \end{array} \right| = \int \frac{\cos x}{\cos^2 x}dx = \int \frac{\cos x}{1 - \sin^2 x}dx = \int \frac{1}{1 - t^2}dt =$$

$$\frac{1}{1 - t^2} = \underline{\frac{1}{(1 - t)(1 + t)} = \frac{A}{1 - t} + \frac{B}{1 + t}} \quad / \cdot (1 - t)(1 + t)$$

$$1 = A(1 + t) + B(1 - t)$$

$$\left.\begin{array}{l} t: \ \ 0 = A - B \\ \underline{k: \ \ 1 = A + B} \end{array}\right\} \Rightarrow A = \tfrac{1}{2}, B = \tfrac{1}{2}$$

$$= \int \left(\frac{\frac{1}{2}}{1 - t} + \frac{\frac{1}{2}}{1 + t}\right)dt = -\tfrac{1}{2}\int \frac{-1}{1 - t}dt + \tfrac{1}{2}\int \frac{-1}{1 + t}dt = -\tfrac{1}{2}\ln|1 - t| + \tfrac{1}{2}\ln|1 + t| + c = \tfrac{1}{2}\ln\left|\frac{1 + t}{1 - t}\right| + c = \underline{\underline{\tfrac{1}{2}\ln\left|\frac{1 + \sin x}{1 - \sin x}\right| + c}}$$

b) $$\int \frac{\mathbf{\sin^3 x}}{\mathbf{1-\cos^3 x}}\,\mathbf{dx} = \left|\begin{array}{ll} \text{licha vuci sinx} \Rightarrow & t = \cos x \\ & dt = -\sin x dx \\ & -dt = \sin x dx \end{array}\right| = \int \frac{\sin^2 x \sin x}{1-\cos^3 x}dx = \int \frac{(1-\cos^2 x)\sin x}{1-\cos^3 x}dx =$$

$$= \int \frac{1-t^2}{1-t^3}(-dt) = -\int \frac{(1-t)(1+t)}{(1-t)(1+t+t^2)}dt = \int \frac{-t-1}{t^2+t+1}dt = \int \frac{-\frac{1}{2}(2t+1)-\frac{1}{2}}{t^2+t+1}dt =$$

$$= -\tfrac{1}{2}\int \frac{2t+1}{t^2+t+1}dt - \tfrac{1}{2}\int \frac{1}{t^2+t+1}dt = -\tfrac{1}{2}\ln\left|t^2+t+1\right| - \tfrac{1}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}dt =$$

$$= -\tfrac{1}{2}\ln(t^2+t+1) - \tfrac{1}{2}\cdot\tfrac{4}{3}\int \frac{1}{\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1}dt = \left|\begin{array}{l} s = \frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{2t+1}{\sqrt{3}} \\ ds = \frac{2}{\sqrt{3}}dt \Rightarrow \frac{\sqrt{3}}{2}ds = dt \end{array}\right| = -\tfrac{1}{2}\ln(t^2+t+1) - \tfrac{2}{3}\cdot\tfrac{\sqrt{3}}{2}\int \frac{1}{s^2+1}ds =$$

$$= -\tfrac{1}{2}\ln(t^2+t+1) - \tfrac{1}{\sqrt{3}}\,\mathrm{arctg}(s) + c = -\tfrac{1}{2}\ln(t^2+t+1) - \tfrac{1}{\sqrt{3}}\,\mathrm{arctg}\,\tfrac{2t+1}{\sqrt{3}} + c =$$

$$= \underline{\underline{-\tfrac{1}{2}\ln(\cos^2 x + \cos x + 1) - \tfrac{1}{\sqrt{3}}\,\mathrm{arctg}\,\tfrac{2\cos x+1}{\sqrt{3}} + c}}$$

DÚ 9

1. Vypočtete délku křivky $x = e^t$, $y = \frac{2}{3}e^{\frac{3}{2}t}$ pro $t \in \langle \ln 3, \ln 8 \rangle$.

$x' = e^t$

$y' = e^{\frac{3}{2}t}$

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\,dt = \int_{\ln 3}^{\ln 8} \sqrt{e^{2t} + e^{3t}}\,dt = \int_{\ln 3}^{\ln 8} \sqrt{(e^t)^2(1 + e^t)}\,dt = \int_{\ln 3}^{\ln 8} e^t\sqrt{1 + e^t}\,dt =$$

$$= \left| \begin{array}{ll} s = 1 + e^t & t = \ln 3 \rightarrow s = 4 \\ ds = e^t dt & t = \ln 8 \rightarrow s = 9 \end{array} \right| = \int_4^9 \sqrt{s}\,ds = \int_4^9 s^{\frac{1}{2}}\,ds = \frac{2}{3}\left[s^{\frac{3}{2}} \right]_4^9 = \frac{2}{3}\left[s\sqrt{s} \right]_4^9 = \frac{2}{3}(27 - 8) = \frac{38}{3} = \underline{\underline{12\tfrac{2}{3}}}$$

2. Vypočtete délku křivky $y = \ln x$ v intervalu $x \in \left\langle \sqrt{3}, \sqrt{8} \right\rangle$.

$y' = \frac{1}{x}$

$$L = \int_a^b \sqrt{1 + (y')^2}\,dx = \int_{\sqrt{3}}^{\sqrt{8}} \sqrt{1 + \frac{1}{x^2}}\,dx = \int_{\sqrt{3}}^{\sqrt{8}} \frac{1}{x}\sqrt{x^2 + 1}\,dx = \left| \begin{array}{lll} t = \sqrt{x^2 + 1} & tdt = x^2\frac{1}{x}dx & x = \sqrt{3} \rightarrow t = 2 \\ t^2 = x^2 + 1 & tdt = (t^2 - 1)\frac{1}{x}dx & x = \sqrt{8} \rightarrow t = 3 \\ 2tdt = 2xdx & \frac{t}{t^2 - 1}dt = \frac{1}{x}dx & \end{array} \right| =$$

$$= \int_2^3 t\frac{t}{t^2 - 1}\,dt = \int_2^3 \frac{t^2}{t^2 - 1}\,dt = \left| \text{vydelime} \right| = \int_2^3 \left(1 + \frac{1}{t^2 - 1} \right) dt = \left| \text{rozklad na parc. zlomky} \right| = \int_2^3 \left(1 - \frac{\frac{1}{2}}{t+1} + \frac{\frac{1}{2}}{t-1} \right) dt =$$

$$= \left[t - \frac{1}{2}\ln|t + 1| + \frac{1}{2}\ln|t - 1| \right]_2^3 = \left[t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| \right]_2^3 = \left(3 + \frac{1}{2}\ln\frac{2}{4} \right) - \left(2 + \frac{1}{2}\ln\frac{1}{3} \right) = 1 + \frac{1}{2}\ln\frac{\frac{1}{2}}{\frac{1}{3}} = \underline{\underline{1 + \frac{1}{2}\ln\frac{3}{2}}}$$

3. Vypočtěte objem tělesa vzniklého rotací křivky $y = \sqrt{x\cos x}$ kolem osy x v intervalu $\langle 0, \frac{\pi}{2} \rangle$.

$$V = \pi\int_a^b f^2(x)\,dx = \pi\int_0^{\frac{\pi}{2}} x\cos x\,dx = \left| \begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{array} \right| = \pi\left\{ [x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx \right\} = \pi[x\sin x + \cos x]_0^{\frac{\pi}{2}} =$$

$$= \pi\left[\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} \right) - (0.\sin 0 + \cos 0) \right] = \pi\left(\frac{\pi}{2} - 1 \right) = \underline{\underline{\frac{\pi^2 - 2\pi}{2}}}$$

4. Určete průměr funkce $y = \sin x$ na intervalu $\langle 0, \pi \rangle$.

$$av_{[a,b]}f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$av_{[0,\pi]}(\sin x) = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}[-\cos x]_0^{\pi} = -\frac{1}{\pi}(\cos\pi - \cos 0) = -\frac{1}{\pi}(-1 - 1) = \underline{\underline{\frac{2}{\pi}}}$$

DÚ 10

1. Vypočtete obsah plochy pod křivkou

a) $\mathbf{y = e^{-x}}$ **v intervalu** $\mathbf{\langle 0,\infty)}$

$$S = \int_0^\infty e^{-x}dx = -\left[e^{-x}\right]_0^\infty = -\left(e^{-\infty} - e^0\right) = -(0-1) = \underline{\underline{1}}$$

b) $\mathbf{y = \frac{1}{\sqrt{x}}}$ **v intervalu** $\mathbf{(0,1\rangle}$

$$S = \int_0^1 \tfrac{1}{\sqrt{x}}dx = \int_0^1 x^{-\frac{1}{2}}dx = \left[2\sqrt{x}\right]_0^1 = \underline{\underline{2}}$$

c) $\mathbf{y = \frac{1}{x^4}}$ **v intervalu** $\mathbf{(0,1\rangle}$

$$S = \int_0^1 \tfrac{1}{x^4}dx = \int_0^1 x^{-4}dx = -\tfrac{1}{3}\left[\tfrac{1}{x^3}\right]_0^1 = -\tfrac{1}{3}\left(1 - \lim_{x\to 0^+}\tfrac{1}{x^3}\right) = -\tfrac{1}{3}(1-\infty) = \underline{\underline{\infty}}$$

2. Vypočtete rovnici

$\underline{\mathbf{\sqrt{x^3}\,.\,\sqrt[4]{x^3}\,.\,\sqrt[8]{x^3}\,.\,\sqrt[16]{x^3}\,.\ldots = 8}}$ $\quad x \geq 0$

$x^{\frac{3}{2}}\cdot x^{\frac{3}{4}}\cdot x^{\frac{3}{8}}\cdot x^{\frac{3}{16}}\cdot\ldots = 8$

$x^{\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\ldots} = 8$ $\qquad \frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\ldots = \sum_{n=0}^{\infty}\frac{3}{2}\cdot\left(\frac{1}{2}\right)^n = \frac{\frac{3}{2}}{1-\frac{1}{2}} = 3$

$x^3 = 8$

$\underline{\underline{x = 2}}$

3. Určete součet řad

a) $$\sum_{n=0}^{\infty}\frac{2^n-5}{6^n} = \sum_{n=0}^{\infty}\frac{2^n}{6^n} - \sum_{n=0}^{\infty}\frac{5}{6^n} = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n - \sum_{n=0}^{\infty}5\left(\frac{1}{6}\right)^n = \frac{1}{1-\frac{1}{3}} - \frac{5}{1-\frac{1}{6}} = \frac{3}{2} - 6 = \underline{\underline{-\frac{9}{2}}}$$

b) $$\frac{7}{4}+\frac{7}{8}+\frac{7}{16}+\frac{7}{32}+\frac{7}{64}+\ldots = \sum_{n=0}^{\infty}\frac{7}{4}\left(\frac{1}{2}\right)^n = \frac{\frac{7}{4}}{1-\frac{1}{2}} = \frac{\frac{7}{4}}{\frac{1}{2}} = \underline{\underline{\frac{7}{2}}}$$

c) $$\frac{12}{9}+\frac{12}{27}+\frac{12}{81}+\frac{12}{243}+\ldots = \sum_{n=0}^{\infty}\frac{12}{9}\left(\frac{1}{3}\right)^n = \frac{\frac{12}{9}}{1-\frac{1}{3}} = \frac{\frac{12}{9}}{\frac{2}{3}} = \underline{\underline{2}}$$

d) $$\sum_{n=1}^{\infty}\frac{3^n}{n} = \frac{3}{1}+\frac{9}{2}+\frac{27}{3}+\frac{81}{4}+\frac{243}{5}+\ldots$$

sčítáme stále větší členy => řada diverguje

(není splněna nutná podmínka konvergence, neboť $\lim_{n\to\infty}\frac{3^n}{n} = \left\|\frac{\infty}{\infty}\right\| \overset{\text{L.P.}}{=} \lim_{n\to\infty}\frac{3^n\ln 3}{1} = \infty \neq 0$)

e) $$\sum_{n=1}^{\infty}\left(\sqrt{n} - 2\sqrt{n+1} + \sqrt{n+2}\right)$$

$$s_n = \left(\sqrt{1}-2\sqrt{2}+\sqrt{3}\right)+\left(\sqrt{2}-2\sqrt{3}+\sqrt{4}\right)+\left(\sqrt{3}-2\sqrt{4}+\sqrt{5}\right)+\left(\sqrt{4}-2\sqrt{5}+\sqrt{6}\right)+\ldots+$$
$$+\left(\sqrt{n-2}-2\sqrt{n-1}+\sqrt{n}\right)+\left(\sqrt{n-1}-2\sqrt{n}+\sqrt{n+1}\right)+\left(\sqrt{n}-2\sqrt{n+1}+\sqrt{n+2}\right) = 1-\sqrt{2}-\sqrt{n+1}+\sqrt{n+2}$$

$$s = \lim_{n\to\infty}\left(1-\sqrt{2}-\sqrt{n+1}+\sqrt{n+2}\right) = 1-\sqrt{2}+\lim_{n\to\infty}\left(\sqrt{n+2}-\sqrt{n+1}\right) = \|\infty-\infty\| = 1-\sqrt{2}+\lim_{n\to\infty}\frac{n+2-(n+1)}{\sqrt{n+2}+\sqrt{n+1}} =$$
$$= 1-\sqrt{2}+\lim_{n\to\infty}\frac{1}{\sqrt{n+2}+\sqrt{n+1}} = \left\|\frac{1}{\infty}\right\| = 1-\sqrt{2}+0 = 1-\sqrt{2} \quad\Rightarrow\quad \sum_{n=1}^{\infty}\left(\sqrt{n}-2\sqrt{n+1}+\sqrt{n+2}\right) = \underline{\underline{1-\sqrt{2}}}$$

f) $\displaystyle\sum_{n=1}^{\infty}\frac{\mathbf{2}}{\mathbf{n(n+2)}}$

$$\underline{\frac{2}{n(n+2)}=\frac{A}{n}+\frac{B}{n+2}}\quad /\cdot n(n+2)$$

$$2=A(n+2)+Bn$$

$$\left.\begin{array}{l} n:\quad 0=A+B \\ \underline{k:\quad 2=2A\quad} \end{array}\right\}\Rightarrow A=1,\ B=-1$$

$$\sum_{n=1}^{\infty}\frac{2}{n(n+2)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

$$s_n=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$$

$$s=\lim_{n\to\infty}\left(\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{3}{2}-0-0=\frac{3}{2}\quad\Rightarrow\quad\sum_{n=1}^{\infty}\frac{2}{n(n+2)}=\underline{\underline{\frac{3}{2}}}$$

g) $\displaystyle\sum_{n=1}^{\infty}\frac{\mathbf{4n^2+12n+6}}{\mathbf{(n^2+n)(n^2+5n+6)}}$

$$\frac{4n^2+12n+6}{(n^2+n)(n^2+5n+6)}=\frac{4n^2+12n+6}{n(n+1)(n+2)(n+3)}==\underline{\frac{A}{n}+\frac{B}{n+1}+\frac{C}{n+2}+\frac{D}{n+3}}\quad /\cdot n(n+1)(n+2)(n+3)$$

$$4n^2+12n+6=A(n+1)(n+2)(n+3)+Bn(n+2)(n+3)+Cn(n+1)(n+3)+Dn(n+1)(n+2)$$

$$4n^2+12n+6=A(n^3+6n^2+11n+6)+B(n^3+5n^2+6n)+C(n^3+4n^2+3n)+D(n^3+3n^2+2n)$$

$$\left.\begin{array}{l} n^3:\quad 0=A+B+C+D \\ n^2:\quad 4=6A+5B+4C+3D \\ n:\quad 12=11A+6B+3C+2D \\ \underline{k:\quad 6=6A} \end{array}\right\}\Rightarrow A=1, B=1, C=-1, D=-1$$

$$\sum_{n=1}^{\infty}\frac{4n^2+12n+6}{(n^2+n)(n^2+5n+6)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right)$$

$$s_n=\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}\right)+\ldots+$$

$$+\left(\frac{1}{n-3}+\frac{1}{n-2}-\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n-2}+\frac{1}{n-1}-\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n-1}+\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}\right)+\left(\frac{1}{n}+\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right)=$$

téměř každé číslo se v s_n vyskytuje dvakrát se znaménkem – a dvakrát se znaménkem +

$$\Rightarrow s_n=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{2}{n+2}-\frac{1}{n+3}=\frac{7}{3}-\frac{1}{n+1}-\frac{2}{n+2}-\frac{1}{n+3}$$

$$s=\lim_{n\to\infty}\left(\frac{7}{3}-\frac{1}{n+1}-\frac{2}{n+2}-\frac{1}{n+3}\right)=\frac{7}{3}-0-0-0=\frac{7}{3}\quad\Rightarrow\quad\sum_{n=1}^{\infty}\frac{4n^2+12n+6}{(n^2+n)(n^2+5n+6)}=\underline{\underline{\frac{7}{3}}}$$

DÚ 11

1. Vypočtěte součty řad

a) $\boldsymbol{\sum_{n=1}^{\infty}\frac{5n}{2^n}}$

(1) $$s_n = \tfrac{5}{2} + \tfrac{10}{4} + \tfrac{15}{8} + \tfrac{20}{16} + \ldots + \tfrac{5(n-1)}{2^{n-1}} + \tfrac{5n}{2^n} \quad /\cdot \tfrac{1}{2}$$

(2) $$\tfrac{1}{2} s_n = \tfrac{5}{4} + \tfrac{10}{8} + \tfrac{15}{16} + \ldots\ldots\ldots\ldots + \tfrac{5(n-1)}{2^n} + \tfrac{5n}{2^{n+1}}$$

(1) - (2) $$\tfrac{1}{2} s_n = \tfrac{5}{2} + \tfrac{5}{4} + \tfrac{5}{8} + \tfrac{5}{16} + \ldots + \tfrac{5}{2^n} - \tfrac{5n}{2^{n+1}} \quad /\cdot 2$$

$$s_n = 5 + \tfrac{5}{2} + \tfrac{5}{4} + \tfrac{5}{8} + \ldots + \tfrac{5}{2^{n-1}} - \tfrac{5n}{2^n}$$

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(5 + \tfrac{5}{2} + \tfrac{5}{4} + \tfrac{5}{8} + \ldots + \tfrac{5}{2^{n-1}} - \tfrac{5n}{2^n}\right) = 5 + \tfrac{5}{2} + \tfrac{5}{4} + \tfrac{5}{8} + \ldots - \lim_{n\to\infty} \tfrac{5n}{2^n} = \left\|\tfrac{\infty}{\infty}\right\| \overset{\text{L.P.}}{=} \tfrac{5}{1-\frac{1}{2}} - \lim_{n\to\infty} \tfrac{5}{2^n \ln 2} = 10 - 0 = \underline{\underline{10}}$$

b) $\boldsymbol{\sum_{n=1}^{\infty}\frac{n}{3^n}}$

(1) $$s_n = \tfrac{1}{3} + \tfrac{2}{9} + \tfrac{3}{27} + \tfrac{4}{81} + \ldots + \tfrac{n-1}{3^{n-1}} + \tfrac{n}{3^n} \quad /\cdot \tfrac{1}{3}$$

(2) $$\tfrac{1}{3} s_n = \tfrac{1}{9} + \tfrac{2}{27} + \tfrac{3}{81} + \ldots\ldots\ldots\ldots + \tfrac{n-1}{3^n} + \tfrac{n}{3^{n+1}}$$

(1) - (2) $$\tfrac{2}{3} s_n = \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \ldots + \tfrac{1}{3^n} - \tfrac{n}{3^{n+1}} \quad /\cdot \tfrac{3}{2}$$

$$s_n = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{18} + \tfrac{1}{54} + \ldots + \tfrac{3}{2.3^{n-1}} - \tfrac{n}{2.3^n}$$

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{18} + \tfrac{1}{54} + \ldots + \tfrac{3}{2.3^{n-1}} - \tfrac{n}{2.3^n}\right) = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{18} + \tfrac{1}{54} + \ldots - \lim_{n\to\infty} \tfrac{n}{2.3^n} = \left\|\tfrac{\infty}{\infty}\right\| \overset{\text{L.P.}}{=} \tfrac{\frac{1}{2}}{1-\frac{1}{3}} - \lim_{n\to\infty} \tfrac{1}{2.3^n \ln 3} =$$

$$= \tfrac{3}{4} - 0 = \underline{\underline{\tfrac{3}{4}}}$$

2. Rozhodněte o konvergenci či divergenci řad

a) $\boldsymbol{\sum_{n=1}^{\infty}\left(\frac{n}{n!}\right)^3}$

$$\lim_{n\to\infty} \tfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{n+1}{(n+1)!}\right)^3}{\left(\frac{n}{n!}\right)^3} = \lim_{n\to\infty}\left(\frac{(n+1)\cdot n!}{n\cdot(n+1)!}\right)^3 = \lim_{n\to\infty}\left(\frac{1}{n}\right)^3 = \lim_{n\to\infty}\frac{1}{n^3} = \left\|\frac{1}{\infty}\right\| = 0 < 1 \Rightarrow \text{řada } \mathbf{K}$$

b) $\boldsymbol{\sum_{n=1}^{\infty}\frac{\sqrt{n+1}}{n}}$

$$\frac{\sqrt{n+1}}{n} > \frac{1}{n} \ \text{ pro } \forall\, n \geq 1 \quad \oplus \quad \sum_{n=1}^{\infty}\frac{1}{n} \ \text{ diverguje} \quad \Rightarrow \ \text{řada } \sum_{n=1}^{\infty}\frac{\sqrt{n+1}}{n} \ \mathbf{D}$$

c) $\boldsymbol{\sum_{n=1}^{\infty}\left(\frac{n+1}{5n-6}\right)^{2n}}$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n+1}{5n-6}\right)^{2n}} = \lim_{n\to\infty}\left[\left(\frac{n+1}{5n-6}\right)^{2n}\right]^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n+1}{5n-6}\right)^2 = \lim_{n\to\infty}\frac{n^2+2n+1}{25n^2-60n+36} =$$

$$= \left\|\frac{\infty}{\infty}\right\| \overset{\text{L.P.}}{=} \lim_{n\to\infty}\frac{2n+2}{50n-60} = \left\|\frac{\infty}{\infty}\right\| \overset{\text{L.P.}}{=} \lim_{n\to\infty}\frac{2}{50} = \frac{2}{50} < 1 \Rightarrow \ \text{řada } \mathbf{K}$$

d) $\boldsymbol{\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^5}}$

$$\frac{\sin^2 n}{n^5} < \frac{1}{n^5} \quad \oplus \quad \sum_{n=1}^{\infty}\frac{1}{n^5} \ \text{ konverguje} \quad \Rightarrow \ \text{řada } \sum_{n=1}^{\infty}\frac{\sin^2 n}{n^5} \ \ \mathbf{K}$$

e) $\boldsymbol{\sum_{n=1}^{\infty}\frac{2n}{\sqrt{n^2+1}}}$

$$\int_1^{\infty}\frac{2x}{\sqrt{x^2+1}}\,dx = \left|\begin{matrix} t = x^2+1 & x = 1 \to t = 2 \\ dt = 2xdx & x = \infty \to t = \infty \end{matrix}\right| = \int_2^{\infty}\frac{1}{\sqrt{t}}\,dt = \int_2^{\infty} t^{-\frac{1}{2}}dt = 2\left[\sqrt{t}\right]_2^{\infty} =$$

$$= 2\left(\infty - \sqrt{2}\right) = \infty \ \ \mathbf{D} \quad \Rightarrow \ \text{řada } \sum_{n=1}^{\infty}\frac{2n}{\sqrt{n^2+1}} \ \ \mathbf{D}$$

f) $\boldsymbol{\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}}$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n^2}\right]^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1 \Rightarrow \ \text{řada } \mathbf{D}$$

g) $\boldsymbol{\sum_{n=1}^{\infty}\frac{n^2}{n!}}$

$$\lim_{n\to\infty}\tfrac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} = \lim_{n\to\infty}\frac{(n+1)^2 n!}{n^2(n+1)n!} = \lim_{n\to\infty}\frac{n+1}{n^2} = \left\|\frac{\infty}{\infty}\right\| \overset{\text{L.P.}}{=} \lim_{n\to\infty}\frac{1}{2n} = 0 < 1 \Rightarrow \ \text{řada } \mathbf{K}$$

3. Dokažte, že řada $\sum\limits_{n=1}^{\infty}\frac{1}{n^{10}}$ konverguje.

$$\int_1^{\infty}\frac{1}{x^{10}}\,dx=\int_1^{\infty}x^{-10}dx=-\frac{1}{9}\left[\frac{1}{x^{9}}\right]_1^{\infty}=-\frac{1}{9}(0-1)=\frac{1}{9}\quad\mathbf{K}\ \Rightarrow\ \text{řada}\ \sum_{n=1}^{\infty}\frac{1}{n^{10}}\quad\mathbf{K}$$

4. Dokažte, že řada $\sum\limits_{n=2}^{\infty}\frac{(-1)^n}{\ln^5 n}$ konverguje.

$$\left.\begin{array}{l}\left\{\frac{1}{\ln^5 n}\right\}\text{ je nerostoucí posloupnost }\checkmark\\[4pt] \frac{1}{\ln^5 n}>0\ \text{ pro }\forall\, n\ge 2\ \checkmark\\[4pt] \lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{1}{\ln^5 n}=\left\|\frac{1}{\infty}\right\|=0\ \checkmark\end{array}\right\}\Rightarrow\ \text{řada}\ \sum_{n=1}^{\infty}\frac{(-1)^n}{\ln^5 n}\quad\mathbf{K}$$

5. Rozhodněte, zda řada konverguje relativně, absolutně nebo nekonverguje.

a) $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n-\ln n}$

$$\left.\begin{array}{l}n>\ln n\,\text{pro }\forall\, n\ge 1\Rightarrow n-\ln n\text{ roste }\Rightarrow\\[4pt] \Rightarrow\left\{\frac{1}{n-\ln n}\right\}\text{ je nerostoucí posloupnost }\checkmark\\[4pt] n>\ln n\,\text{pro }\forall\, n\ge 1\Rightarrow\frac{1}{n-\ln n}>0\ \checkmark\\[4pt] \lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{1}{n-\ln n}=\left\|\frac{1}{\infty}\right\|=0\ \checkmark\end{array}\right\}\Rightarrow\ \text{řada}\ \sum_{n=1}^{\infty}\frac{(-1)^n}{n-\ln n}\quad\mathbf{K}$$

- $\sum\limits_{n=1}^{\infty}\frac{1}{n-\ln n}\qquad \frac{1}{n-\ln n}>\frac{1}{n}\quad\oplus\quad\sum\limits_{n=1}^{\infty}\frac{1}{n}$ diverguje $\Rightarrow$ řada $\sum\limits_{n=1}^{\infty}\frac{1}{n-\ln n}$ **D**

celkem: řada $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n-\ln n}$ **K relativně**

b) $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{e^n}$

$$\left.\begin{array}{l}\left\{\frac{1}{e^n}\right\}\text{ je nerostoucí posloupnost }\checkmark\\[4pt] \frac{1}{e^n}>0\ \text{ pro }\forall\, n\ge 1\ \checkmark\\[4pt] \lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{1}{e^n}=\left\|\frac{1}{\infty}\right\|=0\ \checkmark\end{array}\right\}\Rightarrow\ \text{řada}\ \sum_{n=1}^{\infty}\left(-\frac{1}{e}\right)^n\quad\mathbf{K}$$

- $\sum\limits_{n=1}^{\infty}\frac{1}{e^n}\qquad \lim\limits_{n\to\infty}\sqrt[n]{a_n}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{1}{e^n}}=\lim\limits_{n\to\infty}\frac{1}{e}<1\Rightarrow$ řada $\sum\limits_{n=1}^{\infty}\frac{1}{e^n}$ **K**

celkem: řada $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{e^n}$ **K absolutně**

DÚ 12

1. Pro která x konvergují mocninné řady:

a) $\sum\limits_{n=1}^{\infty} n(x-3)^n$

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1$$

$x = 2$ ... $\sum\limits_{n=1}^{\infty} n(2-3)^n = \sum\limits_{n=1}^{\infty} n(-1)^n$ ... nekonverguje

$x = 4$ ... $\sum\limits_{n=1}^{\infty} n(4-3)^n = \sum\limits_{n=1}^{\infty} n$ ... diverguje

celkem: řada konverguje pro $\underline{\underline{x \in (2,4)}}$

b) $\sum\limits_{n=1}^{\infty} \frac{5^n}{n!} x^n$

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{5^n}{n!}}{\frac{5^{n+1}}{(n+1)!}}\right| = \lim_{n\to\infty}\frac{5^n(n+1)n!}{n!\cdot 5\cdot 5^n} = \lim_{n\to\infty}\frac{n+1}{5} = \infty$$ ⇒ řada konverguje pro $\underline{\underline{x \in (-\infty,\infty)}}$

c) $\sum\limits_{n=1}^{\infty} \frac{n^3}{2^n}(x+4)^n$

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{n^3}{2^n}}{\frac{(n+1)^3}{2^{n+1}}}\right| = \lim_{n\to\infty}\frac{n^3\cdot 2\cdot 2^n}{2^n(n+1)^3} = \lim_{n\to\infty}\frac{2n^3}{(n+1)^3} = 2$$

$x = -6$ ... $\sum\limits_{n=1}^{\infty}\frac{n^3}{2^n}(-6+4)^n = \sum\limits_{n=1}^{\infty}\frac{n^3}{2^n}(-2)^n = \sum\limits_{n=1}^{\infty}(-1)^n n^3$ ... nekonverguje

$x = -2$ ... $\sum\limits_{n=1}^{\infty}\frac{n^3}{2^n}(-2+4)^n = \sum\limits_{n=1}^{\infty}\frac{n^3}{2^n}2^n = \sum\limits_{n=1}^{\infty} n^3$ ... diverguje

celkem: řada konverguje pro $\underline{\underline{x \in (-6,-2)}}$

2) Určete obor konvergence a součet řady $\sum\limits_{n=1}^{\infty} nx^n$. Pomocí výsledku pak určete součet $\sum\limits_{n=1}^{\infty}\frac{n}{10^n}$.

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1$$

$x = -1$ ... $\sum\limits_{n=1}^{\infty} n(-1)^n$ ... nekonverguje

$x = 1$ ... $\sum\limits_{n=1}^{\infty} n$ ... diverguje

obor konvergence je tedy $\underline{\underline{x \in (-1,1)}}$

$$\sum_{n=1}^{\infty} nx^n = x\cdot\sum_{n=1}^{\infty} nx^{n-1} = x\cdot\sum_{n=1}^{\infty}(x^n)' = x\cdot\left(\sum_{n=1}^{\infty}x^n\right)' = x\cdot\left(x+x^2+x^3+\ldots\right)' = x\cdot\left(\frac{x}{1-x}\right)' = x\cdot\frac{1\cdot(1-x)-x\cdot(-1)}{(1-x)^2} \Rightarrow$$

$$\sum_{n=1}^{\infty} nx^n = \underline{\underline{\frac{x}{(1-x)^2}}} \quad \text{pro } x \in (-1,1)$$

$$\sum_{n=1}^{\infty}\frac{n}{10^n} = \left|\text{predchozi vypocet pro } x = \tfrac{1}{10}\right| = \frac{\frac{1}{10}}{(1-\frac{1}{10})^2} = \frac{\frac{1}{10}}{\frac{81}{100}} = \underline{\underline{\frac{10}{81}}}$$

3) Určete obor konvergence a součet řady $\sum_{n=1}^{\infty}\frac{x^{n+3}}{n+1}$. Pomocí výsledku pak určete součet $\sum_{n=2}^{\infty}\frac{1}{n.2^{n+2}}$.

$$r=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n+2}}\right|=\lim_{n\to\infty}\frac{n+2}{n+1}=1$$

$x = -1$ ... $\sum_{n=1}^{\infty}\frac{(-1)^{n+3}}{n+1}$ ... konverguje (dle Leibnitzova kritéria)

$x = 1$ ... $\sum_{n=1}^{\infty}\frac{1}{n+1}$ ... diverguje (podle např. integrálního kritéria)

obor konvergence je tedy $\underline{\underline{x\in\langle -1,1)}}$

$$\sum_{n=1}^{\infty}\frac{x^{n+3}}{n+1}=x^2\cdot\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}=x^2\cdot\sum_{n=1}^{\infty}\int x^n dx=x^2\cdot\int\sum_{n=1}^{\infty}x^n dx=x^2\cdot\int\left(x+x^2+x^3+\ldots\right)dx=x^2\cdot\int\frac{x}{1-x}dx=$$

$$=\left|\text{vydelime}\right|=x^2\cdot\int\left(-1+\frac{1}{1-x}\right)dx=x^2\left(-x-\ln|1-x|\right)+c=-x^2\left(x+\ln|1-x|\right)+c\ \text{ pro }\ x\in(-1,1)$$

Ze zadání je vidět, že součet řady pro x = 0 je 0 (dostaneme totiž řadu samých nul). Dosazením do právě získaného výsledku dostáváme:

$$0=-0^2\left(0+\ln|1-0|\right)+c\quad\Rightarrow\quad c=0\quad\Rightarrow\quad\sum_{n=1}^{\infty}\frac{x^{n+3}}{n+1}=-x^2\left(x+\ln|1-x|\right)\ \text{ pro }\ x\in(-1,1)$$

zadaná řada však konverguje ještě v bodě x = − 1:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+3}}{n+1}=\lim_{x\to -1^+}\left[-x^2\left(x+\ln|1-x|\right)\right]=-(-1+\ln|1-(-1)|=1-\ln 2$$

celkem tedy:

$$\sum_{n=1}^{\infty}\frac{x^{n+3}}{n+1}=\underline{\underline{-x^2\left(x+\ln|1-x|\right)}}\ \text{ pro }\ x\in\langle -1,1)$$

$$\sum_{n=2}^{\infty}\frac{1}{n\cdot 2^{n+2}}=\left|\text{pozor na spodni "mez"}\right|=\sum_{n=1}^{\infty}\frac{1}{(n+1)\cdot 2^{n+3}}=\left|\text{predchozi vypocet pro }x=\tfrac{1}{2}\right|=-\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}+\ln\left|1-\tfrac{1}{2}\right|\right)=$$

$$=-\tfrac{1}{4}\left(\tfrac{1}{2}+\ln\tfrac{1}{2}\right)=-\tfrac{1}{8}\left(1+2\ln 2^{-1}\right)=-\tfrac{1}{8}\left(1-2\ln 2\right)=\tfrac{1}{8}\left(2\ln 2-1\right)=\underline{\underline{\tfrac{2\ln 2-1}{8}}}\doteq 0{,}0483$$

4. Funkci cosx rozviňte do mocninné řady se středem v 0.

$x_0 = 0$

$f(x)=\cos x\quad f'(x)=-\sin x\quad f''(x)=-\cos x\quad f'''(x)=\sin x\quad f^{(4)}(x)=\cos x\quad f^{(5)}(x)=-\sin x\quad f^{(6)}(x)=-\cos x$

$f(x_0)=1\quad f'(x_0)=0\quad f''(x_0)=-1\quad f'''(x_0)=0\quad f^{(4)}(x_0)=1\quad f^{(5)}(x_0)=0\quad f^{(6)}(x_0)=-1$

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots$$

$$\cos x=1+\frac{0}{1!}(x-0)+\frac{-1}{2!}(x-0)^2+\frac{0}{3!}(x-0)^3+\frac{1}{4!}(x-0)^4+\frac{0}{5!}(x-0)^5+\frac{-1}{6!}(x-0)^6+\ldots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots=\underline{\underline{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}}}\quad(\text{pozn.: plati pro }x\in(-\infty,\infty))$$

DÚ 13

1. Řešte diferenciální rovnice:

a) $\underline{\mathbf{1 + y^2 = xy'}}$

$$\frac{1}{x} = \frac{1}{1+y^2}\frac{dy}{dx}$$

$$\int \frac{1}{x}dx = \int \frac{1}{1+y^2}dy$$

$$\ln|x| = \operatorname{arctg} y + c_1$$

$$\operatorname{arctg} y = \ln|x| + c$$

$$\underline{\underline{y = \operatorname{tg}(\ln|x| + c)}}$$

b) $\underline{\mathbf{e^{-y}(y' - 1) = 1}}$

$$y' - 1 = e^y$$

$$\frac{dy}{dx} = e^y + 1$$

$$\int \frac{1}{e^y + 1}dy = \int dx$$

...

$$-\ln(e^y + 1) + y = x + c_1 \quad / \cdot (-1)$$

$$\ln(e^y + 1) - y = -x + c_2$$

$$\ln(e^y + 1) = y - x + c_2$$

$$e^{\ln(e^y+1)} = e^{y-x+c_2}$$

$$e^y + 1 = c.e^{y-x}$$

$$\underline{\underline{e^y = c.e^{y-x} - 1}}$$

$$\int \frac{1}{e^y + 1}dy = \left| \begin{array}{ll} t = e^y + 1 & dt = (t-1)dy \\ dt = e^y dy & \frac{dt}{t-1} = dy \end{array} \right| = \int \frac{1}{t}\frac{dt}{t-1} =$$

$$= \int \frac{1}{t(t-1)}dt = |\text{rozklad na parc. zlomky}| =$$

$$= \int \left(-\frac{1}{t} + \frac{1}{t-1}\right)dt = -\ln|t| + \ln|t-1| =$$

$$= -\ln|e^y + 1| + \ln|e^y + 1 - 1| = -\ln(e^y + 1) + \ln(e^y) =$$

$$= -\ln(e^y + 1) + y$$

c) $\underline{\mathbf{y^2 + x^2y' = xyy'}}$

$$xyy' - x^2y' = y^2$$

$$y'(xy - x^2) = y^2$$

$$y' = \frac{y^2}{xy - x^2}$$

$$y' = \frac{\left(\frac{y}{x}\right)^2}{\frac{y}{x} - 1}$$

$$u = \tfrac{y}{x}$$

$$ux = y$$

$$u'x + u = y'$$

$$u'x + u = \frac{u^2}{u-1}$$

$$u'x = \frac{u^2}{u-1} - u$$

$$u'x = \frac{u^2 - u^2 + u}{u-1}$$

$$\frac{du}{dx}x = \frac{u}{u-1} \quad /\cdot \frac{u-1}{u}$$

$$\int \frac{u-1}{u}du = \int \frac{1}{x}dx$$

$$\int \left(1 - \frac{1}{u}\right)du = \int \frac{1}{x}dx$$

$$u - \ln|u| = \ln|x| + c_1$$

$$\tfrac{y}{x} - \ln\left|\tfrac{y}{x}\right| = \ln|x| + c_1$$

$$\tfrac{y}{x} - \ln|y| + \ln|x| = \ln|x| + c_1$$

$$\tfrac{y}{x} - \ln|y| = c_1$$

$$\ln|y| = \tfrac{y}{x} + c_2$$

$$e^{\ln|y|} = e^{\frac{y}{x} + c_2}$$

$$|y| = c_3.e^{\frac{y}{x}}$$

$$\underline{\underline{y = c.e^{\frac{y}{x}}}}$$

Rovnici jsme násobili výrazem u-1:

$u - 1 \neq 0 \rightarrow u \neq 1 \rightarrow \tfrac{y}{x} \neq 1 \rightarrow y \neq x$

Není y = x řešením původní rovnice?

$$x^2 + x^2(x)' = xx(x)'$$

$$2x^2 = x^2$$

Ne, y = x není řešením dané rovnice.

Rovnici jsme dělili výrazem u:

$u \neq 0 \rightarrow u \neq 0 \rightarrow \tfrac{y}{x} \neq 0 \rightarrow y \neq 0$

Není y = 0 řešením původní rovnice?

$$0^2 + x^2(0)' = x.0.(0)'$$

$$0 = 0$$

Ano, y = 0 je řešením a je obsaženo v obecném řešení pro c = 0.

d) $x^2y' = y^2$

$$y' = \left(\frac{y}{x}\right)^2$$

$$u = \frac{y}{x}$$

$$ux = y$$

$$u'x + u = y'$$

$$u'x + u = u^2$$

$$u'x = u^2 - u$$

$$\frac{du}{dx}x = u(u-1) \quad / : u(u-1)$$

$$\int \frac{1}{u(u-1)}du = \int \frac{1}{x}dx$$

$$\int\left(-\frac{1}{u} + \frac{1}{u-1}\right)du = \int \frac{1}{x}dx$$

$$-\ln|u| + \ln|u-1| = \ln|x| + c_1$$

$$\ln\left|\frac{u-1}{u}\right| = \ln|x| + c_1$$

$$e^{\ln\left|\frac{u-1}{u}\right|} = e^{\ln|x|+c_1}$$

$$\left|\frac{u-1}{u}\right| = c_2|x|$$

$$\frac{u-1}{u} = c_3x$$

$$\frac{\frac{y}{x}-1}{\frac{y}{x}} = c_3x$$

$$\frac{y-x}{y} = c_3x$$

$$y - x = c_3xy$$

$$y + cxy = x$$

$$y(1+cx) = x$$

$$\underline{\underline{y = \frac{x}{1+cx}\,,\quad y = 0}}$$

Rovnici jsme dělili výrazem u-1:
$u - 1 \neq 0 \rightarrow u \neq 1 \rightarrow \frac{y}{x} \neq 1 \rightarrow y \neq x$

Není y = x řešením původní rovnice?

$$x^2(x)' = x^2$$

$$x^2 = x^2$$

Ano, y = x je řešením a je obsaženo v obecném řešení pro c = 0.

Rovnici jsme dělili také výrazem u:
$u \neq 0 \rightarrow u \neq 0 \rightarrow \frac{y}{x} \neq 0 \rightarrow y \neq 0$

Není y = 0 řešením původní rovnice?

$$x^2(0)' = 0^2$$

$$0 = 0$$

Ano, y = 0 je řešením, ale není obsaženo v obecném řešení. Proto jej musíme dopsat.

e) $y' = y + x$

Jedná se o rovnici lineární; nejprve vyřešíme příslušnou homogenní rovnici:

$$y' - y = 0$$

$$\frac{dy}{dx} = y$$

$$\int \frac{1}{y}dx = \int dx$$

$$\ln|y| = x + c_1$$

$$e^{\ln|y|} = e^{x+c_1}$$

$$|y| = c_2e^x$$

$$y = c.e^x$$

řešení původní rovnice má pak tvar:
$y = c.e^x + c(x).e^x$

samotné $c(x).e^x$ je řešením původní rovnice:

$$\left(c(x).e^x\right)' = c(x).e^x + x$$

$$c'(x).e^x + c(x).e^x = c(x).e^x + x$$

$$c'(x).e^x = x$$

$$c'(x) = x.e^{-x}$$

$$c(x) = \int x.e^{-x}dx = \begin{vmatrix} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{vmatrix} = -xe^{-x} - \int -e^{-x}dx =$$

$$= -xe^{-x} - e^{-x} = e^{-x}(-x-1)$$

celkem tedy:

$$y = c.e^x + e^{-x}(-x-1).e^x$$

$$\underline{\underline{y = c.e^x - x - 1}}$$

f) $(x+1)y' - 2y = (x+1)^4$

$$y' - \frac{2}{x+1}y = (x+1)^3 \quad \text{… lineární rovnice}$$

$$y' - \frac{2}{x+1}y = 0 \quad \text{… příslušná homogenní}$$

$$\frac{dy}{dx} = \frac{2}{x+1}y$$

$$\int \frac{1}{y}dy = \int \frac{2}{x+1}dx$$

$$\ln|y| = 2\ln|x+1| + c_1$$

$$\ln|y| = \ln|x+1|^2 + c_1$$

$$e^{\ln|y|} = e^{\ln(x+1)^2 + c_1}$$

$$|y| = c_2(x+1)^2$$

$$y = c.(x+1)^2$$

řešení původní rovnice má pak tvar:
$y = c.(x+1)^2 + c(x).(x+1)^2$

samotné $c(x).(x+1)^2$ je řešením původní rovnice:

$$(x+1).\left(c(x).(x+1)^2\right)' - 2c(x).(x+1)^2 = (x+1)^4 \quad / : (x+1)$$

$$c'(x).(x+1)^2 + c(x).2(x+1) - 2c(x).(x+1) = (x+1)^3$$

$$c'(x).(x+1)^2 = (x+1)^3$$

$$c'(x) = x + 1$$

$$c(x) = \int (x+1)dx$$

$$c(x) = \frac{x^2}{2} + x$$

celkem tedy:

$$y = c.(x+1)^2 + (\tfrac{x^2}{2} + x).(x+1)^2 \Rightarrow \underline{\underline{y = (x+1)^2\left(\tfrac{x^2}{2} + x + c\right)}}$$