MB102 Matematika II
písemky

písemka 1

1. Najděte polynom, pro který platí P(1) = 6, P(2) = 1, P`(-1) = 5.

$P(x) = ax^2 + bx + c$ $\qquad P'(x) = 2ax + b$
$6 = a + b + c$ $\qquad 5 = -2a + b$
$\underline{1 = 4a + 2b + c}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 4 & 2 & 1 & 1 \\ -2 & 1 & 0 & 5 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -3 & -23 \\ 0 & 3 & 2 & 17 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 3 & 23 \\ 0 & 0 & -5 & -35 \end{array}\right)$$

$c = 7$
$b = 1$
$a = -2$

$$\underline{\underline{f(x) = -2x^2 + x + 7}}$$

2. Rozložte na parciální zlomky $\mathbf{\frac{4x}{x^3 - x^2 + 3x - 3}}$**.**

$$\frac{4x}{x^3 - x^2 + 3x - 3} = \frac{4x}{(x-1)(x^2+3)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 3} \quad / \cdot (x-1)(x^2+3)$$

$x^3 - x^2 + 3x - 3 = A(x^2 + 3) + (Bx + C)(x - 1)$

$x^3 - x^2 + 3x - 3 = Ax^2 + 3A + Bx^2 - Bx + Cx - C$

$x^2: \ 0 = A + B \rightarrow A = -B$

$x: \ 4 = -B + C$

$\underline{k: \ 0 = 3A - C}$

$4 = -B + C$

$\underline{0 = -3B - C}$

$4 = -4B \rightarrow B = -1 \Rightarrow A = 1, C = 3$

$$\underline{\underline{\frac{4x}{x^3 - x^2 + 3x - 3} = \frac{1}{x-1} + \frac{-x + 3}{x^2 + 3}}}$$

3. Vypočtete limity:

a) $$\lim_{x\to\infty}\left(\sqrt{4x^2 + 7x} - 2x\right) = \|\infty - \infty\| = \lim_{x\to\infty} \frac{\sqrt{4x^2 + 7x} - 2x}{1} \cdot \frac{\sqrt{4x^2 + 7x} + 2x}{\sqrt{4x^2 + 7x} + 2x} = \lim_{x\to\infty} \frac{4x^2 + 7x - 4x^2}{\sqrt{4x^2 + 7x} + 2x} =$$

$$= \lim_{x\to\infty} \frac{7x}{\sqrt{4x^2 + 7x} + 2x} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to\infty} \frac{7}{\sqrt{4 + \frac{7}{x}} + 2} = \frac{7}{2 + 2} = \underline{\underline{\frac{7}{4}}}$$

b) $$\lim_{x\to 2} \frac{2x - 1}{x^2 - 4x + 4} = \left\|\frac{3}{0}\right\| = \lim_{x\to 2} \frac{2x - 1}{(x-2)^2} = \left\|\frac{3}{+0} \text{ pro limitu zprava i zleva}\right\| = \underline{\underline{\infty}}$$

4. Zderivujte:

a) $\mathbf{y = \ln\frac{x^2 - 3}{x^2 + 3}}$

$$y' = \frac{1}{\frac{x^2 - 3}{x^2 + 3}} \cdot \frac{2x \cdot (x^2 + 3) - (x^2 - 3) \cdot 2x}{(x^2 + 3)^2} = \frac{x^2 + 3}{x^2 - 3} \cdot \frac{2x^3 + 6x - 2x^3 + 6x}{(x^2 + 3)^2} = \frac{12x}{(x^2 - 3)(x^2 + 3)} = \underline{\underline{\frac{12x}{x^4 - 9}}}$$

b) $\mathbf{y = \sqrt[4]{x \cdot \ln 2x} = (x \cdot \ln 2x)^{\frac{1}{4}}}$

$$y' = \frac{1}{4}(x \cdot \ln 2x)^{-\frac{3}{4}} \cdot \left(1 \cdot \ln 2x + x \cdot \frac{1}{2x} \cdot 2\right) = \underline{\underline{\frac{\ln 2x + 1}{4 \cdot \sqrt[4]{(x \cdot \ln 2x)^3}}}}$$

5. Najděte rovnici tečny ke grafu funkce y = x.ln(x – 2) v bodě A = [3, ?].

$A = [3, 0]$

$$f'(x) = 1 \cdot \ln(x-2) + x \cdot \frac{1}{x - 2}$$

$$k = f'(3) = 0 + 3 \cdot \frac{1}{1} = 3$$

$t: y = kx + q$
$y = 3x + q$
$0 = 3 \cdot 3 + q$
$q = -9$

$\underline{\underline{t: y = 3x - 9}}$

písemka 2

I. Vyšetřete průběh funkce $y = \frac{x}{x^2+1}$.

1) $D(f) = \mathbf{R}$

$f(-x) = \frac{-x}{(-x)^2+1} = -\frac{x}{x^2+1} \Rightarrow$ lichá funkce

2) Kladná, záporná

$x^2+1 \ldots$ vždy >0

	$(-\infty, 0)$	$(0, \infty)$
x	−	+
y	−	+

3) Rostoucí, klesající, extrémy

$$y' = \frac{x^2+1-x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}$$

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
1 − x	+	+	−
1 + x	−	+	+
y′	−	+	−
	↘ min	↗ max	↘

4) Konvexní, konkávní, inflexní body

$$y'' = \frac{-2x\cdot(x^2+1)^2-(1-x^2)\cdot 2(x^2+1)2x}{(x^2+1)^4} =$$

$$= \frac{-2x(x^2+1)-(1-x^2)4x}{(x^2+1)^3} = \frac{-2x^3-2x-4x+4x^3}{(x^2+1)^3} =$$

$$= \frac{2x^3-6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3} = \frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3}$$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
2x	−	−	+	+
$x+\sqrt{3}$	−	+	+	+
$x-\sqrt{3}$	−	−	−	+
y″	−	+	−	+
	∩ inf	∪ inf	∩ inf	∪

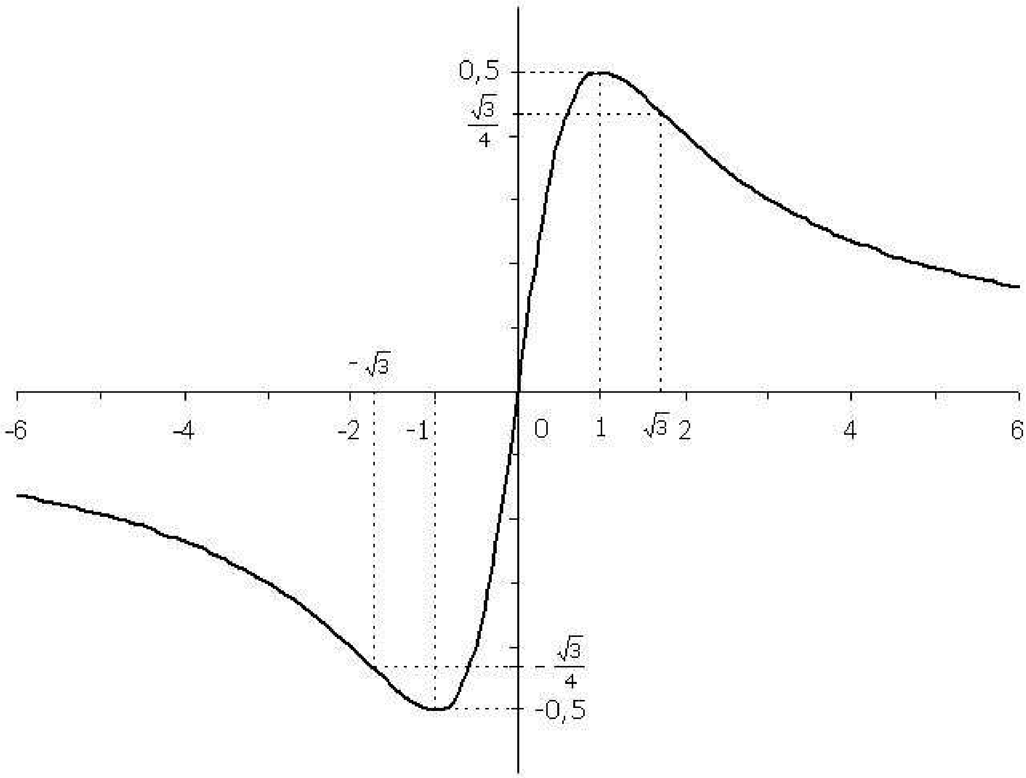
5) Asymptoty

BS ... neexistují

SS $a = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\frac{x}{x^2+1}}{x} = \lim_{x\to\pm\infty} \frac{1}{x^2+1} = \left\|\frac{1}{\infty}\right\| = 0$

$b = \lim_{x\to\pm\infty}[f(x)-ax] = \lim_{x\to\pm\infty} \frac{x}{x^2+1} = \left\|\frac{\infty}{\infty}\right\| \overset{L.P.}{=} \lim_{x\to\pm\infty} \frac{1}{2x} = \left\|\frac{1}{\infty}\right\| = 0$

$\Rightarrow$ asymptota SS pro $\pm\infty$ je $y = 0$



II. Zintegrujte

a) $\int \mathbf{(x^2+5)e^x dx} = \begin{vmatrix} u = x^2+5 & u' = 2x \\ v' = e^x & v = e^x \end{vmatrix} = (x^2+5)e^x - \int 2xe^x dx = \begin{vmatrix} u = 2x & u' = 2 \\ v' = e^x & v = e^x \end{vmatrix} =$

$= (x^2+5)e^x - \left(2xe^x - \int 2e^x dx\right) = (x^2+5)e^x - 2xe^x + 2e^x + c = \underline{\underline{e^x(x^2-2x+7)+c}}$

b) $\int \mathbf{\frac{3}{x(1+\ln x)^4} dx} = \int \frac{1}{x}\frac{3}{(1+\ln x)^4} dx = \begin{vmatrix} t = 1+\ln x \\ dt = \frac{1}{x}dx \end{vmatrix} = \int \frac{3}{t^4} dt = \int 3t^{-4} dt = 3\frac{t^{-3}}{-3} + c = -\frac{1}{t^3} + c =$

$\underline{\underline{= -\frac{1}{(1+\ln x)^3} + c}}$

písemka 3

1. $\displaystyle\int \frac{\mathbf{x^2-7x-34}}{\mathbf{x^3+5x^2+7x-13}}\,\mathbf{dx} = \left|\text{rozlozime na parcialni zlomky}\right| =$

$$\frac{x^2-7x-34}{x^3+5x^2+7x-13} = \frac{x^2-7x-34}{(x-1)(x^2+6x+13)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+6x+13} \qquad /\cdot(x-1)(x^2+6x-13)$$

$$x^2-7x-34 = A(x^2+6x+13)+(Bx+C)(x-1)$$

$$x^2-7x-34 = A(x^2+6x+13)+B(x^2-x)+C(x-1)$$

$$\left.\begin{array}{ll} x^2: & 1 = A+B \\ x: & -7 = 6A-B+C \\ k: & -34 = 13A-C \end{array}\right\} \Rightarrow A=-2,\ B=3,\ C=8$$

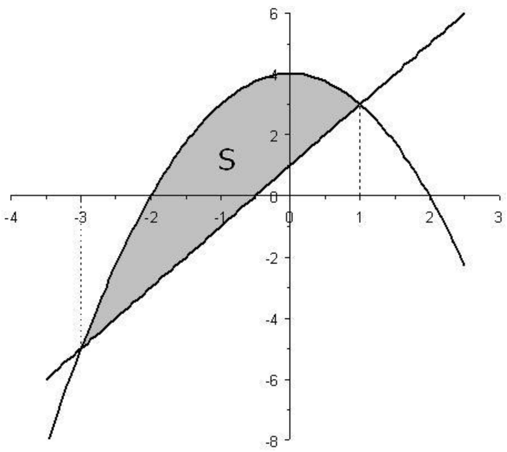
$$= \int\left(\frac{-2}{x-1}+\frac{3x+8}{x^2+6x+13}\right)dx = -2\ln|x-1| + \int\frac{\frac{3}{2}(2x+6)-1}{x^2+6x+13}dx =$$

$$= -2\ln|x-1| + \tfrac{3}{2}\int\frac{2x+6}{x^2+6x+13}dx - \int\frac{1}{(x+3)^2+4}dx = -2\ln|x-1| + \tfrac{3}{2}\ln|x^2+6x+13| - \tfrac{1}{4}\int\frac{1}{\left(\frac{x+3}{2}\right)^2+1}dx =$$

$$= \left|\begin{array}{l} t=\frac{x+3}{2} \\ dt=\frac{1}{2}dx \\ 2dt=dx \end{array}\right| = -2\ln|x-1| + \tfrac{3}{2}\ln(x^2+6x+13) - \tfrac{2}{4}\int\frac{1}{t^2+1}dt = -2\ln|x-1| + \tfrac{3}{2}\ln(x^2+6x+13) - \tfrac{1}{2}\operatorname{arctg} t + c =$$

$$= \underline{\underline{-2\ln|x-1| + \tfrac{3}{2}\ln(x^2+6x+13) - \tfrac{1}{2}\operatorname{arctg}\tfrac{x+3}{2} + c}}$$

2. Vypočtěte obsah plochy ohraničené křivkami $\mathbf{y = -x^2+4}$, $\mathbf{y = 2x+1}$.



$$-x^2+4 = 2x+1$$

$$x^2+2x-3=0$$

$$x=-3 \quad x=1$$

$$S = \int_{-3}^{1}\left[(-x^2+4)-(2x+1)\right]dx = \int_{-3}^{1}(-x^2-2x+3)dx =$$

$$= \left[-\tfrac{x^3}{3}-x^2+3x\right]_{-3}^{1} = \left(-\tfrac{1}{3}-1+3\right)-(9-9-9) =$$

$$= \tfrac{5}{3}+9 = \underline{\underline{\tfrac{32}{3}}}$$

3. Vypočtěte objem tělesa vzniklého rotací funkce $\mathbf{y = 3x\sqrt{\ln x}}$ kolem osy x v intervalu $\mathbf{\langle 1,3\rangle}$.

$$V = \pi\int_1^3 9x^2\ln x\,dx = \left|\begin{array}{ll} u'=9x^2 & u=3x^3 \\ v=\ln x & u'=\frac{1}{x} \end{array}\right| = \pi\left\{\left[3x^3\ln x\right]_1^3 - \int_1^3 3x^2dx\right\} = \pi\left[3x^3\ln x - x^3\right]_1^3 =$$

$$= \pi\left[(81\ln 3-27)-(3\ln 1-1)\right] = \underline{\underline{\pi(81\ln 3-26)}}$$

4. Vypočtěte délku křivky $\mathbf{x=\frac{1}{2}t^2}$, $\mathbf{y=\frac{1}{3}t^3}$ pro $\mathbf{t\in\langle 0,\sqrt{3}\rangle}$.

$$\begin{array}{l} x'=t \\ y'=t^2 \end{array} \qquad L = \int_0^{\sqrt{3}}\sqrt{t^2+t^4}\,dt = \int_0^{\sqrt{3}} t\sqrt{1+t^2}\,dt = \left|\begin{array}{ll} s=1+t^2 & t=0\to s=1 \\ ds=2tdt & \\ \frac{1}{2}ds=tdt & t=\sqrt{3}\to s=4 \end{array}\right| = \int_1^4\sqrt{s}\,\tfrac{1}{2}ds = \tfrac{1}{2}\int_1^4 s^{\frac{1}{2}}ds = \tfrac{1}{2}\left[\tfrac{2}{3}s^{\frac{3}{2}}\right]_1^4 =$$

$$= \tfrac{1}{3}\left[s\sqrt{s}\right]_1^4 = \tfrac{1}{3}(8-1) = \underline{\underline{\tfrac{7}{3}}}$$

5. Vypočtěte průměr funkce $\mathbf{f(x)=x^3}$ na intervalu $\mathbf{\langle -2,4\rangle}$.

$$av_{[-2,4]}f(x) = \frac{1}{4-(-2)}\int_{-2}^{4}x^3dx = \tfrac{1}{6}\left[\tfrac{x^4}{4}\right]_{-2}^{4} = \tfrac{1}{6}(64-4) = \underline{\underline{10}}$$

písemka 4

1. Určete součet řady $\sum_{n=1}^{\infty}\frac{3n}{2^n}$.

(1) $s_n = \frac{3}{2}+\frac{6}{4}+\frac{9}{8}+\frac{12}{16}+\dots+\frac{3(n-1)}{2^{n-1}}+\frac{3n}{2^n} \quad /\cdot\frac{1}{2}$

(2) $\frac{1}{2}s_n = \quad \frac{3}{4}+\frac{6}{8}+\frac{9}{16}+\dots\dots\dots\dots+\frac{3(n-1)}{2^n}+\frac{3n}{2^{n+1}}$

(1) - (2) $\frac{1}{2}s_n = \frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\dots+\frac{3}{2^n}-\frac{3n}{2^{n+1}} \quad /\cdot 2$

$s_n = 3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\dots+\frac{3}{2^{n-1}}-\frac{3n}{2^n}$

$$s=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\dots+\frac{3}{2^{n-1}}-\frac{3n}{2^n}\right)=3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\dots-\lim_{n\to\infty}\frac{3n}{2^n}=\left\|\frac{\infty}{\infty}\right\|\overset{L.P.}{=}\frac{3}{1-\frac{1}{2}}-\lim_{n\to\infty}\frac{3}{2^n\ln 2}=6-0=\underline{\underline{6}}$$

2. Určete součet řady $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$.

$$s_n=\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}\right)+\dots+\left(\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}}\right)+\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=1-\frac{1}{\sqrt{n+1}}$$

$$s=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right)=1-0=\underline{\underline{1}}$$

3. Rozhodněte o konvergenci či divergenci následujících řad

a) $\sum_{n=1}^{\infty}\frac{(n!)^2}{n}$ $\quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{[(n+1)!]^2}{n+1}}{\frac{(n!)^2}{n}}=\lim_{n\to\infty}\frac{n}{n+1}\left(\frac{(n+1)n!}{n!}\right)^2=\lim_{n\to\infty}\frac{n}{n+1}(n+1)^2=\lim_{n\to\infty}n(n+1)=\infty\Rightarrow$ řada **D**

b) $\sum_{n=1}^{\infty}\left(\frac{n-1}{3n+2}\right)^n$ $\quad \lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n-1}{3n+2}\right)^n}=\lim_{n\to\infty}\frac{n-1}{3n+2}=\left\|\frac{\infty}{\infty}\right\|\overset{L.P.}{=}\lim_{n\to\infty}\frac{1}{3}=\frac{1}{3}<1\Rightarrow$ řada **K**

c) $\sum_{n=1}^{\infty}\frac{n-1}{n^5}=\sum_{n=1}^{\infty}\left(\frac{1}{n^4}-\frac{1}{n^5}\right)$ $\quad \frac{1}{n^4}-\frac{1}{n^5}<\frac{1}{n^4}\ \oplus\ \sum_{n=1}^{\infty}\frac{1}{n^4}$ konverguje $\Rightarrow$ řada **K**

4. Dokažte, že řada $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln^2 n}$ konverguje.

$\left\{\frac{1}{\ln^2 n}\right\}$ je nerostoucí posloupnost ✓

$\frac{1}{\ln^2 n}>0$ pro $\forall\, n\geq 2$ ✓

$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{\ln^2 n}=\left\|\frac{1}{\infty}\right\|=0$ ✓

$\Rightarrow$ řada **K**

5. Rozhodněte, zda řada $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\cdot\ln^3 n}$ konverguje absolutně, relativně nebo nekonverguje.

$\left\{\frac{1}{n\cdot\ln^3 n}\right\}$ je nerostoucí posloupnost ✓

$\frac{1}{n\cdot\ln^3 n}>0$ pro $\forall\, n\geq 2$ ✓

$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{n\cdot\ln^3 n}=\left\|\frac{1}{\infty}\right\|=0$ ✓

$\Rightarrow$ řada $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\cdot\ln^3 n}$ **K**

- $\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln^3 n}$

$$\int_2^{\infty}\frac{1}{x\cdot\ln^3 x}dx=\left|\begin{matrix}t=\ln x & x=2\to t=\ln 2\\ dt=\frac{1}{x}dx & x=\infty\to t=\infty\end{matrix}\right|=\int_{\ln 2}^{\infty}\frac{1}{t^3}dt=-\frac{1}{2}\left[\frac{1}{t^2}\right]_{\ln 2}^{\infty}=-\frac{1}{2}\left(0-\frac{1}{\ln^2 2}\right)=\frac{1}{2\ln^2 2}\Rightarrow\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln^3 n}\ \mathbf{K}$$

celkem: řada $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\cdot\ln^3 n}$ **K absolutně.**