DÚ 11

1. Určete objem tělesa, které vznikne rotací obrazce omezeného křivkami $y = x\sqrt{\sin x}$, $y = 0$ a $x = \frac{\pi}{2}$ kolem osy x.

2. Určete délku křivky $y = \frac{1}{3}(x-3)\sqrt{x}$ mezi průsečíky s osou x.

3. Vypočtete $\int_0^\infty \frac{dx}{(x+1)^4}$.

4. Rozhodněte o konvergenci či divergenci rady $\sum_{n=1}^\infty \frac{1}{n(\ln n + 2)^2}$.

Řešení:

Př. 1

$$V = \pi\int_0^{\frac{\pi}{2}} \left(x\cdot\sqrt{\sin x}\right)^2 dx = \pi\cdot\int_0^{\frac{\pi}{2}} x^2\sin x\,dx = \begin{vmatrix} u = x^2 & v' = \sin x \\ u' = 2x & v = -\cos x \end{vmatrix} = \pi\cdot\left(\left[-x^2\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 2x\cos x\,dx\right) =$$

$$= \begin{vmatrix} u = 2x & v' = \cos x \\ u' = 2 & v = \sin x \end{vmatrix} = \pi\cdot\left(\left[-x^2\cos x\right]_0^{\frac{\pi}{2}} + \left[2x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\sin x\,dx\right) =$$

$$= \pi\cdot\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\frac{\pi}{2}} = \pi\cdot\left[(-0+\pi+0)-(-0+0+2)\right] = \underline{\underline{\pi(\pi-2)}}$$

Př. 2 $y = \frac{1}{3}(x-3)\sqrt{x}$ $\quad D(f) = \langle 0,\infty)$

průsečíky s osou x: $\frac{1}{3}(x-3)\sqrt{x} = 0$

$x = 3 \quad x = 0$

	$(0,3)$	$(3,\infty)$
$\sqrt{x}$	+	+
$x-3$	−	+

$$y = \frac{1}{3}(x-3)x^{\frac{1}{2}} = \frac{1}{3}\left(x^{\frac{3}{2}} - 3x^{\frac{1}{2}}\right)$$

$$y' = \frac{1}{3}\left(\frac{3}{2}x^{\frac{1}{2}} - 3\cdot\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$y' = \frac{1}{2}\sqrt{x} - \frac{1}{2\sqrt{x}} = \frac{1}{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)$$

$$l = \int_0^3 \sqrt{1 + \left(\frac{1}{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\right)^2}\,dx = \int_0^3 \sqrt{1 + \frac{1}{4}\left(x - 2 + \frac{1}{x}\right)}\,dx =$$

$$= \frac{1}{2}\int_0^3 \sqrt{x + \frac{1}{x} + 2}\,dx = \frac{1}{2}\int_0^3 \sqrt{\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2}\,dx = \frac{1}{2}\int_0^3 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx = \frac{1}{2}\int_0^3 \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx =$$

$$= \frac{1}{2}\left[\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right]_0^3 = \left[\frac{1}{3}(\sqrt{x})^3 + \sqrt{x}\right]_0^3 = \frac{1}{3}\sqrt{27} + \sqrt{3} = \frac{1}{3}3\sqrt{3} + \sqrt{3} = \underline{\underline{2\sqrt{3}}}$$

Př. 3

$$\int_0^\infty \frac{dx}{(x+1)^4} = \begin{vmatrix} t = x+1 & x = 0 \to t = 1 \\ dt = dx & x = \infty \to t = \infty \end{vmatrix} = \int_1^\infty \frac{1}{t^4}\,dt = \left[\frac{t^{-3}}{-3}\right]_1^\infty = -\frac{1}{3}\left[\frac{1}{t^3}\right]_1^\infty = -\frac{1}{3}(0-1) = \underline{\underline{\frac{1}{3}}}$$

Př. 4 $\sum_{n=1}^\infty \frac{1}{n(\ln n + 2)^2}$

$$\int_1^\infty \frac{1}{x(\ln x + 2)^2}\,dx = \begin{vmatrix} t = \ln x + 2 \\ dt = \frac{1}{x}dx \end{vmatrix} = \int_2^\infty \frac{1}{t^2}\,dt = -\left[\frac{1}{t}\right]_2^\infty = -\left(0 - \frac{1}{2}\right) = \underline{\underline{\frac{1}{2}}} \Rightarrow \text{řada } \underline{\underline{K}}$$

$x = 1 \to t = 2$

$x = \infty \to t = \infty$