DÚ 14

1) $P(x)=ax^2+a^2x+b$, $P(1)=1$, $P(2)=-1$, $P(3)=1$, $a,b\in\mathbb{R}=?$ existují takové?

$(1)\quad 1=a+a^2+b$ $\qquad (2)-(1):\ -2=3a+a^2$ $\qquad$ dosadíme do (1): $a_1=-1$

$(2)\quad -1=4a+2a^2+b$ $\qquad a^2+3a+2=0$ $\qquad b_1=1-a_1-a_1^2=1+1-1=1$

$(3)\quad 1=9a+3a^2+b$ $\qquad a_1=-1,\ a_2=-2$ $\qquad a_2=-2$

$\qquad b_2=1-a_2-a_2^2=1+2-4=-1$

$a=-1, b=1$ dosadíme do (3): $1=9\cdot(-1)+3\cdot(-1)^2+1$

$1=-9+3+1$

$1=-5$ ... neplatí $\Rightarrow a=-1,\ b=1$ nevyhovují

$a=-2, b=-1$ —— // —— : $1=9\cdot(-2)+3\cdot(-2)^2-1$

$1=-18+12-1$

$1=-7$ ... neplatí $\Rightarrow a=-2, b=-1$ nevyhovují

<u>závěr</u>: <u>a,b taková neexistují</u>

2) a) $\lim\limits_{n\to\infty}\sqrt{\ln n}=\|\sqrt{\infty}\|=\underline{\underline{\infty}}$

b) $\lim\limits_{n\to\infty}\sqrt[n^2]{n!}=\|\sqrt[\infty]{\infty}\|=\lim\limits_{n\to\infty}(n!)^{\frac{1}{n^2}}=\|\infty^0\text{ ... neurčitý výraz}\|=e^{\ln\lim\limits_{n\to\infty}(n!)^{\frac{1}{n^2}}}=e^{\lim\limits_{n\to\infty}\ln(n!)^{\frac{1}{n^2}}}=$

$=e^{\lim\limits_{n\to\infty}\frac{1}{n^2}\cdot\ln n!}=e^0=\underline{\underline{1}}$

$\to\lim\limits_{n\to\infty}\frac{1}{n^2}\cdot\ln n!=\lim\limits_{n\to\infty}\frac{\ln n!}{n^2}=\lim\limits_{n\to\infty}\frac{\ln[n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot2\cdot1]}{n^2}=$

$=\lim\limits_{n\to\infty}\left(\frac{\ln n}{n^2}+\frac{\ln(n-1)}{n^2}+\frac{\ln(n-2)}{n^2}+\ldots+\frac{\ln 2}{n^2}+\frac{\ln 1}{n^2}\right)=0+0+0+\ldots+0+0=\underline{\underline{0}}$

3) a) $\lim\limits_{x\to0}\frac{1+2^{\frac{1}{x}}}{3+2^{\frac{1}{x}}}=\lim\limits_{x\to0}\frac{1+2^{\frac{1}{x}}}{3+2^{\frac{1}{x}}}$

$$\left.\begin{aligned}&\to\lim_{x\to0^-}\frac{1+2^{\frac{1}{x}}}{3+2^{\frac{1}{x}}}=\left\|\frac{1+2^{-\infty}}{3+2^{-\infty}}=\frac{1+0}{3+0}\right\|=\underline{\underline{\tfrac{1}{3}}}\\&\to\lim_{x\to0^+}\frac{1+2^{\frac{1}{x}}}{3+2^{\frac{1}{x}}}=\left\|\frac{1+2^{\infty}}{3+2^{\infty}}=\frac{\infty}{\infty}\right\|\overset{L.P.}{=}\lim_{x\to0^+}\frac{2^{\frac{1}{x}}\cdot\ln2\cdot(-\frac{1}{x^2})}{2^{\frac{1}{x}}\cdot\ln2\cdot(-\frac{1}{x^2})}=\underline{\underline{1}}\end{aligned}\right\}\to\lim_{x\to0}\frac{1+2^{\frac{1}{x}}}{3+2^{\frac{1}{x}}}\ \nexists$$

b) $\lim\limits_{x\to\infty}e^{\sqrt{x^2-x}}=\|e^\infty\|=\underline{\underline{\infty}}$

4) $\left(\sqrt[4]{2x-1}\right)' = ?$ z definice

$$A^4-B^4=(A-B)(A^3+A^2B+AB^2+B^3)$$

$$\left(\sqrt[4]{2x-1}\right)' = \lim_{\delta\to 0}\frac{f(x+\delta)-f(x)}{\delta} = \lim_{\delta\to 0}\frac{\overbrace{\sqrt[4]{2(x+\delta)-1}}^{A}-\overbrace{\sqrt[4]{2x-1}}^{B}}{\delta} =$$

$$=\lim_{\delta\to 0}\frac{\left(\sqrt[4]{2x+2\delta-1}-\sqrt[4]{2x-1}\right)}{\delta}\cdot\frac{\left(\sqrt[4]{2x+2\delta-1}\right)^3+\left(\sqrt[4]{2x+2\delta-1}\right)^2\left(\sqrt[4]{2x-1}\right)+\left(\sqrt[4]{2x+2\delta-1}\right)\left(\sqrt[4]{2x-1}\right)^2+\left(\sqrt[4]{2x-1}\right)^3}{\text{———}\,//\,\text{———}} =$$

$$=\lim_{\delta\to 0}\frac{\overbrace{(2x+2\delta-1)-(2x-1)}^{2\cancel{\delta}}}{\cancel{\delta}\cdot\left(\quad\text{—}\,//\,\text{—}\quad\right)} =$$

$$=\lim_{\delta\to 0}\frac{2}{\left(\sqrt[4]{2x+2\delta-1}\right)^3+\left(\sqrt[4]{2x+2\delta-1}\right)^2\left(\sqrt[4]{2x-1}\right)+\left(\sqrt[4]{2x+2\delta-1}\right)\left(\sqrt[4]{2x-1}\right)^2+\left(\sqrt[4]{2x-1}\right)^3} =$$

$$=\frac{2}{\left(\sqrt[4]{2x-1}\right)^3+\underbrace{\left(\sqrt[4]{2x-1}\right)^2\left(\sqrt[4]{2x-1}\right)}_{\left(\sqrt[4]{2x-1}\right)^3}+\underbrace{\left(\sqrt[4]{2x-1}\right)\left(\sqrt[4]{2x-1}\right)^2}_{\left(\sqrt[4]{2x-1}\right)^3}+\left(\sqrt[4]{2x-1}\right)^3} = \frac{2}{4\cdot\left(\sqrt[4]{2x-1}\right)^3} = \underline{\underline{\frac{1}{2\sqrt[4]{(2x-1)^3}}}}$$

5) Vzdálenost bodu $A=[2,0]$ od paraboly $p: y = x^2-x+1$.

na parabole zvolíme obecný bod $X=[x,y]$. Pak $|XA| = \sqrt{(x-2)^2+(y-0)^2} = \sqrt{x^2-4x+4+(x^2-x+1-0)^2} =$

$$=\sqrt{x^2-4x+4+x^4+x^2+1-2x^3+2x^2-2x} =$$

$$=\sqrt{x^4-2x^3+4x^2-6x+5}$$

nyní hledáme minimum funkce

$$f(x)=\sqrt{x^4-2x^3+4x^2-6x+5}$$

$$f'(x) = \frac{1}{2\sqrt{x^4-2x^3+4x^2-6x+5}}\cdot(4x^3-6x^2+8x-6) = 0 \quad /:2$$

$$2x^3-3x^2+4x-3=0$$

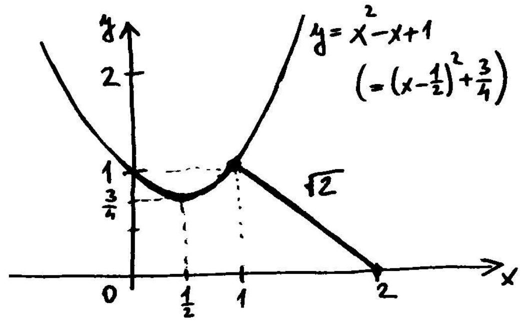
$$(x-1)(2x^2-x+3)=0$$

$$x=1 \quad \text{nebo} \quad x_{2,3}=\frac{1\pm\sqrt{1-24}}{4}\ \ldots\text{nelze}$$

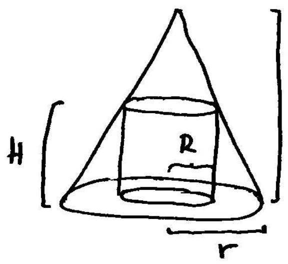
$\Rightarrow$ nejmenší vzdálenost bodu $[2,0]$ od paraboly $y=x^2-x+1$ je v bodě $x=1 \Rightarrow y = 1^2-1+1 = 1$

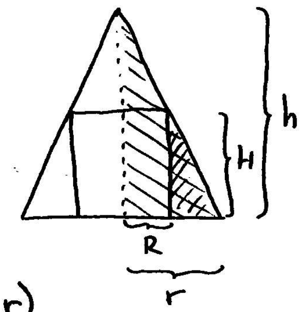
a vzdálenost je:

$$|AX| = \sqrt{(1-2)^2+(1-0)^2} = \underline{\underline{\sqrt{2}}}$$



6) Max. objem válce vepsaného do rotačního kužele o poloměru podstavy r a výšce h.





$$V = \pi R^2 \cdot H \qquad \frac{h}{r} = \frac{H}{r-R} \quad \text{(plyne z podobnosti vyšrafovaných Δ-ů)}$$

$$H = \frac{h}{r}(r-R)$$

$$V = \pi R^2 \cdot \frac{h}{r}(r-R)$$

$$V = \frac{\pi h}{r}\cdot(rR^2 - R^3)$$

$$V' = \frac{\pi h}{r}(2rR - 3R^2) = 0$$

$$2rR - 3R^2 = 0$$

$$R(2r - 3R) = 0$$

$$R = 0 \quad \text{nebo} \quad 2r - 3R = 0$$

nelze, $R \in (0, r)$

$$\underline{R = \tfrac{2}{3}r} \quad (\Rightarrow H = \tfrac{1}{3}h)$$

jedná se o max nebo min (v $\frac{2}{3}r$)?

$$V'' = \frac{\pi h}{r}(2r - 6R)$$

$$V''\left(\tfrac{2}{3}r\right) = \frac{\pi h}{r}\left(2r - 6\cdot\tfrac{2}{3}r\right) = \frac{\pi h}{r}\cdot(2r - 4r) = -2\pi h < 0 \Rightarrow \text{v } R = \tfrac{2}{3}r \text{ je } \underline{\text{max}}$$

objem max. válce je pak:
$$V_{max} = \frac{\pi h}{r}(rR^2 - R^3) = \frac{\pi h}{r}\left(r\cdot\left(\tfrac{2}{3}r\right)^2 - \left(\tfrac{2}{3}R\right)^3\right) =$$
$$= \frac{\pi h}{r}\left(\frac{4}{9}r^3 - \frac{8}{27}r^3\right) = \frac{\pi h}{r}\cdot\frac{4}{27}r^3 = \underline{\underline{\frac{4}{27}\pi r^2 h}}$$

pozn. uvědomíme-li, že objem celého kužele je $V_k = \frac{1}{3}\pi r^2 h$, pak můžeme říct, že:

„Maximální objem válce vepsaného do rotačního kužele tvoří $\frac{4}{9}$ objemu tohoto kužele."
(tedy necelou polovinu :))

7) a)
$$\int (9+x^2)^{3/2}\,dx = \int \frac{(9+x^2)^{3/2}}{1}\cdot\frac{(9+x^2)^{1/2}}{(9+x^2)^{1/2}}\,dx = \int \frac{(9+x^2)^2}{\sqrt{9+x^2}}\,dx =$$

$$= \int \frac{x^4 + 18x^2 + 81}{\sqrt{9+x^2}}\,dx = (Ax^3 + Bx^2 + Cx + D)\sqrt{9+x^2} + k\cdot\int\frac{1}{\sqrt{9+x^2}}\,dx \quad \Big/\,'$$

$$\frac{x^4 + 18x^2 + 81}{\sqrt{9+x^2}} = (3Ax^2 + 2Bx + C)\sqrt{9+x^2} + \frac{1\cdot\cancel{2}x}{\cancel{2}\sqrt{9+x^2}}\cdot(Ax^3 + Bx^2 + Cx + D) + k\cdot\frac{1}{\sqrt{9+x^2}} \quad \Big/ \cdot\sqrt{9+x^2}$$

$$x^4 + 18x^2 + 81 = (3Ax^2 + 2Bx + C)(9+x^2) + x(Ax^3 + Bx^2 + Cx + D) + k$$

$$x^4 + 18x^2 + 81 = 27Ax^2 + 3Ax^4 + 18Bx + 2Bx^3 + 9C + Cx^2 + Ax^4 + Bx^3 + Cx^2 + Dx + k$$

x^4: $1 = 4A \rightarrow A = \frac{1}{4}$
x^3: $0 = 3B \rightarrow B = 0$
x^2: $18 = 27A + 2C \rightarrow C = \frac{45}{8}$
x^1: $0 = 18B + D \rightarrow D = 0$
x^0: $81 = 9C + k \quad k = \frac{243}{8}$

$$\int\frac{1}{\sqrt{9+x^2}}\,dx = \ln\left|x + \sqrt{9+x^2}\right|$$

celkem:

$$\int (9+x^2)^{3/2}\,dx = \left(\frac{1}{4}x^3 + \frac{45}{8}x\right)\sqrt{9+x^2} + \frac{243}{8}\ln\left|x + \sqrt{9+x^2}\right| + c$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx =$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx = \frac{1}{4}\left(x - \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x\right) + C =$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \Rightarrow \cos^2 2x = \frac{1+\cos 4x}{2}$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

c) $$\int \frac{1}{\sqrt{1-x}}dx = \left| \begin{array}{l} t = 1-x \\ dt = -dx \\ -dt = dx \end{array} \right| = \int \frac{1}{\sqrt{t}}(-dt) = -\int t^{-\frac{1}{2}}dt = -\frac{t^{1/2}}{\frac{1}{2}} = -2\sqrt{t} + C = -2\sqrt{1-x} + C$$

d) $$\int \frac{\sqrt{9-4x^2}}{x}dx = \left| \begin{array}{l} t = \sqrt{9-4x^2}\ \Big|^2 \\ t^2 = 9-4x^2 \to x^2 = \frac{9-t^2}{4} \\ 2t\,dt = -8x\,dx \quad \big|:(-8) \\ -\frac{1t}{4}dt = x\,dx \end{array} \right| = \int \frac{\sqrt{9-4x^2}}{x^2}\cdot x\,dx = \int \frac{t}{\frac{9-t^2}{4}}\left(-\frac{1t}{4}dt\right) = \int \frac{t^2}{t^2-9}dt =$$

$$t^2 : (t^2-9) = 1 + \frac{9}{t^2-9}$$
$$\frac{-(t^2-9)}{9}$$

$$\frac{9}{t^2-9} = \frac{9}{(t-3)(t+3)} = \frac{A}{t-3} + \frac{B}{t+3} \quad \Big|\cdot(t-3)(t+3)$$

$$9 = A(t+3) + B(t-3)$$

$$t^1: \quad 0 = A + B \quad |\cdot 3$$

$$t^0: \quad 9 = 3A - 3B$$

$$9 = 6A \quad A = \frac{3}{2}, \quad B = -\frac{3}{2}$$

$$= \int \left(1 + \frac{+\frac{3}{2}}{t-3} + \frac{-\frac{3}{2}}{t+3}\right)dt = t + \frac{3}{2}\ln|t-3| - \frac{3}{2}\ln|t+3| + C = t + \frac{3}{2}\ln\left|\frac{t-3}{t+3}\right| + C =$$

$$= \sqrt{9-4x^2} + \frac{3}{2}\ln\left|\frac{\sqrt{9-4x^2}-3}{\sqrt{9-4x^2}+3}\right| + C$$

e) $\int x \cdot \arcsin x \, dx = \begin{vmatrix} u' = x & v = \arcsin x \\ u = \frac{x^2}{2} & v' = \frac{1}{\sqrt{1-x^2}} \end{vmatrix} = \frac{x^2}{2} \arcsin x - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} \, dx$

$$\int \frac{\frac{1}{2}x^2}{\sqrt{1-x^2}} \, dx = (Ax+B)\sqrt{1-x^2} + k \cdot \int \frac{1}{\sqrt{1-x^2}} \, dx \quad \Big/ \, '$$

$$\frac{\frac{1}{2}x^2}{\sqrt{1-x^2}} = A\sqrt{1-x^2} + (Ax+B)\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} + k \cdot \frac{1}{\sqrt{1-x^2}} \quad \Big/ \cdot \sqrt{1-x^2}$$

$$\tfrac{1}{2}x^2 = A(1-x^2) + (Ax+B)(-x) + k$$

$$\tfrac{1}{2}x^2 = A - \underline{Ax^2} - \underline{Ax^2} - Bx + k$$

$x^2: \ \frac{1}{2} = -2A \ \rightarrow \ A = -\frac{1}{4}$

$x^1: \ 0 = -B \ \rightarrow \ B = 0$

$x^0: \ 0 = A + k \ \rightarrow \ k = \frac{1}{4}$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x$$

celkem:

$$\int x \cdot \arcsin x \, dx = \frac{x^2}{2} \arcsin x - \left(-\frac{1}{4}x\sqrt{1-x^2} + \frac{1}{4}\arcsin x\right) + c = \left(\frac{x^2}{2} - \frac{1}{4}\right)\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + c$$

f) $\int \frac{x^2+x-1}{(x^2+1)^2} \, dx = *$

$$\frac{x^2+x-1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \quad \Big/ \cdot (x^2+1)^2$$

$$x^2+x-1 = (Ax+B)(x^2+1) + Cx + D$$

$$x^2+x-1 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

$x^3: \ 0 = A \ \rightarrow \ A = 0$

$x^2: \ 1 = B \ \rightarrow \ B = 1$

$x^1: \ 1 = A + C \ \rightarrow \ C = 1$

$x^0: \ -1 = B + D \ \rightarrow \ D = -2$

pozn.: při výpočtu integrálu ⌂ využijeme vztah:

$$\int \frac{1}{(x^2+a^2)^n} \, dx = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} \cdot \int \frac{1}{(x^2+a^2)^{n-1}} \, dx$$

resp.:

$$\int \frac{1}{(x^2+a^2)^{n+1}} \, dx = \frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2n} \cdot \int \frac{1}{(x^2+a^2)^n} \, dx$$

pro $a = 1$:

$$\int \frac{1}{(x^2+1)^{n+1}} \, dx = \frac{x}{2n(x^2+1)^n} + \frac{2n-1}{2n} \int \frac{1}{(x^2+1)^n} \, dx$$!!

sem dosadíme za n číslo 1.

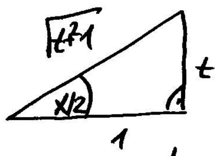
$$* = \int \frac{1}{x^2+1} + \frac{x-2}{(x^2+1)^2} \, dx = \int \left(\frac{1}{x^2+1} + \frac{x}{(x^2+1)^2} - \frac{2}{(x^2+1)^2}\right) dx = \begin{vmatrix} t = x^2+1 \\ dt = 2x\,dx \\ \frac{1}{2}dt = x\,dx \end{vmatrix} =$$

$$= \operatorname{arctg} x + \int \frac{1}{t^2} \tfrac{1}{2} dt - 2 \overbrace{\int \frac{1}{(x^2+1)^2} \, dx}^{⌂} = \operatorname{arctg} x - \frac{1}{2t} - 2\left(\frac{x}{2(x^2+1)} + \frac{1}{2} \underbrace{\int \frac{1}{x^2+1} \, dx}_{\operatorname{arctg} x}\right) =$$

$$= -\frac{1}{2(x^2+1)} - \frac{x}{x^2+1} + c = \frac{-1-2x}{2x^2+2} + c = -\frac{2x+1}{2(x^2+1)} + c$$

8) $\displaystyle\int \frac{1}{3-2\cos x}\,dx = \left|\begin{array}{l} \text{funkce: není lichá vůči } \sin x \\ \qquad \text{—"—} \qquad \cos x \\ \text{není sudá vůči } \sin x \text{ a } \cos x \text{ zároveň} \end{array}\right\} \Rightarrow$ univerzální substituce $t = \operatorname{tg}\frac{x}{2}$

$$\left|\begin{array}{llll} t = \operatorname{tg}\frac{x}{2} & & \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} & \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \\ \operatorname{arctg} t = \frac{x}{2} & & \sin x = 2\frac{t}{\sqrt{t^2+1}}\cdot\frac{1}{\sqrt{t^2+1}} & \cos x = \left(\frac{1}{\sqrt{t^2+1}}\right)^2 - \left(\frac{t}{\sqrt{t^2+1}}\right)^2 \\ \frac{1}{1+t^2}\,dt = \frac{1}{2}dx & \sin\frac{x}{2} = \frac{t}{\sqrt{t^2+1}} & \sin x = \frac{2t}{t^2+1} & \cos x = \frac{1-t^2}{t^2+1} \\ \frac{2}{1+t^2}\,dt = dx & \cos\frac{x}{2} = \frac{1}{\sqrt{t^2+1}} & & \end{array}\right| =$$



$$= \int \frac{1}{3-2\frac{1-t^2}{t^2+1}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{2}{3(1+t^2)-2(1-t^2)}\,dt = \int \frac{2}{3+3t^2-2+2t^2}\,dt = \int \frac{2}{1+5t^2}\,dt =$$

$$= \frac{2}{5}\int \frac{1}{t^2+\frac{1}{5}}\,dt = \frac{2}{5}\frac{1}{\sqrt{\frac{1}{5}}}\operatorname{arctg}\frac{t}{\sqrt{\frac{1}{5}}} + c = \frac{2\sqrt{5}}{5}\operatorname{arctg}\sqrt{5}t + c = \underline{\underline{\frac{2}{\sqrt{5}}\operatorname{arctg}\left(\sqrt{5}\operatorname{tg}\frac{x}{2}\right) + c}}$$

8) a) $\displaystyle\int_0^1 \ln(x^2+1)\,dx = \left|\begin{array}{ll} u' = 1 & v = \ln(x^2+1) \\ u = x & v' = \frac{1}{x^2+1}\cdot 2x \end{array}\right| = \left[x\ln(x^2+1)\right]_0^1 - \int_0^1 \frac{2x^2}{x^2+1}\,dx =$

$$\begin{array}{l} 2x^2 : (x^2+1) = 2 - \frac{2}{x^2+1} \\ \underline{-(2x^2+2)} \\ \quad -2 \end{array}$$

$$= \left[x\ln(x^2+1)\right]_0^1 - \int_0^1\left(2 - \frac{2}{x^2+1}\right)dx = \left[x\ln(x^2+1) - 2x + 2\operatorname{arctg} x\right]_0^1 =$$

$$= \left[\ln 2 - 2 - 2\cdot\frac{\pi}{4}\right] - \left[0 - 0 + 2\cdot 0\right] = \underline{\underline{\ln 2 - 2 - \frac{\pi}{2}}}$$

b) $\displaystyle\int_1^3 \ln\left(x+\sqrt{x^2-1}\right)dx = \left|\begin{array}{ll} u' = 1 & v = \ln\left(x+\sqrt{x^2-1}\right) \\ u = x & v' = \frac{1}{x+\sqrt{x^2-1}}\cdot\left(1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}\right) = \frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}} \end{array}\right| =$

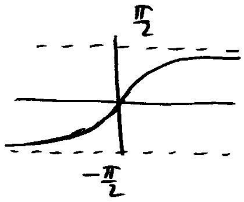
$$= \left[x\ln\left(x+\sqrt{x^2-1}\right)\right]_1^3 - \int_1^3 \frac{x}{\sqrt{x^2-1}}\,dx = \left|\begin{array}{ll} t = x^2-1 & x=1 \to t=0 \\ dt = 2x\,dx & x=3 \to t=8 \\ \frac{1}{2}dt = x\,dx & \end{array}\right| =$$

$$= 3\ln(3+\sqrt{8}) - 0 - \int_0^8 \underbrace{\frac{1}{\sqrt{t}}}_{t^{-1/2}}\frac{1}{2}\,dt = 3\ln(3+\sqrt{8}) - \cancel{\tfrac{1}{2}}\left[\frac{t^{1/2}}{\cancel{1/2}}\right]_0^8 = \underline{\underline{3\ln(3+\sqrt{8}) - \sqrt{8}}}$$

c) $\int_2^{\infty} \frac{dx}{x\ln^2 x} = \left| \begin{matrix} t=\ln x & x=2 \to t=\ln 2 \\ dt=\frac{1}{x}dx & x=\infty \to t=\infty \end{matrix} \right| = \int_{\ln 2}^{\infty} \overset{t^{-2}}{\frac{1}{t^2}}\, dt = \left[\frac{t^{-1}}{-1}\right]_{\ln 2}^{\infty} = -\left[\frac{1}{t}\right]_{\ln 2}^{\infty} =$

$= -\left(0-\frac{1}{\ln 2}\right) = \underline{\underline{\frac{1}{\ln 2}}}$

d) $\int_{-\infty}^{\infty} \frac{dx}{1+4x^2} = \frac{1}{4}\int_{-\infty}^{\infty} \frac{dx}{x^2+\frac{1}{4}} = \left[\frac{1}{4}\cdot\frac{1}{\sqrt{\frac{1}{4}}}\,\mathrm{arctg}\,\frac{x}{\sqrt{\frac{1}{4}}}\right]_{-\infty}^{\infty} = \frac{1}{2}\left[\mathrm{arctg}\, 2x\right]_{-\infty}^{\infty} =$



$= \frac{1}{2}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \underline{\underline{\frac{\pi}{2}}}$

9) Rozhodněte o konvergenci či divergenci řad:

a) $\sum_{n}^{\infty} \frac{n}{2^{2n}}$

podílové kritérium: $q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{2(n+1)}}}{\frac{n}{2^{2n}}} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \lim_{n\to\infty} \frac{2^{2n}}{2^{2n+2}} \overset{L.P.}{=}$

$= \lim_{n\to\infty} \frac{1}{1} \cdot \lim \frac{\cancel{2^{2n}}}{\cancel{2^{2n}}\cdot 2^2} = 1 \cdot \frac{1}{4} = \frac{1}{4} < 1 \Rightarrow$ <u>řada konverguje</u>

b) $\sum_{n}^{\infty} n\left(\frac{3}{4}\right)^n$

podílové kritérium: $q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\left(\frac{3}{4}\right)^{\cancel{n+1}}}{n\cancel{\left(\frac{3}{4}\right)^n}} = \lim_{n\to\infty} \frac{n+1}{n}\cdot\frac{3}{4} = 1\cdot\frac{3}{4} = \frac{3}{4} < 1 \Rightarrow$

$\Rightarrow$ <u>řada konverguje</u>

10) Rozviňte do mocninné řady funkci arctg x v bodě 0 a určete, pro která x konverguje.

$f(x) = \mathrm{arctg}\, x \qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x^2)^2}\cdot 2x \qquad f''(0) = 0$

$f'''(x) = -\frac{2(1+x^2)^{\cancel{2}} - 2x\cdot 2\cancel{(1+x^2)}\cdot 2x}{(1+x^2)^{\cancel{4}}} = -\frac{2+2x^2-8x^2}{(1+x^2)^3} = \frac{6x^2-2}{(1+x^2)^3} \qquad f'''(0) = -2$

$f^{(IV)}(x) = \frac{12x(1+x^2)^{\cancel{3}} - (6x^2-2)\cdot 3\cancel{(1+x^2)^2}\cdot 2x}{(1+x^2)^{\cancel{6}4}} = \frac{12x+12x^3-36x^3+12x}{(1+x^2)^4} = \frac{24x-24x^3}{(1+x^2)^4} \qquad f^{(IV)}(0) = 0$

$= 24\frac{x-x^3}{(1+x^2)^4}$

$$f^{(v)}(x) = 24\,\frac{(1-3x^2)(1+x^2)^{\cancel{4}} - (x-x^3)\cdot 4\cancel{(1+x^2)^3}\cdot 2x}{(1+x^2)^{\cancel{8}5}} = 24\,\frac{1+x^2-3x^2-3x^4-8x^2+8x^4}{(1+x^2)^5} = 24\,\frac{5x^4-10x^2+1}{(1+x^2)^5}$$

$$f^{(v)}(0) = 24$$

$$f(x_0) + \frac{f'(x_0)}{1!}x + \frac{f''(x_0)}{2!}x^2 + \frac{f'''(x_0)}{3!}x^3 + \frac{f^{(iv)}(x_0)}{4!}x^4 + \frac{f^{(v)}(x_0)}{5!}x^5 + \dots =$$

$$= 0 + \frac{1}{1}x + 0 + \frac{-2}{6}x^3 + 0 + \frac{24}{120}x^5 + \dots =$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \underline{\underline{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}}$$

<u>obor konvergence</u> :

podílové kritérium :

$$q = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n \frac{x^{2(n+1)+1}}{2(n+1)+1}}{(-1)^{n+1}\frac{x^{2n+1}}{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)\,x^{2n+3}}{(2n+3)\,x^{2n+1}}\right| = \lim_{n\to\infty}\frac{2n+1}{2n+3}\cdot|x^2| = x^2$$

řada konverguje tam, kde $q < 1$

$$x^2 < 1 \Rightarrow x \in (-1,1)$$

krajní body :

$x=-1$: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{2n+1}$ ... K dle Leibnitzova kriteria, protože :

$\left\{\frac{1}{2n+1}\right\}$ ... nerostoucí ✓

$\frac{1}{2n+1} > 0 \ \forall n \in \mathbb{N}$ ✓

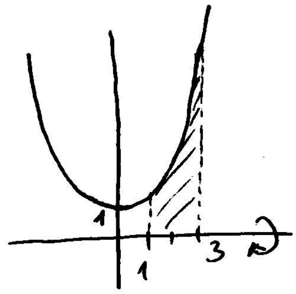
$\lim_{n\to\infty}\frac{1}{2n+1} = 0$ ✓

$x=1$: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ ... K —— || ——

obor konvergence je tedy <u>$x \in \langle -1, 1\rangle$</u>

celkově tedy platí: $\underline{\underline{\displaystyle \operatorname{arctg} x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \quad \text{pro } x \in \langle -1,1\rangle}}$

11) Objem tělesa vzniklého rotací paraboly $y = 2x^2+1$ kolem osy x, $x \in \langle 1,3 \rangle$.



$$V = \pi \int_1^3 f^2(x)\,dx = \pi \int_1^3 (2x^2+1)^2\,dx = \pi \int_1^3 (4x^4+4x^2+1)\,dx =$$

$$= \pi \left[4\frac{x^5}{5} + 4\frac{x^3}{3} + x \right]_1^3 =$$

$$= \pi \left[\left(4\cdot\frac{243}{5} + 4\cdot\frac{27}{3} + 3\right) - \left(\frac{4}{5} + \frac{4}{3} + 1\right) \right] =$$

$$= \pi\,\frac{2916+540+45-12-20-15}{15} = \frac{3454}{15}\pi = \underline{\underline{230\tfrac{4}{15}\pi}}$$

12) Vyšetřete průběh funkce $y = \ln\left(\frac{x^2-1}{x-2}\right)$.

1) $D(f)$: $\frac{x^2-1}{x-2} > 0$

$\frac{(x+1)(x-1)}{x-2} > 0$

$x+1$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
	$-$	$\oplus$	$-$	$\oplus$
		-1	1	2

$D(f) = (-1,1) \cup (2,\infty)$

$f(-x) = \ln\left(\frac{(-x)^2-1}{-x-2}\right) = \ln\frac{x^2-1}{-x-2}$ $\Rightarrow$ fce není ani sudá ani lichá

fce není ani periodická

2) Kladná, záporná

$y = \ln\frac{x^2-1}{x-2} \lessgtr 0$

$\frac{x^2-1}{x-2} \lessgtr 1$

$\frac{x^2-1}{x-2} - 1 \lessgtr 0$

$\frac{x^2-1-x+2}{x-2} \lessgtr 0$

$\frac{x^2-x+1}{x-2} \lessgtr 0$

$D = (-1)^2 - 4\cdot1\cdot1 = -3 \Rightarrow$ nerozložitelný

x^2-x+1 ... vždy $\oplus$

$x-2$	$-$	$+$
		2
y	$-$	$+$
	pod osou	nad osou

3) Rostoucí, klesající

$$y' = \frac{\cancel{x-2}}{x^2-1}\cdot\frac{2x\cdot(x-2)-(x^2-1)\cdot 1}{(x-2)^{2}} = \frac{x^2-4x+1}{\underbrace{(x^2-1)(x-2)}_{x^3-2x^2-x+2}}$$

n.b.: $\pm1, 2,$ $\frac{4\pm\sqrt{16-4}}{2} = 2\pm\sqrt{3}$

x^2-4x+1	$+$	$+$	$-$	$-$	$-$	$+$
$x-1$	$-$	$-$	$-$	$+$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$-$	$+$	$+$
		-1	$2-\sqrt{3}$	1	2	$2+\sqrt{3}$
y'	$-$	$+$	$-$	$+$	$-$	$+$
	$\searrow$	$\nearrow$ max	$\searrow$	$\nearrow$	$\searrow$ min	$\nearrow$

je v $D(f)$ je v $D(f)$

4) Konvexní, konkávní

$$y'' = \frac{(2x-4)(x^3-2x^2-x+2)-(x^2-4x+1)(3x^2-4x-1)}{(x^2-1)^2(x-2)^2} =$$

$$= \frac{2x^4-4x^3-2x^2+4x-4x^3+8x^2+4x-8-3x^4+4x^3+x^2+12x^3-16x^2-4x-3x^2+4x+1}{(x^2-1)^2(x-2)^2} =$$

$$= \frac{-x^4+8x^3-12x^2+8x-7}{\underbrace{(x^2-1)^2(x-2)^2}_{\text{vždy } \oplus}} \xrightarrow{\text{pc}}$$

− asi 1,5 + asi 6,2 −

⊕ D(f)

y'': -1 − 1 2 + − ; ∩ ∪ ∩

5) Asymptoty

a) bez směrnice

$$\lim_{x\to -1^+} \ln\frac{x^2-1}{x-2} = \lim_{x\to -1^+} \ln\frac{(x+1)(x-1)}{x-2} = \lim_{x\to-1^+} \ln(x+1) + \lim_{x\to-1^+}\ln\frac{x-1}{x-2} = -\infty + \ln\frac{2}{3} = -\infty$$

$$\lim_{x\to 1^-} \ln\frac{x^2-1}{x-2} = \lim_{x\to 1^-} \frac{(x+1)(x-1)}{x-2} = \left\| \ln\frac{2\cdot(-0)}{-1} = \ln(+0) \right\| = -\infty$$

asymptota $x = \pm 1$

$$\lim_{x\to 2^+} \frac{x^2-1}{x-2} = \left\| \ln\frac{3}{+0} = \ln(+\infty) \right\| = \infty$$

… asymptota $x = 2$

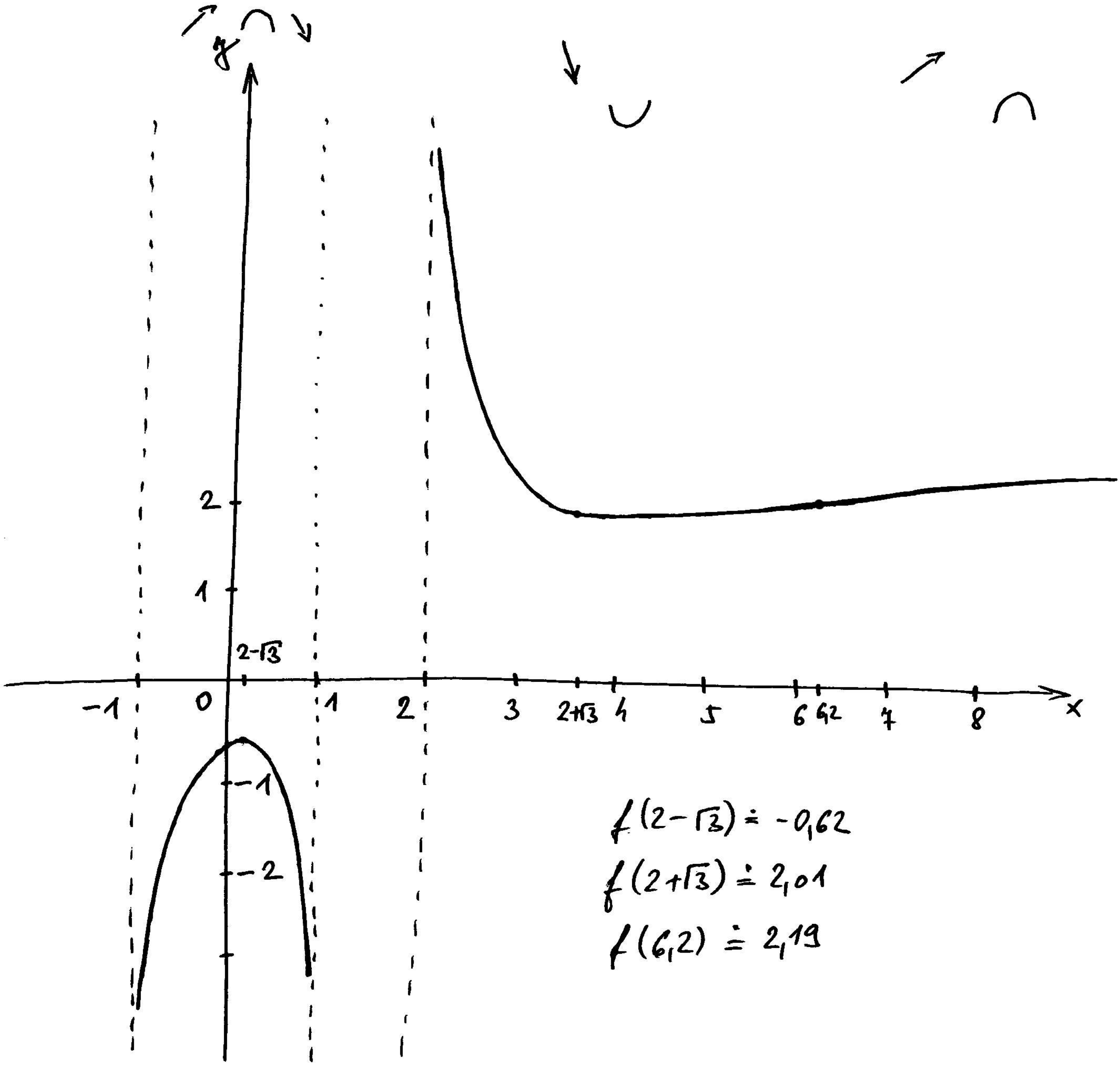
b) se směrnicí

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\ln\frac{x^2-1}{x-2}}{x} = \left\| \lim_{x\to\infty} \frac{x^2-1}{x-2} = \infty \Rightarrow \lim_{x\to\infty} \ln\frac{x^2-1}{x-2} = \infty \Rightarrow \frac{\infty}{\infty} \right\| =$$

$$\overset{\text{L.P.}}{=} \lim_{x\to\infty} \frac{\frac{x^2-4x+1}{(x^2-1)(x-2)}}{1} = \lim_{x\to\infty} \frac{x^2-4x+1}{x^3-2x^2-x+2} = 0$$

$$b = \lim_{x\to\infty} (f(x) - a\cdot x) = \lim_{x\to\infty} \ln\frac{x^2-1}{x-2} = \infty$$

asymptotu se směrnicí nemá



$f(2-\sqrt{3}) \doteq -0{,}62$

$f(2+\sqrt{3}) \doteq 2{,}01$

$f(6{,}2) \doteq 2{,}19$