DÚ8

př 1 $V = 64\,cm^3$, válec, rozměry, aby S_{min}

a) uzavřená

$V = \pi r^2 \cdot v$ $\qquad$ $v = \frac{V}{\pi r^2}$

$S = 2\pi r^2 + 2\pi r v$

$S = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2}$

$S = 2\pi r^2 + \frac{2V}{r}$

$S' = 4\pi r - \frac{2V}{r^2}$

$4\pi r - \frac{2V}{r^2} = 0 \;/\cdot r^2$

$2\pi r^3 - V = 0$

$r = \sqrt[3]{\frac{V}{2\pi}} \Rightarrow d = 2\sqrt[3]{\frac{V}{2\pi}} = \sqrt[3]{\frac{4V}{\pi}}$

$v = \frac{V}{\pi r^2} = \frac{V}{\pi \sqrt[3]{\frac{V^2}{4\pi^2}}} = \sqrt[3]{\frac{V^3 4\pi^2}{\pi^3 V^2}} = \sqrt[3]{\frac{4V}{\pi}}$

$\left.\right\} v = d = \sqrt[3]{\frac{4V}{\pi}} \doteq \underline{\underline{4{,}3354\,cm}}$

$S'' = 4\pi + \frac{4V}{r^3}$

$S''\left(\sqrt[3]{\frac{4V}{\pi}}\right) = 4\pi + \frac{4V}{\frac{4V}{\pi}} = 5\pi > 0 \Rightarrow$

$\Rightarrow \underline{\underline{\min}}$

b) otevřená

$V = \pi r^2 \cdot v$ $\qquad$ $v = \frac{V}{\pi r^2}$

$S = \pi r^2 + 2\pi r \cdot v$

$S = \pi r^2 + 2\pi r \frac{V}{\pi r^2}$

$S = \pi r^2 + \frac{2V}{r}$

$S' = 2\pi r - \frac{2V}{r^2}$

$2\pi r - \frac{2V}{r^2} = 0 \;/\cdot r^2$

$\pi r^3 - V = 0$

$r^3 = \sqrt[3]{\frac{V}{\pi}}$

$v = \frac{V}{\pi r^2} = \frac{V}{\pi \sqrt[3]{\frac{V^2}{\pi^2}}} = \sqrt[3]{\frac{V^3 \pi^2}{\pi^3 V^2}} = \sqrt[3]{\frac{V}{\pi}}$

$\left.\right\} r = v = \sqrt[3]{\frac{V}{\pi}} \doteq \underline{\underline{2{,}7311\,cm}}$

$S'' = 2\pi + \frac{4V}{r^3}$

$S''\left(\sqrt[3]{\frac{V}{\pi}}\right) = 2\pi + \frac{4V}{\frac{V}{\pi}} = 6\pi > 0 \Rightarrow$

$\Rightarrow \underline{\underline{\min}}$

př 2 $y = \frac{x^2 - x + 2}{2x^2 + x - 1} = \frac{x^2 - x + 2}{2(x-0{,}5)(x+1)}$

nerozložitelný ($D = -7$)

1) $D(f) = \mathbb{R} \setminus \{0{,}5;\, -1\}$

2) kladná, záporná

$x^2 - x + 2$ … vždy $\oplus$

		-1		$0{,}5$	
$x - 0{,}5$	$-$		$-$		$+$
$x + 1$	$-$		$+$		$+$
y	$\oplus$ nad osou		$\ominus$ pod osou		$\oplus$ nad osou

3) Rostoucí, klesající

$y' = \frac{(2x-1)(2x^2+x-1) - (x^2-x+2)(4x+1)}{(2x^2+x-1)^2} = \frac{4x^3 + 2x^2 - 2x - 2x^2 - x + 1 - 4x^3 - x^2 + 4x^2 + x - 8x - 2}{(2x^2+x-1)^2} =$

$= \frac{3x^2 - 10x - 1}{(2x^2+x-1)^2}$ ← vždy

$\longrightarrow x_{1,2} = \frac{10 \pm \sqrt{100+12}}{6} = \frac{10 \pm \sqrt{112}}{6} = \frac{10 \pm \sqrt{7 \cdot 16}}{6} = \frac{10 \pm 4\sqrt{7}}{6} = \frac{5 \pm 2\sqrt{7}}{3} \doteq$ 3,4 ; −0,1

		$\frac{5-2\sqrt{7}}{3}$		$\frac{5+2\sqrt{7}}{3}$	
$x - \frac{5-2\sqrt{7}}{3}$	$-$		$+$		$+$
$x - \frac{5+2\sqrt{7}}{3}$	$-$		$-$		$+$
y'	$+$ ↗	max	$-$ ↘	min	$+$ ↗

4) Konvexní, konkávní

$$y'' = \frac{(6x-10)(2x^2+x-1)^{\cancel{2}} - (3x^2-10x-1)2\,\cancel{(2x^2+x-1)}\cdot(4x+1)}{(2x^2+x-1)^{\cancel{4}\,3}} =$$

$$= \frac{12x^3+6x^2-6x-20x^2-10x+10-24x^3-6x^2+80x^2+20x+8x+2}{8(x-0{,}5)^3(x+1)^3} = \frac{12(-x^3+5x^2+x+1)}{8(x-0{,}5)^3(x+1)^2}$$

polynom $-x^3+5x^2+x+1$ → jeho znaménko zjistíme tak, že si necháme na pc vyjet graf nebo celou konvexnost a konkávnost vynechme :)

		-1		$0{,}5$		přibližně $5{,}23$	
$(x-0{,}5)^3$	$-$		$-$		$+$		$+$
$(x+1)^2$	$-$		$+$		$+$		$+$
$-x^3+5x^2+x+1$	$+$		$+$		$+$		$-$
y''	$+$		$-$		$+$		$-$
	∪		∩		∪	infl.	∩

5) Asymptoty

a) bez směrnice

$$\lim_{x\to -1^+} \frac{x^2-x+2}{2x^2+x-1} = \lim_{x\to -1^+} \frac{x^2-x+2}{2(x-0{,}5)} \cdot \lim_{x\to -1^+} \frac{1}{(x+1)} = \frac{4}{-3}\cdot\infty = -\infty$$

$$\lim_{x\to -1^-} \text{—||—} = \lim_{x\to -1^-} \text{—||—} \cdot \lim_{x\to -1^-} \text{—||—} = \frac{4}{-3}\cdot(-\infty) = \infty$$

... asymptota bez směrnice $x = -1$

$$\lim_{x\to 0{,}5^+} \frac{x^2-x+2}{2x^2+x-1} = \lim_{x\to 0{,}5^+} \frac{x^2-x+2}{2(x+1)} \cdot \lim_{x\to 0{,}5^+} \frac{1}{x-0{,}5} = \frac{1{,}75}{3}\cdot\infty = \infty$$

$$\lim_{x\to 0{,}5^-} \text{—||—} = \lim_{x\to 0{,}5^-} \text{—||—} \cdot \lim_{x\to 0{,}5^-} \text{—||—} = \frac{1{,}75}{3}\cdot(-\infty) = -\infty$$

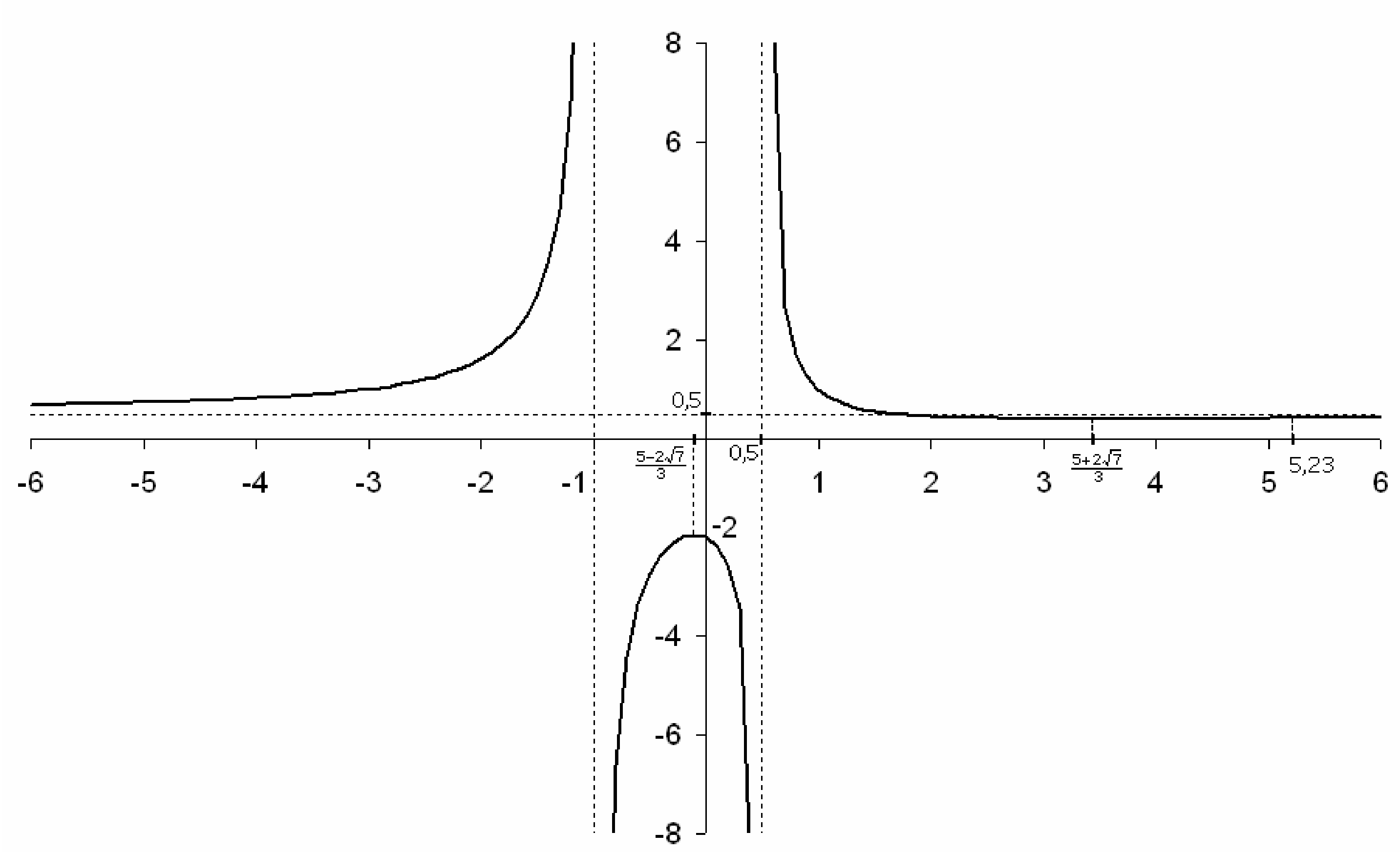
... asymptota bez směrnice $x = 0{,}5$

b) se směrnicí

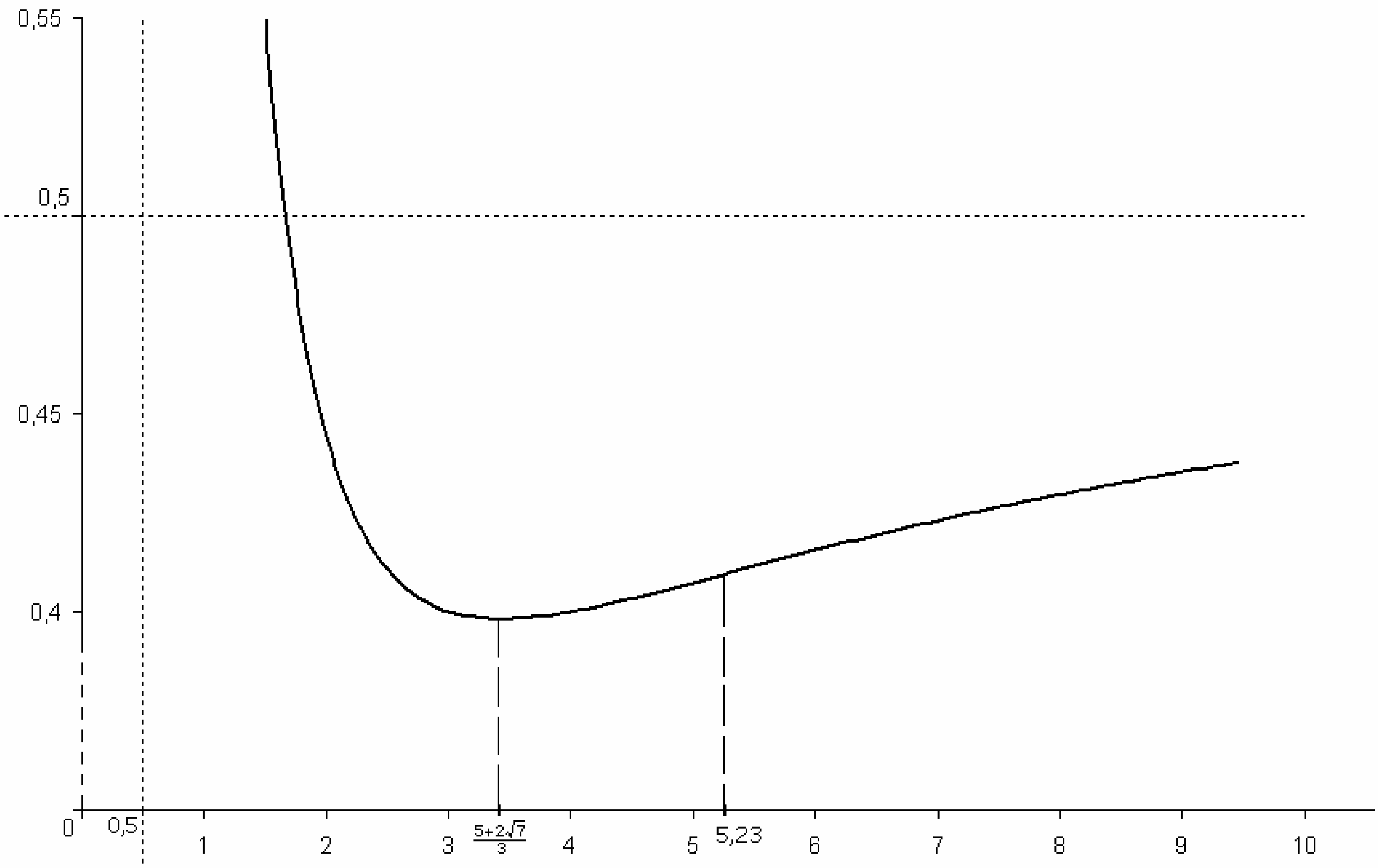
$$a = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2-x+2}{2x^3+x^2-1} = 0$$

$$b = \lim_{x\to\pm\infty} (f(x) - a\cdot x) = \lim_{x\to\pm\infty} \frac{x^2-x+2}{2x^2+x-1} \overset{\frac{\infty}{\infty}\,L.P.}{=} \lim_{x\to\pm\infty} \frac{2x-1}{4x^2+1} \overset{\frac{\infty}{\infty}\,L.P.}{=} \lim_{x\to\pm\infty} \frac{2}{4} = \frac{1}{2}$$

asymptota se směrnicí pro $\pm\infty$ ... $y = \frac{1}{2}$



V kladných hodnotách protne graf funkce asymptotu y = 0,5 a pak se k ní zespodu přibližuje. To není v celém grafu příliš patrné; lépe je to vidět na detailu:



(př. 3)

a) $y = \sin \frac{1}{x^2}$

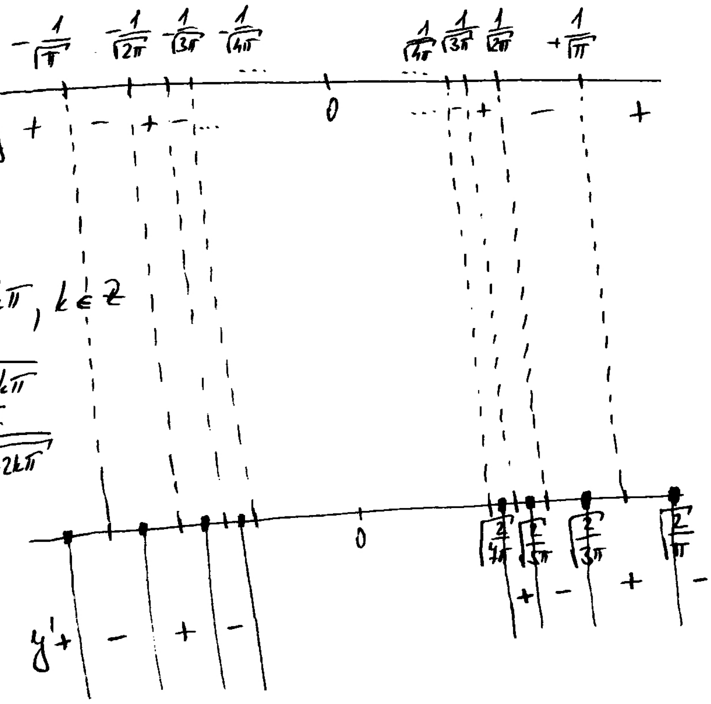
1) $D(f) = \mathbb{R} \setminus \{0\}$, $\sin \frac{1}{(-x)^2} = \sin \frac{1}{x^2} \Rightarrow$ sudá fce

2) Kladná, záporná

nulové body: $\sin \frac{1}{x^2} = 0 \Rightarrow \frac{1}{x^2} = k \cdot \pi$, $k \in \mathbb{Z} - \{0\}$

$x^2 = \frac{1}{k\pi}$

$x = \pm \frac{1}{\sqrt{k\pi}}$



3) Rostoucí, klesající

$y' = \cos \frac{1}{x^2} \cdot \left(-\frac{2}{x^3}\right)$

nulové body: 0; $\cos \frac{1}{x^2} = 0 \Rightarrow \frac{1}{x^2} = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

$x^2 = \frac{1}{\frac{\pi}{2} + k\pi}$

$x = \pm \frac{\sqrt{2}}{\sqrt{\pi + 2k\pi}}$

4) Konvexní, konkávní

$y'' = -\sin \frac{1}{x^2} \cdot \left(-\frac{2}{x^3}\right) \cdot \left(-\frac{2}{x^3}\right) + \cos \frac{1}{x^2} \cdot \left(\frac{6}{x^4}\right) = \frac{1}{x^4}\left(6 \cos \frac{1}{x^2} - \frac{4}{x^2} \sin \frac{1}{x^2}\right)$ ... jindy :)

5) Asymptoty

a) bez směrnice $\lim_{x \to 0^+} \sin \frac{1}{x^2} = \| \sin \infty \ldots \lim \nexists \|$ .... limita $\nexists$ <u>nemá</u> asymptoty bez směrnice

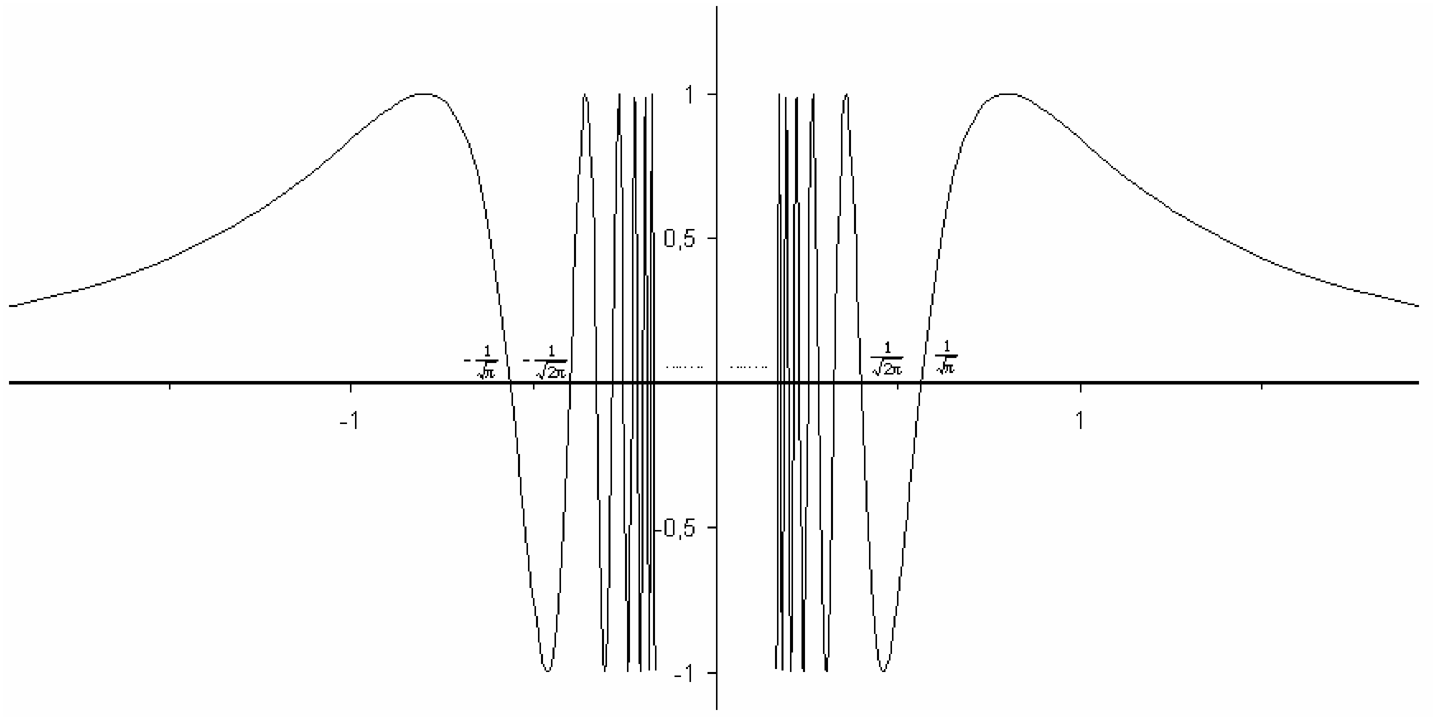
$\lim_{x \to 0^-}$ ——— " ———

b) se směrnicí

$a = \lim_{x \to \pm\infty} \frac{\sin \frac{1}{x^2}}{x} = \lim_{x \to \pm\infty} \sin \frac{1}{x^2} \cdot \frac{1}{x} = \| 0 \cdot 0 \| = 0$

$b = \lim_{x \to \pm\infty} \left(\sin \frac{1}{x^2} - 0 \cdot x\right) = \lim_{x \to \pm\infty} \sin \frac{1}{x^2} = \| \sin \frac{1}{\infty} = \sin 0 \| = 0$

} asymptota pro $\pm\infty$ je $y = 0$ (osa x)



1
0,5
-0,5
-1
-1
1
$-\frac{1}{\sqrt{\pi}}$
$-\frac{1}{\sqrt{2\pi}}$
$\frac{1}{\sqrt{2\pi}}$
$\frac{1}{\sqrt{\pi}}$

b) $y = e^{-\frac{1}{2}t} \sin 2\pi t$

1) $D(f) = \mathbb{R}$, $\sin 2\pi t$ má periodu 1 (to však není perioda fce původní, ta žádnou periodu nemá !!)

2) Kladná, záporná

$y = \underbrace{e^{-\frac{1}{2}t}}_{\text{vždy } \oplus} \sin 2\pi t$

nulové body:

$\sin 2\pi t = 0$

$2\pi t = k\cdot\pi$, $k \in \mathbb{Z}$

$t = \frac{k}{2}$, tj.: $t \in \{ \ldots -2; -1{,}5; -1; -0{,}5; 0; 0{,}5; 1; 1{,}5; 2; \ldots \}$

$-3 \quad -\frac{5}{2} \quad -2 \quad -\frac{3}{2} \quad -1 \quad -\frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 1 \quad \frac{3}{2} \quad 2 \quad \frac{5}{2} \quad 3$

$- \; + \; - \; + \; - \; + \; - \; + \; - \; + \; - \; + \; - \; +$

3) Rostoucí, klesající

$y' = -\frac{1}{2}e^{-\frac{1}{2}t} \sin 2\pi t + e^{-\frac{1}{2}t} \cos 2\pi t \cdot 2\pi = e^{-\frac{1}{2}t}(2\pi \cos 2\pi t - \frac{1}{2} \sin 2\pi t)$

extrém v nejmenším kladném bodě $\underline{a}$ (dle zadání):

$2\pi \cos 2\pi t - \frac{1}{2} \sin 2\pi t = 0 \quad / \cdot 2$

$4\pi \cos 2\pi t = \sin 2\pi t \quad /: \cos 2\pi t$

$4\pi = \operatorname{tg} 2\pi t$

$\operatorname{arctg} 4\pi = 2\pi t$

$t = \frac{\operatorname{arctg} 4\pi}{2\pi} = a \; (\doteq 0{,}24)$

všechny extrémy:

$4\pi = \operatorname{tg}(2\pi t + k\cdot\pi)$, $k \in \mathbb{Z}$

$\operatorname{arctg} 4\pi = 2\pi t + k\pi$

$\frac{\operatorname{arctg} 4\pi - k\pi}{2\pi} = t \Rightarrow t = a + \frac{k}{2}$, $k \in \mathbb{Z}$

přibližné hodnoty $\longrightarrow$ $-1{,}74 \quad -1{,}24 \quad -0{,}74 \quad 0{,}24 \quad 0{,}74 \quad 1{,}24 \quad 1{,}74 \quad 2{,}24$

$\ldots \; - \; + \; - \; + \; - \; + \; - \; + \; \ldots$

$y \quad \ldots \searrow \text{min} \nearrow \text{max} \searrow \text{min} \nearrow \text{max} \searrow \text{min} \nearrow \text{max} \searrow \text{min} \nearrow$

4) Konvexní, konkávní

$y'' = -\frac{1}{2} e^{-\frac{1}{2}t}(2\pi \cos 2\pi t - \frac{1}{2} \sin 2\pi t) + e^{-\frac{1}{2}t}(-\sin 2\pi t \cdot 4\pi^2 - \pi \cos 2\pi t) =$

$= e^{-\frac{1}{2}t}(-\pi \cos 2\pi t + \frac{1}{4} \sin 2\pi t - 4\pi^2 \sin 2\pi t - \pi \cos 2\pi t)$

inflexe v největším záporném $\underline{c}$

$-\pi \cos 2\pi t + \frac{1}{4} \sin 2\pi t - 4\pi^2 \sin 2\pi t - \pi \cos 2\pi t = 0 \quad /: \cos 2\pi t$

$-\pi + \frac{1}{4} \operatorname{tg} 2\pi t - 4\pi^2 \operatorname{tg} 2\pi t - \pi = 0$

$(\frac{1}{4} - 4\pi^2) \operatorname{tg} 2\pi t = 2\pi$

$\operatorname{tg} 2\pi t = \frac{2\pi}{\frac{1}{4} - 4\pi^2}$

$\operatorname{tg} 2\pi t = \frac{8\pi}{1 - 16\pi^2}$

$2\pi t = \operatorname{arctg} \frac{8\pi}{1-16\pi^2}$

$t = \frac{\operatorname{arctg} \frac{8\pi}{1-16\pi^2}}{2\pi} = c \doteq -0{,}025$

všechny inflexní body:

$\operatorname{tg}(2\pi t + k\cdot\pi) = \frac{8\pi}{1-16\pi^2} \quad k \in \mathbb{Z}$

$2\pi t + k\cdot\pi = \operatorname{arctg} \frac{8\pi}{1-16\pi^2}$

$t = \frac{\operatorname{arctg} \frac{8\pi}{1-16\pi^2} - k\pi}{2\pi}$

$t = \frac{\operatorname{arctg} \frac{8\pi}{1-16\pi^2}}{2\pi} - \frac{k}{2}$

$t = c + \frac{k}{2}$, $k \in \mathbb{Z}$

$$\cap^{\text{inf}} \cup^{\text{inf}} \cap^{\text{inf}} \cup^{\text{inf}} \cap^{\text{inf}} \cup^{\text{inf}} \cap^{\text{inf}} \cup^{\text{inf}} \cap$$

$$y'' \quad - \quad + \quad - \quad + \quad - \quad + \quad - \quad + \quad -$$

$$\dots \; -1{,}525 \;\; -1{,}025 \;\; -0{,}525 \;\; -0{,}025 \;\; 0{,}475 \;\; 0{,}975 \;\; 1{,}475 \;\; 1{,}975 \; \dots \longleftarrow \text{přibližné hodnoty}$$

$$\uparrow \quad -0{,}025 = \frac{\operatorname{arctg} \frac{8\pi}{1-16\pi^2}}{2\pi}$$

5) Asymptoty

a) bez směrnice ... nejsou

b) se směrnicí

$$a_\infty = \lim_{t\to+\infty} \frac{e^{-\frac{1}{2}t}\sin 2\pi t}{t} = \lim_{t\to+\infty} e^{-\frac{1}{2}t}\sin 2\pi t \cdot \frac{1}{t} = \| e^{-\infty}\cdot \sin\infty \cdot \frac{1}{\infty} = 0\cdot\langle -1,1\rangle \cdot 0 \| = \underline{\underline{0}}$$

$$a_{-\infty} = \lim_{t\to-\infty} \text{—〃—} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to-\infty} \left(-\tfrac{1}{2}e^{-\frac{1}{2}t}\left(\sin 2\pi t - 4\pi\cos 2\pi t\right)\right) = \|\infty\cdot \text{periodická fce}\| \dots \text{limita} \nexists$$

$$b_\infty = \lim_{t\to+\infty} e^{-\frac{1}{2}t}\sin 2\pi t = \| e^{-\infty}\cdot \sin\dots = 0\cdot\langle -1,1\rangle \| = \underline{\underline{0}}$$

asymptota pro $+\infty$ je $\underline{\underline{y=0}}$