# DÚ 9

pr.1

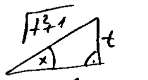
$$\bullet \int \frac{1}{\cos^3 x}\cdot\frac{\cos x}{\cos x}\,dx = \left| \begin{matrix} t=\sin x \\ dt=\cos x\,dx\end{matrix}\right| = \int \frac{dt}{(1-t^2)^2} = \int\left(\frac{\frac14}{t+1}+\frac{\frac14}{(t+1)^2}-\frac{\frac14}{t-1}+\frac{\frac14}{(t-1)^2}\right)dt =$$

rozklad na parc. zlomky

$$= \frac14\ln|t+1| - \frac14\ln|t-1| - \frac14\,\frac{1}{t+1} - \frac14\,\frac{1}{t-1} + c = \frac14\ln\left|\frac{\sin x+1}{\sin x-1}\right| - \frac14\,\frac{2t}{t^2-1}+c =$$

$$= \frac14\left(\ln\left|\frac{\sin x+1}{\sin x-1}\right| + \frac{2\sin x}{\cos^2 x}\right)+c$$

$$\bullet \int \frac{1}{\cos^2 x\,\sin^2 x}\,dx = \left| \begin{matrix} \mathrm{tg}\,x = t \\ \mathrm{arctg}\,t = x \\ \frac{1}{1+t^2}\,dt = dx \end{matrix} \quad \begin{matrix} \sin x = \frac{t}{\sqrt{t^2+1}} \\ \cos x = \frac{1}{\sqrt{t^2+1}} \end{matrix} \right| = \int \frac{1}{\frac{1}{t^2+1}\cdot\frac{t^2}{\cancel{t^2+1}}}\cdot\frac{1}{\cancel{1+t^2}}\,dt = \int \frac{t^2+1}{t^2}\,dt =$$




$$= \left| \begin{matrix} (t^2+1):t^2 = 1+\frac{1}{t^2} \\ -t^2 \\ 1 \end{matrix} \right| = \int\left(1+\frac{1}{t^2}\right)dt = t - \frac{1}{t} + c = \mathrm{tg}\,x - \frac{1}{\mathrm{tg}\,x} + c = \mathrm{tg}\,x - \mathrm{cotg}\,x + c$$

pr.2

$$\bullet \int x^2\cdot\ln x\,dx = \left| \begin{matrix} u=\ln x & u'=\frac{1}{x} \\ v'=x^2 & v=\frac{x^3}{3} \end{matrix}\right| = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + c = \frac{x^3}{3}\left(\ln x - \frac13\right)+c$$

$$\bullet \int x\sqrt{1+x}\,dx = \left| \begin{matrix} t=x+1 \\ dt=dx \end{matrix}\right| = \int (t-1)\sqrt{t}\,dt = \int\left(t^{3/2}-t^{1/2}\right)dt = \frac25 t^{\frac52} - \frac23 t^{\frac32} + c = \sqrt{x+1}\left[\frac25(x+1)^2 - \frac23(x+1)\right]+c$$

$$\bullet \int \frac{x^2}{\sqrt{4-x^2}}\,dx = (Ax+B)\sqrt{4-x^2} + k\cdot\int\frac{1}{\sqrt{4-x^2}}\,dx \quad \Big/\,'$$

$$\frac{x^2}{\sqrt{4-x^2}} = A\sqrt{4-x^2} + (Ax+B)\frac{-x}{\sqrt{4-x^2}} + k\cdot\frac{1}{\sqrt{4-x^2}} \quad \Big/\cdot\sqrt{4-x^2}$$

$$x^2 = A(4-x^2) + (-Ax^2 - Bx) + k$$

$x^2$: $1 = -A - A$ $\qquad A = -\frac12$

$x^1$: $0 = -B$ $\qquad B = 0$

$x^0$: $0 = 4A + k$ $\qquad k = 2$

pozn.: $\int \frac{1}{\sqrt{p^2-x^2}}\,dx = \arcsin\frac{x}{p}$

celkem: $$\int\frac{x^2}{\sqrt{4-x^2}}\,dx = -\frac12 x\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + c$$

$$\bullet \int e^{2x}\sin 2x\,dx = \left| \begin{matrix} u=e^{2x} & v'=\sin 2x \\ u'=2e^{2x} & v=-\frac12\cos 2x \end{matrix}\right| = -\frac12 e^{2x}\cos 2x + \int e^{2x}\cos 2x\,dx = \left| \begin{matrix} u=e^{2x} & v'=\cos 2x \\ u'=2e^{2x} & v=\frac12\sin 2x \end{matrix}\right| =$$

$$= -\frac12 e^{2x}\cos 2x + \frac12 e^{2x}\sin 2x - \int e^{2x}\sin 2x\,dx \Rightarrow \int e^{2x}\sin 2x\,dx = \frac{e^{2x}}{4}\left(\sin 2x - \cos 2x\right)+c$$

pr. 3

• $\int \frac{x^4-x^3-2x^2+x+1}{x^3-x^2+x-6}\,dx = \int\left(x+\frac{-3x^2+7x+1}{x^3-x^2+x-6}\right)dx = \int\left(x+\frac{\frac{1}{3}}{x-2}+\frac{-\frac{10}{3}x}{x^2+x+3}\right)dx =$

$$(x^4-x^3-2x^2+x+1):(x^3-x^2+x-6) = x$$
$$\underline{-(x^4-x^3+x^2-6x)}$$
$$-3x^2+7x+1$$

$$\frac{-3x^2+7x+1}{x^3-x^2+x-6} = \frac{-3x^2+7x+1}{(x-2)(x^2+x+3)} = \frac{A}{x-2}+\frac{Bx+C}{x^2+x+3} \quad /\cdot(x-2)(x^2+x+3)$$
$$-3x^2+7x+1 = A(x^2+x+3)+(Bx+C)(x-2)$$

$$\left.\begin{aligned} x^2&: -3 = A+B \\ x^1&: 7 = A-2B+C \\ x^0&: 1 = 3A-2C \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \tfrac{1}{3} \\ B &= -\tfrac{10}{3} \\ C &= 0 \end{aligned}$$

pozn.: $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\operatorname{arctg}\frac{x}{a}$

$$= \frac{x^2}{2}+\frac{1}{3}\ln|x-2| + \int\frac{-\frac{5}{3}(2x+1)+\frac{5}{3}}{x^2+x+3}dx = \frac{x^2}{2}+\frac{1}{3}\ln|x-2|-\frac{5}{3}\ln|x^2+x+3|+\frac{5}{3}\int\frac{1}{(x+\frac{1}{2})^2+\frac{11}{4}}dx =$$
$$= \frac{x^2}{2}+\frac{1}{3}\ln|x-2|-\frac{5}{3}\ln(x^2+x+3)+\frac{5}{3}\frac{2}{\sqrt{11}}\operatorname{arctg}\frac{2(x+\frac{1}{2})}{\sqrt{11}}+C = \underline{\underline{\frac{x^2}{2}+\frac{1}{3}\ln|x-2|-\frac{5}{3}\ln(x^2+x+3)+\frac{10\sqrt{11}}{33}\operatorname{arctg}\frac{2x+1}{\sqrt{11}}+C}}$$

• $\int\frac{1}{(x^2-4x+4)(x^4+4x^2+4)}dx =$

$$\frac{1}{(x^2-4x+4)(x^4+4x^2+4)} = \frac{1}{(x-2)^2(x^2+2)^2} = \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2+2}+\frac{Ex+F}{(x^2+2)^2}$$
$$\vdots$$
$$A=-\frac{1}{27},\ B=\frac{1}{36},\ C=\frac{4}{108},\ D=\frac{5}{108},\ E=\frac{2}{18},\ F=\frac{1}{18}$$

$$=\int\left(-\frac{\frac{1}{27}}{x-2}+\frac{\frac{1}{36}}{(x-2)^2}+\frac{\frac{5}{108}}{x^2+2}+\frac{\frac{4}{108}x}{x^2+2}+\frac{\frac{2}{18}x}{(x^2+2)^2}+\frac{\frac{1}{18}}{(x^2+2)^2}\right)dx =$$
$$=-\frac{1}{27}\ln|x-2|-\frac{1}{36}\frac{1}{x-2}+\frac{5}{108}\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}+\frac{1}{54}\ln(x^2+2)-\frac{1}{18}\frac{1}{x^2+2}+\frac{1}{18}\left(\frac{x}{4(x^2+2)}+\frac{1}{4}\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}\right)+C =$$
$$\underline{\underline{=-\frac{1}{27}\ln|x-2|-\frac{1}{36}\frac{1}{x-2}+\frac{1}{54}\ln(x^2+2)+\frac{13\sqrt{2}}{432}\operatorname{arctg}\frac{x}{\sqrt{2}}+\frac{x-4}{72}+C}}$$

pozn.: $\int\frac{\frac{1}{36}}{(x-2)^2}dx = \left|\begin{matrix}t=x-2\\ dt=dx\end{matrix}\right| = \frac{1}{36}\int\frac{1}{t^2}dt = \frac{1}{36}\int t^{-2}dt = \frac{1}{36}\frac{t^{-1}}{-1}+C = -\frac{1}{36}\frac{1}{t}+C = -\frac{1}{36}\frac{1}{x-2}+C$

$\int\frac{\frac{4}{108}x}{x^2+2}dx = \left|\begin{matrix}t=x^2+2\\ dt=2x\,dx\end{matrix}\right| = \int\frac{\frac{2}{108}}{t}dt = \frac{1}{54}\ln|t|+C = \frac{1}{54}\ln|x^2+2|+C = \frac{1}{54}\ln(x^2+2)+C$

$\int\frac{\frac{2}{18}x}{(x^2+2)^2}dx = \left|\begin{matrix}t=x^2+2\\ dt=2x\,dx\end{matrix}\right| = \int\frac{\frac{1}{18}}{t^2}dt = \frac{1}{18}\int t^{-2}dt = \frac{1}{18}\frac{t^{-1}}{-1}+C = -\frac{1}{18}\frac{1}{t}+C = -\frac{1}{18}\frac{1}{x^2+2}+C$

$\int\frac{\frac{1}{18}}{(x^2+2)^2}dx =$ | je nutno využít rekurentního vzorce: $\int\frac{1}{(x^2+a^2)^n}dx = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}}+\frac{2n-3}{2(n-1)a^2}\cdot\int\frac{1}{(x^2+a^2)^{n-1}}dx$; zde: $a=\sqrt{2}$, $n=2$ | =

$$=\frac{1}{18}\left(\frac{x}{4(x^2+2)}+\frac{1}{4}\cdot\int\frac{1}{x^2+2}dx\right) = \frac{1}{18}\left(\frac{x}{4(x^2+2)}+\frac{1}{4}\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}\right)+C$$