Množiny

Ekvivalence R r, s, t $a \sim b$

A

Rozklad

$A_i \cap A_j = \emptyset$

$\bigcup A_i = A$

$(A, \circ)$ $\quad a \sim b$ ekvivalence

$\circ$ $=$?

normální podgrupa $G/H = H \backslash G$

splní, že $\circ$ na G/H je dobře definovaná

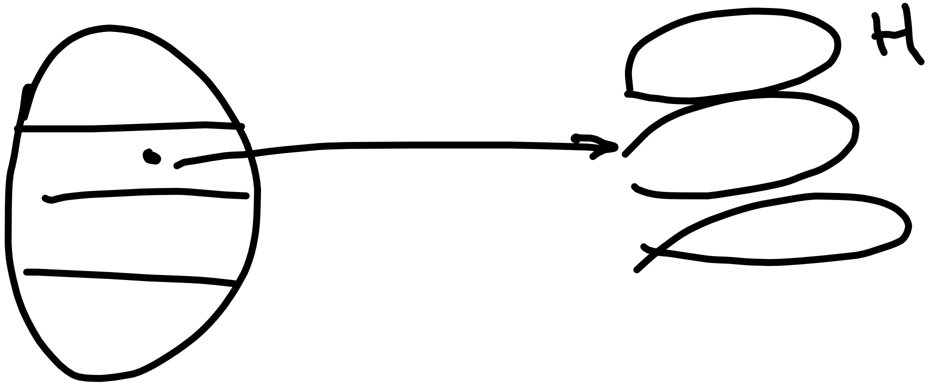
$$f(a \cdot_G b) = f(a) \cdot_H f(b)$$

$$f : G \longrightarrow H$$

$$\ker f = \{ a \in G ; f(a) = e_H \}$$

$$G \longrightarrow G/H$$

$$a \longmapsto aH = Ha$$



$f : G \longrightarrow K$

$H = \ker f$

$G/\ker f \longrightarrow K$

$G/\ker f \cong f(G)$

$$\begin{tikzcd}
G \arrow[rr] \arrow[dr, "\text{surj.}"'] & & K \\
& G/\ker f \arrow[ur, dotted, "\text{! injektivní}"'] &
\end{tikzcd}$$

Př: $GL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R});$ matice regulární řádu n nad $\mathbb{R}\}$

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R});\ \det A = 1\}$$

$SL_n(\mathbb{R}) \lhd GL_n(\mathbb{R})$

$E_n \in SL_n(\mathbb{R})$ ✓

$A, B \in SL_n(\mathbb{R}) \Rightarrow A \cdot B^{-1} \in SL_n(\mathbb{R})$

$\det A = \det B = 1 \Rightarrow \det(A \cdot B^{-1}) = \det(A) \cdot \det(B)^{-1} = 1$

$\Rightarrow$ podgrupa

$\forall G \in GL_n(\mathbb{R})\ \forall A \in SL_n(\mathbb{R}):\ G \cdot A \cdot G^{-1} \in SL_n(\mathbb{R})$ normální

$\det(G \cdot A \cdot G^{-1}) = \det(G) \cdot \det(A) \cdot \det(G)^{-1} = \det A = 1$

Jak vypadá $GL_n(\mathbb{R})/SL_n(\mathbb{R})$?

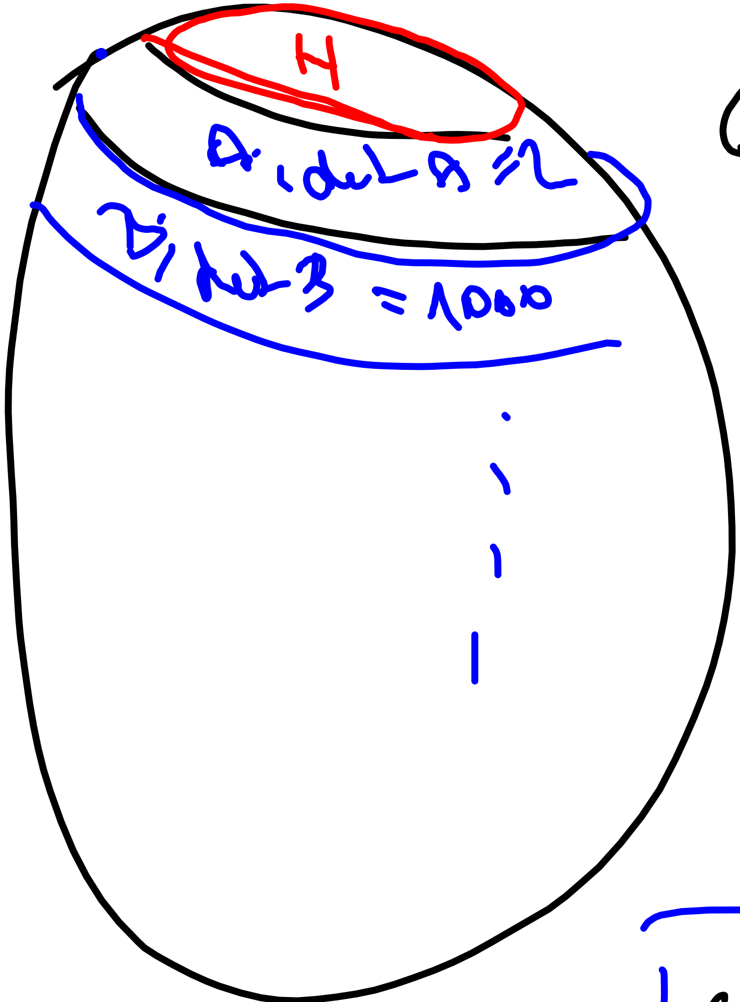
Levé třídy tvaru $a \cdot H$ (tedy $A \cdot SL_n(\mathbb{R})$)

Buď $A \in GL_n(\mathbb{R})$ libovolné

$$A \cdot SL_n(\mathbb{R}) \stackrel{?}{=} B \cdot SL_n(\mathbb{R})$$

$$B^{-1} \cdot A \in SL_n(\mathbb{R}), \text{ tj.}$$

$$\det(B^{-1} \cdot A) = 1 \iff \det A = \det B$$



$GL_n(\mathbb{R})$

$H = SL_n(\mathbb{R})$

$\cong (\mathbb{R}^*, \cdot)$

Hyp: $\boxed{GL_n(\mathbb{R}) / SL_n(\mathbb{R}) \cong (\mathbb{R}^*, \cdot)}$

Chci $f: GL_n(\mathbb{R}) \longrightarrow (\mathbb{R}^*, \cdot)$

$A \longmapsto \det A$

$$G = \{ f : \mathbb{R} \to \mathbb{R} \; ; \; f(x) = ax + b \; ; \; a \neq 0,\ a, b \in \mathbb{R} \}$$

jednotkový: $f(x) = x$

$f(x) = ax + b$

$f \circ g \qquad g(x) = cx + d$

$$(f \circ g)(x) = f(g(x)) = f(cx + d) = a(cx + d) + b$$
$$= acx + ad + b$$

$f^{-1}(x) = cx + d$

$(f \circ f^{-1})(x) = x$

$$\underbrace{ac}_{=1}x + \underbrace{ad + b}_{=0} = x$$

$c = a^{-1} \qquad d = -b a^{-1}$

$f^{-1}(x) = a^{-1}x - a^{-1}b$

T je normální?

$\forall g \in G \ \forall t \in T$: $\quad g(x) = ax+b$

$gtg^{-1} \in T \quad t(x) = \cancel{t}x$

$(gtg^{-1})(x) = g(t(g^{-1}(x))) = g(t(a^{-1}x - ba^{-1})) =$

$= g(ta^{-1}x - tba^{-1}) = a(\quad) + b =$

$= tx \underbrace{-tb+b}_{\text{obecně} \neq 0} \notin T \quad T \ntriangleleft G$

S je normální?

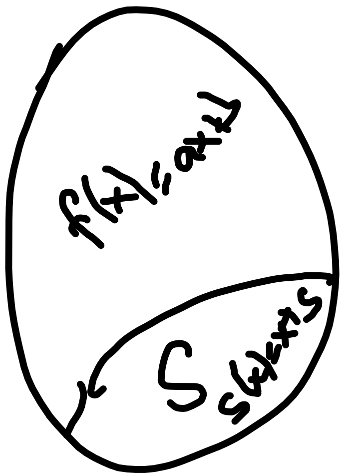
$$g(x) = ax+b$$
$$s(x) = x+s$$

$$(g s g^{-1})(x) = g(s(g^{-1}(x))) = g(s(\underline{a^{-1}x - ba^{-1}}))$$

$$= g(a^{-1}x - ba^{-1} + s) = a(\ldots\ldots) + b =$$

$$= x \underbrace{- b + as + b} \in S \Rightarrow S \triangleleft G$$

G



$$f(x) \sim_S g(x) \qquad f(x) = ax+b, \quad g(x) = cx+d$$

$$g^{-1} \circ f \in S$$

$$(g^{-1} \circ f)(x) = g^{-1}(ax+b) = c^{-1}(ax+b) - dc^{-1}$$
$$\in S$$

$$\Leftrightarrow c^{-1}a = 1 \Leftrightarrow a = c$$

homom.

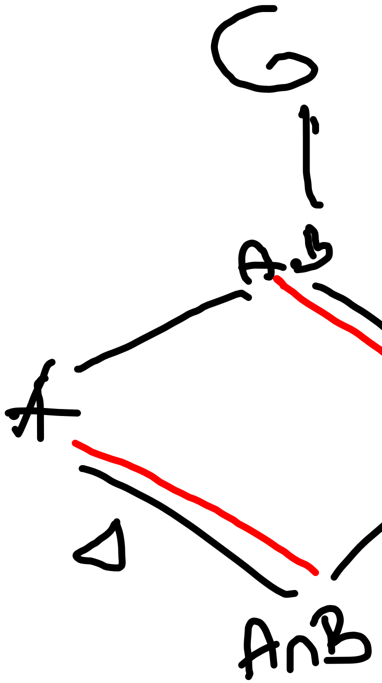
$$G \longrightarrow (\mathbb{R}^*, \cdot)$$

$g(x) \quad \varphi: g \mapsto c$

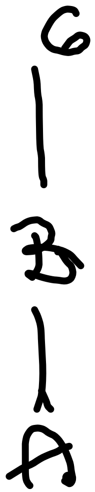
$\quad\ ||$

$cx+d$

Pak $\quad G/S \cong \varphi(G) = (\mathbb{R}^*, \cdot)$



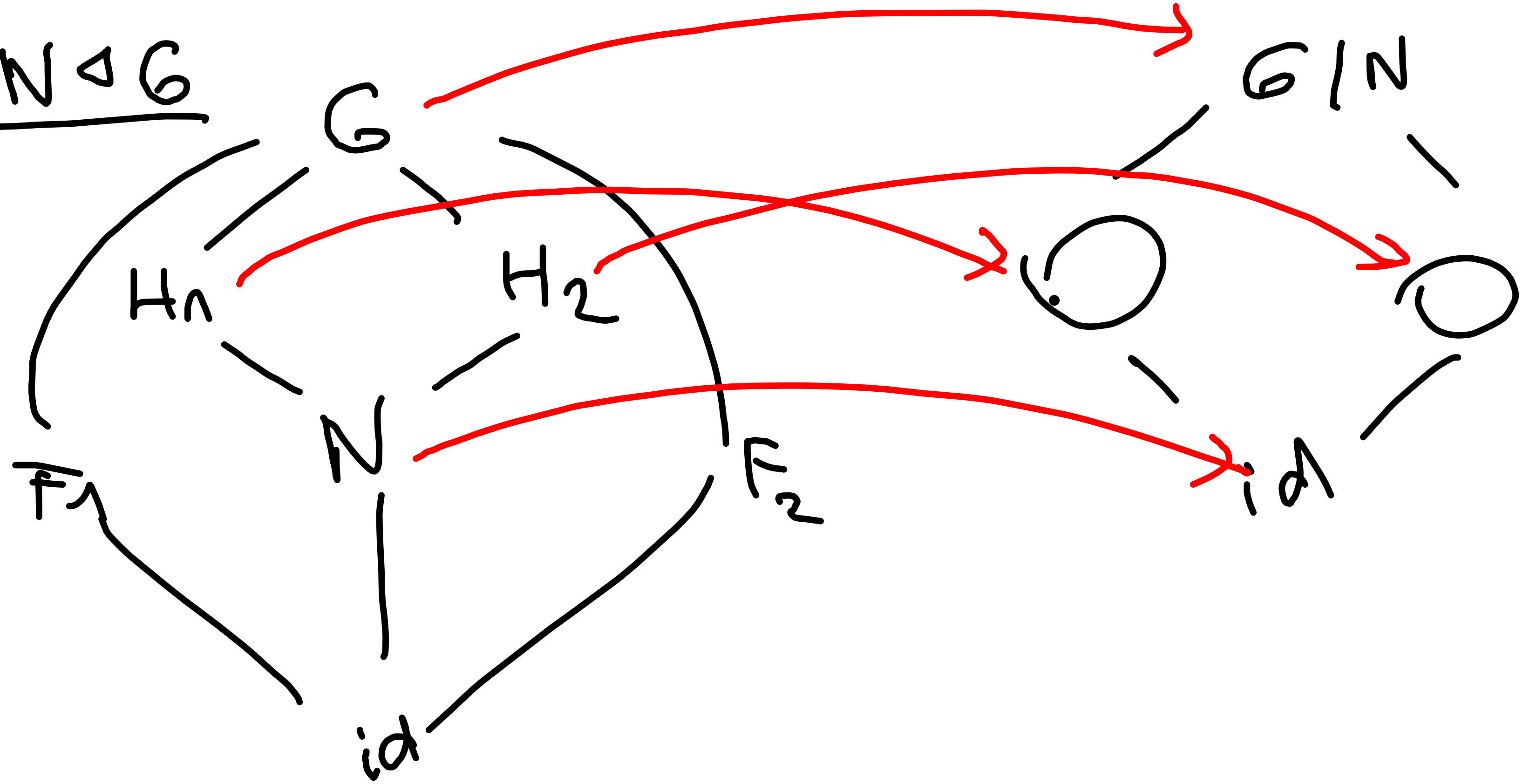
$$AB/B \cong A/A\cap B$$



$$B/A \trianglelefteq G/A$$

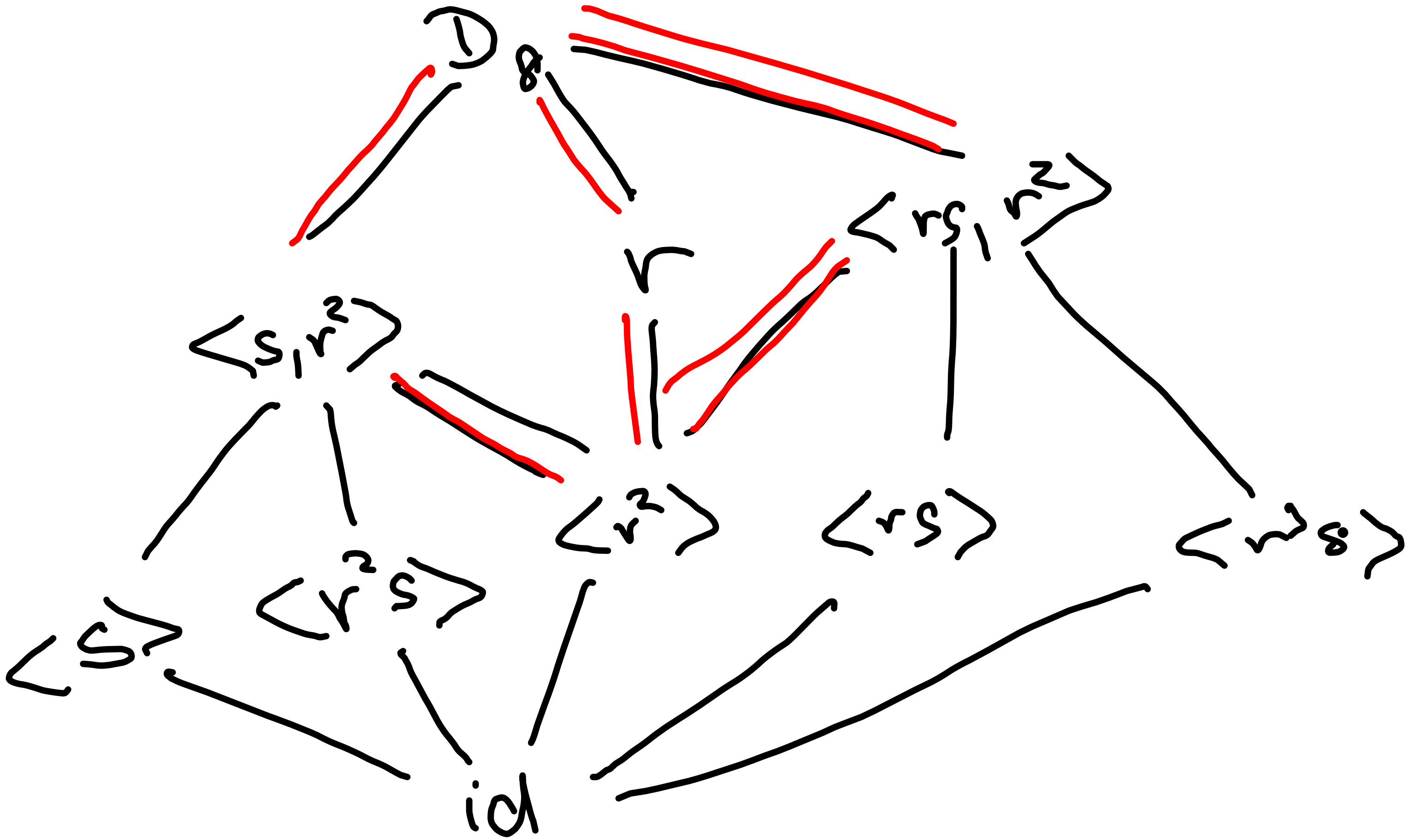
$$(G/A)\big/(B/A) \cong G/B$$

$N \triangleleft G$



$D_8 / \langle r^2 \rangle$

$D_8 = \langle r, s \,;\, r^4 = id,\ s^2 = id,\ r^3 s = s r \rangle$



M, N stejná mohutnost

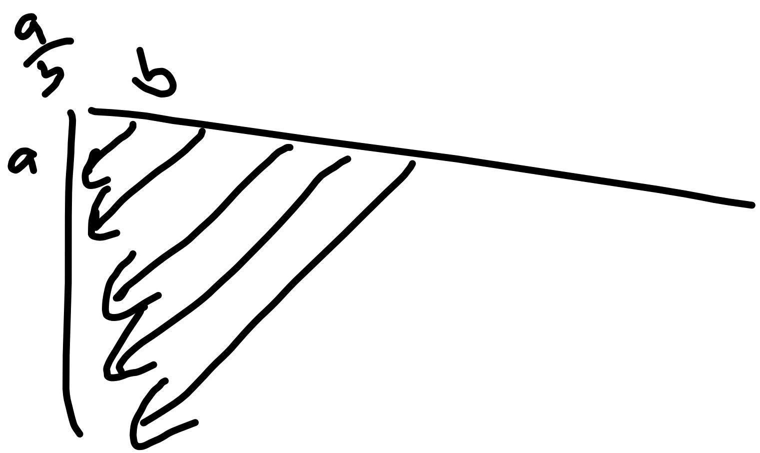
$\Updownarrow$

existuje bijekce mezi M a N

Př ① $\mathbb{Z}$, sudá čísla

$f(a) = 2a$

② $\mathbb{Z}, \mathbb{Q}$



$c \cdot d = 0 \Rightarrow c = 0 \vee d = 0$

$(\mathbb{Z}_4, +, \cdot)$

$[2]_4 \cdot [2]_4 = [0]_4$

n složené $\Rightarrow (\mathbb{Z}_n, +, \cdot)$

není obor integrity

$n = a \cdot b,\ a, b > 1:\ [a]_n \cdot [b]_n = [n]_n = [0]_n$

Př. $0 \cdot c = 0 \quad \forall c \in R$

$0 \cdot c$

$\|$ def. grupy

$(a-a) \cdot c \overset{\text{distr.}}{=} a \cdot c + (-a) \cdot c \overset{\text{grupa + distr.}}{=}$

$= a \cdot c - a \cdot c \overset{\text{grupa}}{=} 0$

$(-a) \cdot c = -(ac)$

Př.: $(\mathbb{Z}_4, +, \cdot)$

$[2]_4 \cdot [3]_4 = [2]$

$[2]_4 \cdot [1]_4 = [2]_4$

$[2]([3] - [1]) = [0]$

$[2] \cdot [2] = [0]$

$(R, +, \cdot)$

jednotky v R jsou prvky, mající inverzi v $(R, \cdot)$

__Př.__ $(\mathbb{Z}, +, \cdot)$ jednotky $\{1, -1\}$

__Př.:__ $(\mathbb{Z}_p, +, \cdot)$ je těleso

$\forall a \in \mathbb{Z}_p \setminus \{0\}\ \exists x : \quad a \cdot x \equiv 1 \pmod{p}$

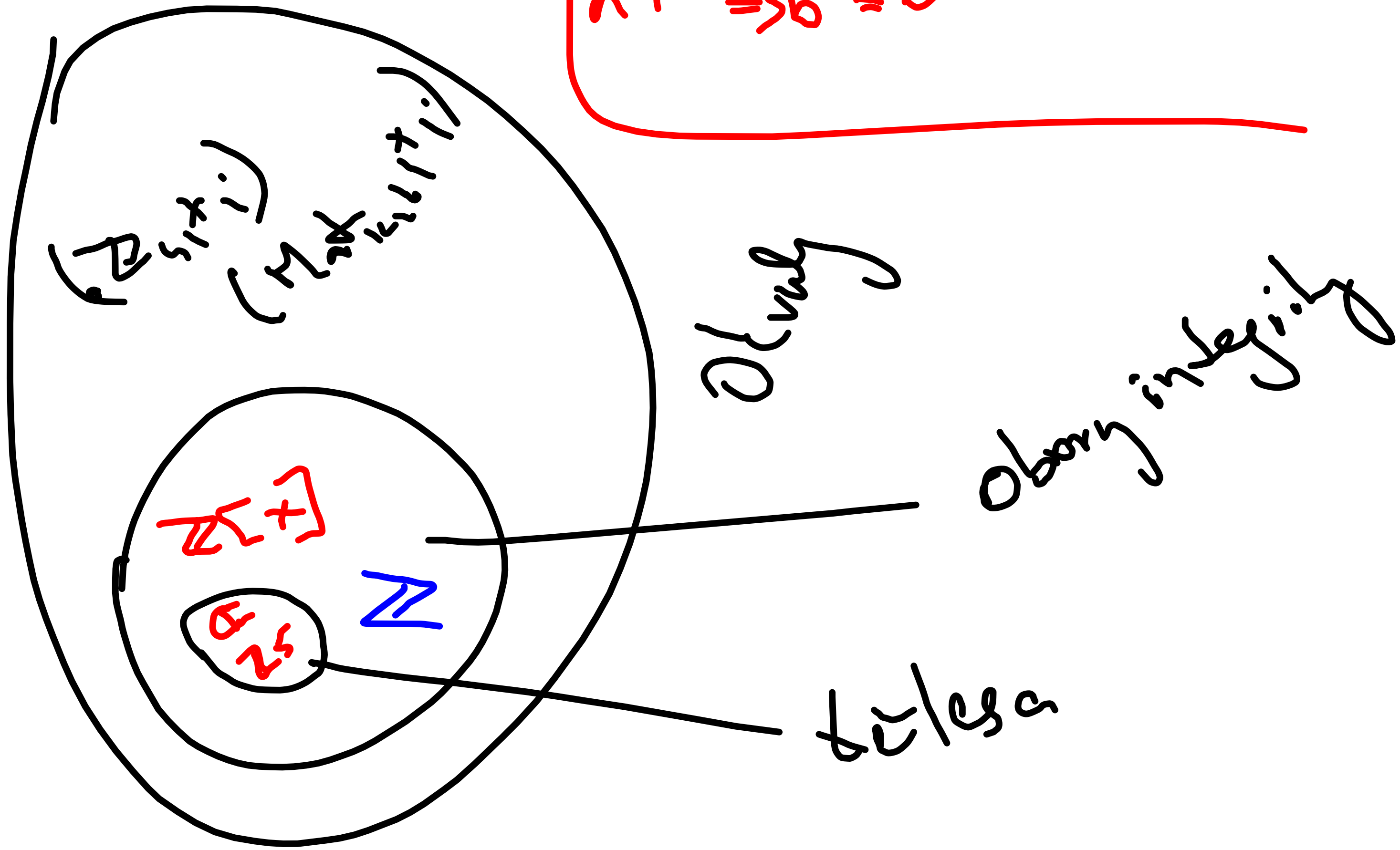
$a^{p-1} \equiv 1 \pmod{p}$

pol. $x = a^{p-2}$

Každé těleso je obor integrity

$a \cdot b = 0 \quad | \cdot a^{-1}$

$a \neq 0 \Rightarrow b = 0$



$f: \mathbb{R} \longrightarrow \mathbb{R}$

$a \in \mathbb{R}$
$a \neq 0$ $\varphi: x \longmapsto a x$

- injektivní

$a \cdot x = a \cdot y \Rightarrow a \cdot (x - y) = 0 \Rightarrow x - y = 0$
$\Rightarrow x = y$

- stejný počet prvků $\Rightarrow \varphi$ je bijekce

$\Rightarrow \varphi$ je surjekce $\Rightarrow \exists x \in \mathbb{R}: a \cdot x = 1$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

polynom ... formální zápis $(a_0, a_1, \dots, a_n, 0, \dots 0)$

polyn. funkce – zobrazení $R \to R$

Př.: $f(x) = x^2 + x \in \mathbb{Z}_2[x]$

$f([0]) = [0]$

$f([1]) = [2] = [0]$, atd.

polynomy $0 = (0, 0, 0, \dots)$

$x^2 + x = (0, 1, 1, 0, \dots)$

mají stejnou pol. funkci

$$f(x)\cdot g(x) = 0$$

$st\, f = m$

$st\, g = n$

$f(x) = a_m x^m + \dots$

$g(x) = b_n x^n + \dots$

$a_m \cdot b_n$ je vedoucí koeficient

(tj. u x^{m+n})

$a_m \neq 0$

$b_n \neq 0$

spec.

$\Rightarrow a_m \cdot b_n = 0$

nelze

v OI

Otázka:

Co jsou invertibilní prvky v $\mathbb{Z}[[x]]$?

Řada $1+x+x^2+x^3+\dots$ je invertibilní

$$\left[\left(1+x+x^2+x^3+\dots\right)(1-x)=1\right]$$