1. $U = \langle u_1, u_2, u_3 \rangle$

$u_1 = (1, 0, -1, 2, 1)$

$u_2 = (1, -1, 0, -1, 2)$

$u_3 = (2, 1, -2, 0, -1)$

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 1 & -1 & 0 & -1 & 2 \\ 2 & 1 & -2 & 0 & -1 \end{pmatrix} \begin{matrix} \\ -1\cdot \mathrm{I} \\ -2\cdot \mathrm{I} \end{matrix} \sim \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & -3 & 1 \\ 0 & 1 & 0 & -4 & -3 \end{pmatrix} \begin{matrix} \\ \\ +\mathrm{II} \end{matrix} \sim$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -7 & -2 \end{pmatrix} \Rightarrow u_1, u_2, u_3 \text{ jsou LN, tedy báze } U$$

tato báze $U$

$$\begin{array}{c} \\ \\ \\ v \rightarrow \end{array} \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -7 & -2 \\ 1 & -2 & 0 & 5 & 5 \end{pmatrix} \begin{matrix} \\ \\ \\ -1\cdot \mathrm{I} \end{matrix} \sim \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -7 & -2 \\ 0 & -2 & 1 & 3 & 4 \end{pmatrix} \begin{matrix} \\ \\ \\ -2\cdot \mathrm{II} \end{matrix} \sim$$

$$\begin{pmatrix} - & & & & \\ 0 & -1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -7 & -2 \\ 0 & 0 & -1 & 9 & 2 \end{pmatrix}$$

... někde je problém, vypadá to, že $v \notin U$

1.) Zkusme jinak

Předpokládejme, že

$v \in U$, tj. $v = a_1 \cdot u_1 + a_2 \cdot u_2 + a_3 \cdot u_3$

← rozepsání po složkách

$$\begin{array}{c}\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -2 \\ -1 & 0 & -2 & 0 \\ 2 & -1 & 0 & 5 \\ 1 & 2 & 1 & 5 \end{array}\right) \\ \begin{array}{cccc} u_1 & u_2 & u_3 & v \end{array}\end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & -2 & 3 \\ 0 & 1 & -1 & 4 \end{array}\right) \Big\} \text{ odporuje!}$$

Tedy $v \notin U$.

2. $u_1 = (1,2,0,1)$
$u_2 = (-2,-3,1,2)$
$u_3 = (0,1,1,4)$
$u_4 = (0,-1,1,1)$
$u_5 = (1,1,1,2)$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ -2 & -3 & 1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix}\begin{matrix} u_1 \\ u_2+2u_1 \\ u_3 \\ u_4 \\ u_5-u_1 \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 2 & 5 \end{pmatrix}\begin{matrix} u_1 \\ u_2+2u_1 \\ u_3-u_2-2u_1 \\ u_4+u_2+2u_1 \\ u_5-u_1+u_2+2u_1 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{matrix} u_1 \\ u_2+2u_1 \\ u_4+u_2+2u_1 \\ u_3-u_2-2u_1 \\ u_5-u_4-u_1 \end{matrix}$$

Tedy: $\dim\langle u_1, u_2, \dots u_5\rangle = 3$

(*) $u_3 - u_2 - 2u_1 = 0$
(**) $u_5 - u_4 - u_1 = 0$

Báze je např. $\{u_1, u_2, u_4\}$
Pak souřadnice $u_1$ vzhledem k této bázi jsou $(1,0,0)$
$u_2 \dots (0,1,0)$
$u_4 \dots (0,0,1)$
$u_3 \overset{(*)}{=} 2u_1 + u_2 \dots (2,1,0)$
$u_5 \overset{(**)}{=} u_1 + u_4 \dots (1,0,1)$

3.) Nechť $A \subseteq U$, $B \subseteq V$, podprostory.
$\varphi: U \Rightarrow V$ lin. zobrazení!

Dk.: 1) $\varphi(A)$ je podprostor $V$
2) $\varphi^{-1}(B)$ je podprostor $U$

1) Stačí ověřit:

i) $\forall a, b \in \varphi(A): \quad a+b \in \varphi(A)$

ii) $\forall r \in \mathbb{K}, \forall a \in \varphi(A): r \cdot a \in \varphi(A)$

Mějme $r \in \mathbb{K}$, $a, b \in \varphi(A)$ libovolné

$a, b \in \varphi(A): \exists x, y \in A: \varphi(x) = a, \varphi(y) = b$

Pak $\underset{\substack{\cap\\ A}}{\varphi(x+y)} \overset{\text{def. lin. zobr.}}{=} \varphi(x) + \varphi(y) = a+b$

$\Rightarrow \underline{a+b \in \varphi(A)}$

$r \cdot a = r \cdot \varphi(\underset{\substack{\cap\\ A}}{x}) \overset{\text{def.}}{=} \varphi(\underset{\substack{\cap\\ A}}{r \cdot x}) \Rightarrow \underline{r \cdot a \in \varphi(A)}$

2) $B \overset{\text{podpr.}}{\subseteq} V \Rightarrow \varphi^{-1}(B) \subseteq U$,

i) $a, b \in \varphi^{-1}(B) \overset{\text{stačí}}{\Rightarrow} a+b \in \varphi^{-1}(B)$

$a \in \varphi^{-1}(B) \Rightarrow \varphi(a) \in B$

$\varphi(a+b) \overset{\text{def.}}{=} \underset{\in B}{\varphi(a)} + \underset{\in B}{\varphi(b)} \in B$

$\Downarrow$

$a+b \in \varphi^{-1}(B)$

ii) $\varphi(r \cdot a) = r \cdot \underset{\substack{\cap\\ B}}{\varphi(a)} \in B$

$\Downarrow$

$ra \in \varphi^{-1}(B)$.

CBD

Název: IV 21-18:38 (4 z 7)

4.) $L: \mathbb{R}^3 \to \mathbb{R}^3$

$L(x_1, x_2, x_3) = (3x_1 + x_2 + 2x_3,\ 4x_1 + 2x_2 + 3x_3,\ 2x_1 + x_3)$

$\mathrm{Ker}(L) = ?,\ \mathrm{Im}(L) = ?$

---

Ker ... vyřešíme homogenní soustavu

$$\begin{pmatrix} 3 & 1 & 2 \\ 4 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \begin{matrix} \cdot 4 \\ \cdot 3 \\ \cdot 6 \end{matrix} \sim \begin{pmatrix} 12 & 4 & 8 \\ 12 & 6 & 9 \\ 12 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 12 & 4 & 8 \\ 0 & 2 & 1 \\ 0 & -4 & -2 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 12 & 4 & 8 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_3 = 1,\ x_2 = -\frac{1}{2},\ x_1 = -\frac{1}{2}$

$(x_1, x_2, x_3) = -\frac{1}{2}(1, 1, -2)$

$\mathrm{Ker}(L) = \langle (1, 1, -2) \rangle$

---

Im ... zobrazíme bázi $\mathbb{R}^3$

kan. báze: $e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)$

$L(e_1) = (3,4,2)$

$L(e_2) = (1,2,0)$

$L(e_3) = (2,3,1)$

$\mathrm{Im}(L) = \langle L(e_1), L(e_2), L(e_3) \rangle =$

$= \langle (3,4,2), (1,2,0), (2,3,1) \rangle$

$$\begin{pmatrix} 3 & 4 & 2 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \begin{matrix} r_2 \\ r_3 - 2r_2 \\ r_1 - 3r_2 \end{matrix}$$

$2r_3 - 4r_2 = r_1 - 3r_2$

$r_1 - 2r_3 + r_2 = 0$

$\mathrm{Im}(L) = \langle \underbrace{(1,2,0), (2,3,1)}_{\text{báze } \mathrm{Im}(L)} \rangle$

# 5.) Matice lineárního zobrazení

$\varphi: \mathbb{R}^n \to \mathbb{R}^m$, kanonické báze !

$$M \in Mat_{m \times n}(\mathbb{R})$$

$$\varphi(u) = M \cdot u$$

$$\begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix}$$

$$\varphi: \mathbb{R}^n \to \underbrace{\mathbb{R}^m}_{n \text{ sloupců}}$$

$$\begin{pmatrix} \vdots \end{pmatrix} = {\scriptstyle m \text{ řádků}} \left\{ \begin{pmatrix} & & \\ & & \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} \right\} n \text{ řádků} \quad \ldots \; M \in Mat_{m \times n}(\mathbb{R})$$

---

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\varphi\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : \quad \varphi\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} : \quad \varphi\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \quad \varphi\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \quad \varphi\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \quad \varphi\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

6. $\mathcal{E} = (\overset{e_1}{1}, \overset{e_2}{x}, \overset{e_3}{x^2}, \overset{e_4}{x^3})$

$\mathcal{B} = (\overset{b_1}{1+x}, \overset{b_2}{1-x}, \overset{b_3}{x^2+x^3}, \overset{b_4}{x^2-x^3})$

M... matice přechodu od $\mathcal{E}$ k $\mathcal{B}$

N... matice přechodu od $\mathcal{B}$ k $\mathcal{E}$

---

Chci $u_{\mathcal{B}} = M \cdot u_{\mathcal{E}}$

spočítejme nejprve souřadnice $e_1 \dots e_4$ vzhledem k $\mathcal{B}$

$e_1 = \frac{b_1 + b_2}{2} = \frac{1}{2} b_1 + \frac{1}{2} b_2$

$e_2 = \frac{b_1 - b_2}{2}$

$e_3 = \frac{b_3 + b_4}{2}$

$e_4 = \frac{b_3 - b_4}{2}$

$$\boxed{\begin{aligned} e_{1\mathcal{B}} &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0) \\ e_{2\mathcal{B}} &= (\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0) \\ e_{3\mathcal{B}} &= (0, 0, \tfrac{1}{2}, \tfrac{1}{2}) \\ e_{4\mathcal{B}} &= (0, 0, \tfrac{1}{2}, -\tfrac{1}{2}) \end{aligned}}$$

nechť $u_{\mathcal{E}} = (1, 2, 3, 5)$

$(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0)$ $(1, -1, 0, 0)$

$(0, 0, \tfrac{5}{2}, -\tfrac{5}{2})$ $(0, 0, \tfrac{3}{2}, \tfrac{3}{2})$

$$\begin{array}{cccc} e_{1\mathcal{B}} & e_{2\mathcal{B}} & e_{3\mathcal{B}} & e_{4\mathcal{B}} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}_{M} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} = u_{\mathcal{B}} = (\tfrac{3}{2}, -\tfrac{1}{2}, 4, -1)$$

---

$u_{1\mathcal{E}} = (1, 1, 0, 0)$

$u_{2\mathcal{E}} = (1, -1, 0, 0)$

$u_{3\mathcal{E}} = (0, 0, 1, 1)$

$u_{4\mathcal{E}} = (0, 0, 1, -1)$

$$N = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$\uparrow u_{1\mathcal{E}}$ $\uparrow u_{2\mathcal{E}}$

Platí (vždy):

$M = N^{-1}$