1.

P_3 = v. p. polynomů stupně nejvýše 3 (nad $\mathbb{R}$)

Báze $\mathcal{E} = (1, x, x^2, x^3)$

$\mathcal{B} = (1+x, 1-x; x^2+x^3, x^2-x^3)$

$u = 5x^3 + 3x^2 - x + 3$

$u_{\mathcal{E}} = (3, -1, 3, 5)$

$u = a_1 \cdot (1+x) + a_2(1-x) + a_3(x^2+x^3) + a_4(x^2-x^3)$

$x^3: \quad 5 = a_3 - a_4 \quad (1)$

$x^2: \quad 3 = a_3 + a_4 \quad (2)$

$x: \quad -1 = a_1 - a_2 \quad (3)$

$1 = x^0: \quad 3 = a_1 + a_2 \quad (4)$

$(1)+(2): \; 8 = 2a_3 \Rightarrow a_3 = 4 \Rightarrow a_4 = -1$

$(3)+(4): \; 2 = 2a_1 \Rightarrow a_1 = 1 \Rightarrow a_2 = 2$

$u_{\mathcal{B}} = (1, 2, 4, -1)$

2) $D : P_3 \to P_2$ zobrazení derivace

$(x^3 \mapsto 3x^2,\ x^2 \mapsto 2x,\ x \mapsto 1,\ 1 \mapsto 0)$

Určete matici D

a) vzhledem k bázím $(1, x, x^2, x^3)$ a $(1, x, x^2)$

$D(u) = A \cdot u$ ↗ sloupcový $\quad u \in P_3$

$D(u) \in P_2$

A - typu 3×4

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & & \dots & \\ & \dots & & a_{34} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_2 \\ 2u_3 \\ 3u_4 \end{pmatrix}$$

← lin. kombinace bázových vektorů P_2, jež jsou obrazem bázových vektorů P_3

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

b) vzhledem k bázím $(\underbrace{1+x}_{d_1}, \underbrace{1-x}_{d_2}, \underbrace{x^2+x^3}_{d_3}, \underbrace{x^2-x^3}_{d_4})$

a $(\underbrace{1+x}_{e_1}, \underbrace{1-x}_{e_2}, \underbrace{x+x^2}_{e_3})$

$* \; D(\underbrace{1+x}_{d_1}) = 1 = \frac{e_1+e_2}{2} = \frac{1}{2}e_1 + \frac{1}{2}e_2$

$** \; D(d_2) = D(1-x) = -1 = -\frac{e_1+e_2}{2} = -\frac{1}{2}e_1 - \frac{1}{2}e_2$

$D(d_3) = D(x^2+x^3) = 2x+3x^2 = 3e_3 + \frac{e_2-e_1}{2} =$

$= -\frac{1}{2}e_1 + \frac{1}{2}e_2 + 3e_3$

$D(d_4) = D(x^2-x^3) = 2x-3x^2 = -3e_3 + 5 \cdot \frac{e_1-e_2}{2} =$

$= \frac{5}{2}e_1 - \frac{5}{2}e_2 - e_3$

$* \; A \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \qquad ** \; A \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 3 & -1 \end{pmatrix}$$

Vlastní vektory, vlastní hodnoty

Motivace: Nechť A je matice $n \times n$.
Existuje skalár $a \in \mathbb{R}$ a vektor $u \in \mathbb{R}^n$ tak,
aby

$$A \cdot u = a \cdot u$$

upravujme

$$A \cdot u = (a \cdot I) \cdot u$$ (I je jednotková matice typu $n \times n$)

$$(A - a \cdot I) \cdot u = O$$

Tedy: nenulové vlastní vektory existují jen pro ty $a \in \mathbb{R}$, pro které je $|A - aI| = 0$.

3) Spočtěte char. polynom, vl. čísla a vl. vektory pro $A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$.

Nechť $\lambda \in \mathbb{R}$. Pak

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} \begin{matrix} \\ -1\cdot \mathrm{III} \\ +(1-\lambda)\mathrm{III} \end{matrix}$$

$$= \begin{vmatrix} 2-\lambda & 0 & \lambda-2 \\ 4-2\lambda & 0 & 14-9\lambda+\lambda^2 \\ 2 & -1 & 8-\lambda \end{vmatrix} = (-1)\cdot(-1)\cdot \begin{vmatrix} 2-\lambda & \lambda-2 \\ 4-2\lambda & 14-9\lambda+\lambda^2 \end{vmatrix} =$$

$$= (2-\lambda)\cdot(14-9\lambda+\lambda^2) - (4-2\lambda)(\lambda-2) =$$

$$= -\lambda^3 + 9\lambda^2 - 14\lambda + 2\lambda^2 - 18\lambda + 28 + 2\lambda^2 - 4\lambda - 4\lambda + 8 =$$

$$= \underbrace{-\lambda^3 + 13\lambda^2 - 40\lambda + 36}_{\text{charakteristický polynom}}$$

Zkusme $\lambda = 2$

$$-\lambda^3 + 13\lambda^2 - 40\lambda + 36 = (\lambda - 2)(-\lambda^2 + 11\lambda - 18)$$

$$\begin{array}{l} -(-\lambda^3 + 2\lambda^2) \\ \hline \quad 11\lambda^2 - 40\lambda + 36 \\ \quad -(11\lambda^2 - 22\lambda) \\ \hline \quad\quad -18\lambda + 36 \\ \quad\quad -(-18\lambda + 36) \\ \hline \quad\quad\quad 0 \end{array}$$

$$-\lambda^2 + 11\lambda - 18 = 0$$

$$\lambda_{2,3} = \frac{-11 \pm \sqrt{11^2 - 4\cdot 18}}{-2} = \frac{11 \mp \sqrt{49}}{2} = \begin{cases} 9 \\ 2 \end{cases}$$

Tedy char. polynom $p(\lambda) = -(\lambda-2)^2(\lambda-9)$

Vlastní čísla jsou $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 9$ a příslušné vlastní vektory dopočítáme.

① $\lambda = 2$ $\begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$u_3 = t$, $u_1 = s$ $\Rightarrow$ $u_2 = 2s + 6t$

tedy vl. vektory tvoří $\langle (0,6,1), (1,2,0) \rangle$

② $\lambda = 9$ $\begin{pmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & 3 \\ 0 & -21 & 21 \\ 0 & 7 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$u_3 = t$, $u_2 = t$, $u_1 = t$

tedy vl. vektory tvoří $\langle (1,1,1) \rangle$

zk: $\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} = 9 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 12 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$\boxed{4}$

$$A = \begin{pmatrix} -13 & 5 & 4 & 2 \\ 0 & -1 & 0 & 0 \\ -30 & 12 & 9 & 5 \\ -12 & 6 & 4 & 1 \end{pmatrix}$$

$$p(\lambda) = \begin{vmatrix} -13-\lambda & 5 & 4 & 2 \\ 0 & -1-\lambda & 0 & 0 \\ -30 & 12 & 9-\lambda & 5 \\ -12 & 6 & 4 & 1-\lambda \end{vmatrix} = (-1-\lambda) \cdot \begin{vmatrix} -13-\lambda & 4 & 2 \\ -30 & 9-\lambda & 5 \\ -12 & 4 & 1-\lambda \end{vmatrix} =$$

$$= -(1+\lambda)\left[(-13-\lambda)(9-\lambda)(1-\lambda) - 240 - 240 + 20(13+\lambda) + 120(1-\lambda) + 24(9-\lambda)\right] =$$

$$= -(1+\lambda) \cdot \left[-\lambda^3 - 3\lambda^2 + (13 \cdot 9 - 9 + 13 + 20 - 120 - 24)\lambda + (-13) \cdot 9 - 240 - 240 + 260 + 120 + 216\right] =$$

$$= -(1+\lambda)(-\lambda^3 - 3\lambda^2 - 3\lambda - 1) = (1+\lambda)(\lambda^3 + 3\lambda^2 + 3\lambda + 1) =$$

$$= (\lambda+1)^4$$

5) $p(\lambda) = (\lambda-6)(\lambda-7)^3$

a) geometrická násobnost vl. čísla 7 je 3

$$A = \begin{pmatrix} 6 & & & \text{cokoliv} \\ 0 & 7 & & \\ 0 & 0 & 7 & \\ 0 & 0 & 0 & 7 \end{pmatrix}, \quad A - 7I = \begin{pmatrix} -1 & & & \text{cokoliv} \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

tedy např. $A = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$, neboť

pak $A - 7I = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ a vl. vektory leží v

$$\langle (0,1,0,0), (0,0,1,0), (0,0,0,1) \rangle$$

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b) geom. násobnost 7 je 2

$$A = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 7 & 0 & 1 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}, \text{ pak } A - 7I = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a $\text{Eigen}(7) = \langle (0,1,0,0), (0,0,1,0) \rangle$

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c) geom. násobnost 7 je 1

$$A = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}, \text{ pak } A - 7I = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a $\text{Eigen}(7) = \langle (0,1,0,0) \rangle$

6)

$$A = \begin{pmatrix} -11 & 5 & 4 & 1 \\ -3 & 0 & 1 & 0 \\ -21 & 11 & 8 & 2 \\ -9 & 5 & 3 & 1 \end{pmatrix}$$

$$p(\lambda) = \lambda^4 + (\mathrm{tr}\,A)\cdot\lambda^3 + c_2\lambda^2 + c_1\lambda + |A|$$

$$|A| = \lambda_1\cdot\lambda_2\cdot\lambda_3\cdot\lambda_4 \ \Big| \overset{?}{=} \ \mathrm{tr}\,A = \lambda_1+\lambda_2+\lambda_3+\lambda_4$$

$$p(1) = p(-1) = 0$$

$$|A| = -3\cdot(-1)\begin{vmatrix} 5 & 4 & 1 \\ 11 & 8 & 2 \\ 5 & 3 & 1 \end{vmatrix} + 1\cdot(-1)\cdot\begin{vmatrix} -11 & 5 & 1 \\ -21 & 11 & 2 \\ -9 & 5 & 1 \end{vmatrix} =$$

$$= -3\cdot\begin{vmatrix} 5 & 4 & 1 \\ 11 & 8 & 2 \\ 0 & -1 & 0 \end{vmatrix} + (-1)\cdot\begin{vmatrix} -11 & 5 & 1 \\ -21 & 11 & 2 \\ 20 & 0 & 0 \end{vmatrix} =$$

$$= (-3)\cdot(-1)\cdot(-1)\cdot\begin{vmatrix} 5 & 1 \\ 11 & 2 \end{vmatrix} + (-1)\cdot 20\cdot\begin{vmatrix} 5 & 1 \\ 11 & 2 \end{vmatrix} =$$

$$= -3\cdot(-1) - 20\cdot(-1) = \underline{\underline{23}}$$

$$\mathrm{tr}\,A = -2$$

$$|A| = 23 = \lambda_1\cdot\lambda_2\cdot\lambda_3\cdot\lambda_4 = -\lambda_3\lambda_4$$

$$\mathrm{tr}\,A = -2 = \lambda_1+\lambda_2+\lambda_3+\lambda_4 = \lambda_3+\lambda_4$$

$$\lambda_3 = -\lambda_4 - 2$$

$$23 = -\lambda_4(-\lambda_4-2) = \lambda_4^2 + 2\lambda_4$$

$$0 = \lambda_4^2 + 2\lambda_4 - 23$$

$$\lambda_4 = \frac{-2 \pm \sqrt{4+4\cdot 23}}{2} = \frac{-2 \pm \sqrt{96}}{2} =$$

$$= -1 \pm \sqrt{24} = -1 \pm 2\sqrt{6}$$

$$\lambda_3 = -1 + 2\sqrt{6} \qquad \lambda_4 = -1 - 2\sqrt{6}$$