1)

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

Chceme najít diagonální matici D a matici P tak, aby $A = PDP^{-1}$

Nalezneme vlastní čísla a vl. vektory A.

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} = \Big[(4-\lambda)(1-\lambda)(8-\lambda) - 12 - 12$$

$$-2\cdot 6\cdot(1-\lambda) + 1\cdot 6(4-\lambda) + 1\cdot 2\cdot(8-\lambda)\Big] =$$

$$= -\lambda^3 + \lambda^2(4+1+8) + \lambda(-4-8-32+12-6-2) + 32-24-12$$

$$+24+16 = -\lambda^3 + 13\lambda^2 - 40\lambda + 36$$

	-1	13	-40	36
2	-1	11	-18	0

$\cdot 2 + 13$

$$-\lambda^3 + 13\lambda^2 - 40\lambda + 36 = (\lambda - 2)(-\lambda^2 + 11\lambda - 18)$$

$$\lambda^2 - 11\lambda + 18 = 0 \Rightarrow \lambda_{2,3} = \frac{11 \pm \sqrt{121-72}}{2} = \frac{11 \pm 7}{2} = \begin{cases} 9 \\ 2 \end{cases}$$

$$p(\lambda) = -(\lambda - 2)^2 \cdot (\lambda - 9)$$

vl. vektory

1) $\lambda_1 = \lambda_2 = 2$

$$\begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_3 &= t \\ x_1 &= s \\ x_2 &= 2s + 6t \end{aligned}$$

$$(x_1, x_2, x_3) = s\cdot(1,2,0) + t\cdot(0,6,1)$$

2) $\lambda_3 = 9$

$$\begin{pmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & -1 \\ 0 & -7 & 7 \\ 10 & -2 & 12 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & -1 \\ 0 & -7 & 7 \\ 0 & -7 & 7 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_3 = t \Rightarrow x_2 = t \Rightarrow x_1 = t$

$$(x_1, x_2, x_3) = t\cdot(1,1,1)$$

Zvolme $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 6 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Pak

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 6 & -1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -7 & -2 & 1 & -6 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{7} & \frac{1}{7} & -\frac{6}{7} \\ 0 & 1 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \\ 0 & 0 & 1 & \frac{2}{7} & -\frac{1}{7} & \frac{6}{7} \end{array}\right)$$

Zkusme

$$P\cdot D\cdot P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{pmatrix} \cdot \left(\frac{1}{7} \begin{pmatrix} 5 & 1 & -6 \\ -2 & 1 & 1 \\ 2 & -1 & 6 \end{pmatrix}\right) =$$

$$= \frac{1}{7}\begin{pmatrix} 2 & 0 & 9 \\ 4 & 12 & 9 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 5 & 1 & -6 \\ -2 & 1 & 1 \\ 2 & -1 & 6 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 28 & -7 & 42 \\ 14 & 7 & 42 \\ 14 & -7 & 56 \end{pmatrix} = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} = A$$

$$|(P\cdot D\cdot P^{-1} - \lambda I)| = 0$$

$$|P^{-1}(PDP^{-1} - \lambda I)P| = 0$$

$$|P^{-1}PDP^{-1}P - P^{-1}(\lambda I)P| = 0$$

$$|D - \lambda I|$$

Zkusme C:

$$\begin{vmatrix} 2-\lambda & 3 & 0 & 4 \\ 3 & -1-\lambda & 2 & 1 \\ 0 & 2 & -\lambda & 3 \\ 4 & 1 & 3 & 4-\lambda \end{vmatrix} \begin{array}{l} I-4\cdot II \\ \\ III-3\cdot II \\ IV-(1+\lambda)II \end{array} \begin{vmatrix} -10-\lambda & 7+4\lambda & -8 & 0 \\ 3 & -1-\lambda & 2 & 1 \\ -9 & 5+3\lambda & -6-\lambda & 0 \\ 3\lambda-8 & -\lambda^2+3\lambda+5 & 3\lambda-5 & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} -10-\lambda & 7+4\lambda & -8 \\ -9 & 5+3\lambda & -6-\lambda \\ 3\lambda-8 & -\lambda^2+3\lambda+5 & 3\lambda-5 \end{vmatrix} = \Big[(10+\lambda)(5+3\lambda)(5-3\lambda)$$

$$+(7+4\lambda)(6+\lambda)(8-3\lambda)+72(-\lambda^2+3\lambda+5)+8(5+3\lambda)(3\lambda-8)+$$

$$+9(7+4\lambda)(3\lambda-5)+(\lambda^2-3\lambda-5)(6+\lambda)(10+\lambda)\Big] =$$

$$= \Big[\lambda^4+\lambda^3(-9-12+6+10-3)+\lambda^2(10-9-21-72$$

$$+32-72+72+108+60-5-18-30)+$$

$$+\lambda(25+192+56-126+216-192+120-180$$

$$+189-50-30-180)+(250+336+360$$

$$-320-315-300)\Big] = \lambda^4-8\lambda^3-36\lambda^2+40\lambda+11$$

	1	-8	-36	40	11
11	1	3	-3	-7	-66
-11	1	-19	moc		

2)

$$A = PDP^{-1}$$

$$A^5 = PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1} =$$

$$= PD^5P^{-1}$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (1-\lambda)\left((2-\lambda)^2-1\right) =$$

$$= (1-\lambda)(\lambda^2-4\lambda+3) = -(\lambda-3)(\lambda-1)^2$$

vl. čísla

1) pro $\lambda = 1$

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{aligned} x_3 &= t \\ x_2 &= s \\ x_1 &= s-t \end{aligned}$$

vl. vektory $\langle (-1,0,1),(1,1,0) \rangle$

2) $\lambda = 3$

$$\begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{aligned} x_3 &= 0 \\ x_2 &= t \\ x_1 &= -t \end{aligned}$$

vl. vektory $\langle (1,-1,0) \rangle$

$$A = PDP^{-1}; \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P^{-1} \ldots \left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \quad P^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 4 & -2 & 2 \\ -2 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} = A \; !$$

$$A^5 = (PDP^{-1})^5 = PD^5P^{-1} = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 243 \end{pmatrix}P^{-1} =$$

$$= \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 243 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 243 \\ 0 & 1 & -243 \\ 1 & 0 & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 244 & -242 & 244 \\ -242 & 244 & -242 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 122 & -121 & 122 \\ -121 & 122 & -121 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-3} = (A^{-1})^3 = \left((PDP^{-1})^{-1}\right)^3 = (PD^{-1}P^{-1})^3 =$$

$$= PD^{-3}P^{-1}$$

$$D^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 27 \end{pmatrix} \qquad D^{-3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix}$$

$$A^{-3} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 & \frac{1}{27} \\ 0 & 1 & -\frac{1}{27} \\ 1 & 0 & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \frac{1}{2} \cdot \frac{1}{27} \cdot \begin{pmatrix} 28 & 26 & -26 \\ 26 & 28 & 26 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{27}\begin{pmatrix} 14 & 13 & -13 \\ 13 & 14 & 13 \\ 0 & 0 & 1 \end{pmatrix}$$

3.

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & & 0 \\ & 2 & \\ 0 & & 9 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 6 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{7} \begin{pmatrix} 5 & 1 & -6 \\ -2 & 1 & 1 \\ 2 & -1 & 6 \end{pmatrix}$$

$$A^k = \left(PDP^{-1}\right)^k = PD^kP^{-1} =$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 2 & 6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 9^k \end{pmatrix} \cdot \frac{1}{7} \begin{pmatrix} 5 & 1 & -6 \\ -2 & 1 & 1 \\ 2 & -1 & 6 \end{pmatrix}$$