An application of Markov chains:
Google - PageRank Algorithm

Problem: list websites in order of rank.

Rank of a page: proportion of time the page is visited.

Assumption: a page with many links to and from pages of high rank should also be ranked high.

Surfer: current page $\rightarrow$ new page:

- Choose a link from the current page randomly: 85%$(= p)$ of the time

- Choose a page from the web randomly: 15%$(= 1 - p)$ of the time.

Notation:

- The web consists of n pages.

- Markov chain: each page is a state.
  $A = [a_{ij}]$ $n \times n$ transition matrix.
  a_{ij} is the probability of moving from page j to page i.

- $C = [c_{ij}]$ $n \times n$ matrix: describes the page links.

  $$c_{ij} = \begin{cases} 1 \text{ if there is a link from page } j \text{ to page } i \\ 0 \text{ otherwise} \end{cases}$$

- s_j = the number of pages to which page j links:

$$s_j = \sum_{i=1}^{n} c_{ij}$$

Find a_{ij}?

- Page j has no links: $s_j = 0$.

  $\Rightarrow$ surfer chooses a page from the web randomly.

$$a_{ij} = \frac{1}{n}$$

- Page j has links to other pages: $s_j \neq 0$.

  2 ways to move from page j to page i:

  1. surfer chooses link to page i on page j: probability is $p\frac{c_{ij}}{s_j}$
  2. surfer chooses a page from the web randomly and the chosen page is page i: probability $(1-p)\frac{1}{n}$.

  So if $s_j \neq 0$, $a_{ij} = p\frac{c_{ij}}{s_j} + \frac{1-p}{n}$.

Summarizing,

$$a_{ij} = \begin{cases} \frac{1}{n} \text{ if } s_j = 0 \\ p\frac{c_{ij}}{s_j} + \frac{1-p}{n} \text{ if } s_j \neq 0 \end{cases}$$

A has a steady-state vector. The components of this vector are used to rank the webpages listed by Google.

Example: suppose the Web consists of ten pages. The links between the pages are given by the matrix C:

$$C = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the formula for a_{ij}, you find that matrix A is given as follows:

$$\begin{bmatrix}
0.015 & 0.440 & 0.185 & 0.2275 & 0.0150 & 0.1 & 0.015 & 0.2275 & 0.440 & 0.015 \\
0.015 & 0.015 & 0.015 & 0.0150 & 0.2275 & 0.1 & 0.440 & 0.0150 & 0.015 & 0.015 \\
0.015 & 0.015 & 0.015 & 0.2275 & 0.0150 & 0.1 & 0.015 & 0.2275 & 0.440 & 0.015 \\
0.015 & 0.015 & 0.185 & 0.0150 & 0.0150 & 0.1 & 0.440 & 0.2275 & 0.015 & 0.015 \\
0.440 & 0.015 & 0.185 & 0.2275 & 0.0150 & 0.1 & 0.015 & 0.0150 & 0.015 & 0.015 \\
0.015 & 0.015 & 0.185 & 0.0150 & 0.0150 & 0.1 & 0.015 & 0.0150 & 0.015 & 0.015 \\
0.015 & 0.015 & 0.185 & 0.0150 & 0.2275 & 0.1 & 0.015 & 0.2275 & 0.015 & 0.015 \\
0.015 & 0.440 & 0.015 & 0.0150 & 0.2275 & 0.1 & 0.015 & 0.0150 & 0.015 & 0.015 \\
0.015 & 0.015 & 0.015 & 0.0150 & 0.0150 & 0.1 & 0.015 & 0.0150 & 0.015 & 0.865 \\
0.440 & 0.015 & 0.015 & 0.2275 & 0.2275 & 0.1 & 0.015 & 0.0150 & 0.015 & 0.015
\end{bmatrix}$$

Steady-state vector:

$$\begin{bmatrix}
0.1583 \\ 0.0774 \\ 0.1072 \\ 0.0860 \\ 0.1218 \\ 0.0363 \\ 0.0785 \\ 0.0769 \\ 0.1282 \\ 0.1295
\end{bmatrix}$$

Reference: Spence L., Insel A., & Friedberg S. (2008). *Elementary Linear Algebra: A matrix approach.* Upper Saddle River, New Jersey: Pearson Prentice Hall.