31) Diferenciální rovnice se separovanými proměnnými

$$y' = f(x)g(x)$$

$$\frac{dy}{dx} = f(x)g(x) \qquad g(y) \neq 0$$

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

$$\Downarrow$$

$$G(y) = F(x) + c \,, \; c \in \mathbb{R}$$

a) $2y - x^3 y' = 0$

$$2y = x^3 y'$$

$$2\frac{1}{x^3} = \frac{1}{y}y' \,, \quad y \neq 0$$

$$2\frac{dx}{x^3} = \frac{1}{y}dy \;\Big|\int$$

$$2 \cdot \frac{x^{-2}}{-2} + c = \ln|y| \,, \; c \in \mathbb{R}$$

$$-\frac{1}{x^2} + c = \ln|y|$$

$$e^{-1/x^2} \cdot e^{c} = |y| \,, \quad c \in \mathbb{R}$$

$$K \cdot e^{-1/x^2} = |y| \,, \quad K \in \mathbb{R}^+$$

$$\underline{\underline{C \cdot e^{-1/x^2} = y}} \,, \; C \in \mathbb{R}$$

pro $c = 0$ obdržíme

proto řešení je pouze

$$y = C \cdot e^{-1/x^2} \,, \; C \in \mathbb{R}.$$

b) $(x+1)dy + xy\,dx = 0$

$(x+1)dy = -xy\,dx \quad | y \neq 0$

$\frac{1}{y}dy = -\frac{x}{x+1}dx \quad | \int$

$\ln|y| = -\int \frac{x}{x+1}dx = -\int\left(1-\frac{1}{x+1}\right)dx = -\left(x-\ln|x+1|\right)+c$

$|y| = e^{-x+\ln|x+1|} \cdot K \quad , \quad K \in \mathbb{R}^+$

$y = C(x+1)e^{-x} \quad , C \in \mathbb{R}$

c) $y - y^2 + xy' = 0$

$xy' = y^2 - y \quad , \quad y \not\equiv 0, \; y \not\equiv 1$

$\frac{dy}{y^2-y} = \frac{dx}{x}$

$\frac{dy}{y(y-1)} = \frac{dx}{x} \quad | \int$

$-\ln|y| + \ln|y-1| = \int \frac{dy}{y(y-1)} = \ln|x| + c$

$e^{-\ln|y|+\ln|y-1|} = e^{\ln|x|} \cdot K, \quad K \in \mathbb{R}^+$

$e^{\ln\left|\frac{y-1}{y}\right|} = e^{\ln|x|} \cdot K$

$\frac{y-1}{y} = Cx \quad , \quad C \in \mathbb{R}$

$y = (1-Cx)^{-1} \quad , C \in \mathbb{R} \quad , \quad y \equiv 0$

d) $y'\cos^2 x = (1+\cos^2 x)\cdot\sqrt{1-y^2}$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{1+\cos^2 x}{\cos^2 x}\,dx \quad \Big|\ y \neq \pm 1$$

$$\frac{dy}{\sqrt{1-y^2}} = \left(\frac{1}{\cos^2 x}+1\right)dx \quad \Big/ \int$$

$$\arcsin y = \operatorname{tg} x + x + C,\quad C\in\mathbb{R}$$

$$\underline{\underline{y = \sin(\operatorname{tg} x + x + C),\ C\in\mathbb{R}}},\quad \underline{\underline{y=\pm 1}}$$

e) $e^{-s}\left(1+\frac{ds}{dt}\right) = 1$

$$1+\frac{ds}{dt} = e^s$$

$$\frac{ds}{dt} = e^s - 1$$

$$\frac{ds}{e^s-1} = dt,\quad e^s \neq 1$$

$$\int \frac{ds}{e^s-1} \left| \begin{array}{l} u = e^s - 1 \\ du = e^s ds \\ du = (u+1)ds \end{array} \right| = \int \frac{1}{u}\cdot\frac{du}{u+1} =$$

$$= \int \frac{1}{u} - \frac{1}{u+1}\,du = \ln|u| - \ln|u+1| =$$

$$= \ln\left|\frac{u}{u+1}\right| = \ln\left|\frac{e^s-1}{e^s}\right|$$

$$\ln\left|\frac{e^s-1}{e^s}\right| = t + C$$

$$\left|\frac{e^s-1}{e^s}\right| = e^t\cdot K,\quad K\in\mathbb{R}^+$$

$$\frac{e^s-1}{e^s} = Ce^t,\quad C\in\mathbb{R}$$

$$1-\frac{1}{e^s} = Ce^t$$

$$\underline{\underline{1-Ce^t = e^{-s}}},\quad C\in\mathbb{R}$$

f) $y' \operatorname{tg} x - y^2 = 1 - 2y$

$$y' \operatorname{tg} x = 1 - 2y + y^2$$

$$\frac{y'}{1-2y+y^2} = \frac{1}{\operatorname{tg} x} \quad \Big/ \ (1-2y+y^2) = (1-y)^2 \neq 0 \Rightarrow y \neq 1$$

$$\frac{dy}{(1-y)^2} = \frac{dx}{\operatorname{tg} x}$$

$$\int \frac{1}{(1-y)^2}\,dy \left| \begin{array}{l} u = 1-y \\ du = -dy \end{array} \right| = -\int \frac{1}{u^2}\,du =$$

$$= +\frac{1}{u} = \frac{1}{1-y}$$

$$\frac{1}{1-y} = \ln|\sin x| + C$$

$$\underline{\underline{1 = (1-y)\left[\ln|\sin x| + C\right], C \in \mathbb{R}}} \qquad \underline{\underline{y = 1}}$$

g) $y' = 6x + 2y + 3$ $\Rightarrow$ substituce $z = 6x + 2y + 3$

$$\frac{dy}{dx} = z$$

$$\frac{dy}{dx} \cdot \frac{dz}{dy} = z \cdot \frac{dz}{dy}$$

$$\frac{dz}{dx} = z \cdot 2$$

$$\frac{dz}{z} = 2dx \quad , z \neq 0$$

$$\ln|z| = 2x + C$$

$$z = C \cdot e^{2x}, \quad C \in \mathbb{R}$$

$$6x + 2y + 3 = C \cdot e^{2x}$$

$$\underline{\underline{y = C \cdot e^{2x} - 3(x+1)}}, \ C \in \mathbb{R}$$

h) $2(1+e^x)yy' = e^x, \quad y(0)=0$

$2\,y\,dy = \frac{e^x}{1+e^x}\,dx$

$\int \frac{e^x}{1+e^x}\,dx \;\left| \begin{array}{l} u = e^x+1 \\ du = e^x dx \end{array} \right| = \int \frac{du}{u} = \ln|u| =$

$= \ln|e^x+1|$

$y^2 = \ln|e^x+1| + C$

$e^{y^2} = K\cdot(e^x+1), \; K\in\mathbb{R}^+$

$y(0)=0:$

$e^0 = K\cdot(1+1)$

$1 = 2K$

$K = 1/2$

$$e^{y^2} = \frac{1}{2}(e^x+1)$$

i) $\sin y \cos x \, dy = \cos y \sin x \, dx, \quad y(0) = \frac{\pi}{4}$

$\mathrm{tg}\, y\, dy = \mathrm{tg}\, x\, dx$

$-\ln|\cos y| = -\ln|\cos x| + C$

$|\cos y| = K\cdot\cos x, \; K\in\mathbb{R}^+$

$\cos y = C\cdot\cos x, \; C\in\mathbb{R}$

$y(0):$ $\cos\frac{\pi}{4} = C\cdot\cos 0$

$\frac{\sqrt{2}}{2} = C$

$\cos y = \frac{\sqrt{2}}{2}\cos x \Rightarrow \frac{2}{\sqrt{2}}\cos y = \cos x \Rightarrow \sqrt{2}\cos y = \cos x$

j) $(x^2+1)(y^2-1)+xyy'=0, \quad y(1)=\sqrt{2}$

$$(x^2+1)(y^2-1)=-xy\frac{dy}{dx}$$

$$-\frac{x^2+1}{x}\,dx=\frac{y}{y^2-1}\,dy$$

$$\int\frac{y}{(y+1)(y-1)}=\int\frac{1/2}{y+1}+\frac{1/2}{y-1}\,dy$$

$$=1/2\ln|y+1|+1/2\ln|y-1|=$$

$$=1/2\ln|y^2-1|$$

$$-\left(x+\frac{1}{x}\right)dx=\frac{y}{(y+1)(y-1)}\,dy$$

$$-\frac{x^2}{2}-\ln|x|+C=\frac{1}{2}\ln|y^2-1|$$

$$-x^2-2\ln|x|+L=\ln|y^2-1|, \quad L\in\mathbb{R}$$

$$e^{-x^2-2\ln|x|}\cdot K=|y^2-1|, \quad K\in\mathbb{R}^+$$

$$e^{-x^2}\cdot x^{-2}\cdot C=y^2-1, \quad C\in\mathbb{R}$$

$$y=\sqrt{e^{-x^2}\cdot x^{-2}\cdot C+1}$$

$y(1)$: $\sqrt{2}=\sqrt{e^{-1}\cdot 1\cdot C+1} \Rightarrow C=e$

$$\underline{\underline{y(x)=\sqrt{e^{1-x^2}\cdot x^{-2}+1}}}$$

k) $y' = \dfrac{x-y+1}{x-y}$

$z = x - y$

$y = x - z$

$y' = 1 - z'$

$1 - z' = \dfrac{z+1}{z}$

$1 - z' = 1 + \dfrac{1}{z}$

$z' = -\dfrac{1}{z}$

$z \, dz = -1 \, dx$

$\dfrac{z^2}{2} = -x + C$

$z^2 + 2x + C = 0$

$\underline{\underline{(x-y)^2 + 2x + C = 0}}$

32) <u>Homogenní diferenciální rovnice</u>

funkce homogenní k-tého stupně

$$f(tx, ty) = t^k f(x,y)$$

$f(x,y) = x^2 + y^2 - xy$ ← homogenní 2. stupně

$f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ ← homogenní 0-tého stupně

homogenní DR lze psát ve tvaru

$$y' = g\left(\frac{y}{x}\right)$$

řešíme substitucí

$$u = \frac{y}{x}$$

$$ux = y$$

$$y' = u + u'x$$

potom dostaneme

$$u + u'x = g(u)$$

máme-li DR

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

ta je homogenní $\Leftrightarrow$ P, Q jsou homogenní funkce téhož stupně

a) $$xy' = \sqrt{x^2 - y^2} + y$$

$$y' = \frac{\sqrt{x^2-y^2}}{x} + \frac{y}{x}$$

$$y' = \sqrt{\frac{x^2-y^2}{x^2}} + \frac{y}{x}$$

$$y' = \sqrt{1-\left(\frac{y}{x}\right)^2} + \frac{y}{x} \quad , \quad u = \frac{y}{x} \quad , \quad y' = u + u'x$$

$$u + u'x = \sqrt{1-u^2} + u$$

$$\frac{du}{\sqrt{1-u^2}} = \frac{dx}{x} \quad , \quad 1-u^2 \neq 0 \Rightarrow u^2 \neq 1 \Rightarrow u = \pm 1$$

$$\arcsin u = \ln|\cancel{x}| + C \quad \cdot \ln|k|$$

nutně $\left|\ln|kx|\right| \leq \frac{\pi}{2}$

$$\arcsin \frac{\cancel{y}}{x} = \ln|kx|$$

$$\underline{\underline{y = x \cdot \sin \ln|kx|}} \,, \quad k \in \mathbb{R} \qquad \underline{\underline{y = \pm x}}$$

b) $(y^2-x^2)dx - 2xy\,dy = 0$

$xy \Rightarrow (tx)\cdot(ty) = t^2xy \leftarrow$ 2. stupeň

$y^2-x^2 \Rightarrow (ty)^2-(tx)^2 = t^2(y^2-x^2) \leftarrow$ 2. stupeň

$(y^2-x^2)dx = 2xy\,dy$

$$\frac{y^2-x^2}{2xy} = \frac{dy}{dx}$$

$$\frac{1}{2}\cdot\left(\frac{y}{x}-\frac{x}{y}\right) = \frac{dy}{dx} \quad , \quad u = \frac{y}{x} \quad , \quad xy \neq 0$$

$$\frac{1}{2}\left(u-\frac{1}{u}\right) = u + u'x$$

$$\frac{1}{2}u - \frac{1}{2}\frac{1}{u} = u + u'x$$

$$-\frac{1}{2}u - \frac{1}{2}\frac{1}{u} = u'x$$

$$\frac{1}{x}dx = \frac{du}{-\frac{1}{2}u-\frac{1}{2u}}$$

$$-\frac{1}{x}dx = \frac{u}{\frac{1}{2}u^2+\frac{1}{2}}du$$

$$-\frac{1}{x}dx = \frac{2u}{u^2+1}du$$

$$-\ln|x| + C = \ln|u^2+1|$$

$$-\ln|kx| = \ln|u^2+1|$$

$$\left|\frac{1}{kx}\right| = u^2+1$$

$$\pm C\cdot\frac{1}{x} = \frac{y^2}{x^2}+1$$

$$\pm Cx = y^2+x^2 \quad , \quad C\in\mathbb{R}$$

c) $xy' + y\ln x = y\ln y$

$xy' = y(\ln y - \ln x)$

$y' = \frac{y}{x} \cdot \ln\frac{y}{x}$, $\frac{y}{x} = u$, $y' = u + u'x$

$u + u'x = u \cdot \ln u$

$u'x = u(-1+\ln u)$

$\frac{du}{u(\ln u - 1)} = \frac{dx}{x}$, $u \neq 0$, $1 \neq \ln u$

$\downarrow$

$y \neq 0$

$\int \frac{du}{u(-1+\ln u)} \left| \begin{matrix} t = -1+\ln u \\ dt = +\frac{1}{u}du \end{matrix} \right| =$

$= \int \frac{+dt}{t} = +\ln|t|$

$\ln|\ln u - 1| = \ln|x| + C$

$|\ln u - 1| = |Kx|$

$\ln u - 1 = Cx$

$\ln u = Cx + 1$

$u = e^{Cx+1}$

$\underline{\underline{y = x \cdot e^{Cx+1}}}$, $C \in \mathbb{R}$,

d) $(xy' - y)\cos\frac{y}{x} = x$

$\left(y' - \frac{y}{x}\right)\cos\frac{y}{x} = 1$, $\frac{y}{x} = u$, $y' = u + u'x$

$(u + u'x - u)\cos u = 1$

$\cos u \, du = \frac{1}{x}dx$

$\sin u = \ln|x| + C$

$\sin\frac{y}{x} = \ln|x| + C$

$\underline{\underline{x = C \cdot e^{\sin\frac{y}{x}}}}$, $C \in \mathbb{R}$

e) $xy' = y \cos \ln \frac{y}{x}$, $\frac{y}{x} > 0$

$y' = \frac{y}{x} \cos \ln \frac{y}{x}$, $\frac{y}{x} = u$, $y' = u + u'x$

$u + u'x = u \cdot \cos \ln u$

$u'x = u(-1 + \cos \ln u)$

$$\frac{u'}{u(-1+\cos \ln u)} = \frac{1}{x}$$

$$\frac{du}{u(-1+\cos \ln u)} = \frac{1}{x}\,dx$$

$$\int \frac{du}{u(-1+\cos \ln u)} \left| \begin{array}{l} t = \ln u \\ dt = \frac{1}{u}du \end{array} \right| = \int \frac{dt}{-1+\cos t} = \left| \begin{array}{l} \operatorname{tg}\frac{t}{2} = v \\ dt = \frac{2}{1+v^2}dv \\ \cos t = \frac{1-v^2}{1+v^2} \end{array} \right|$$

$$= \int \frac{\frac{2}{1+v^2}dv}{-1+\frac{1-v^2}{1+v^2}} = \int \frac{2dv}{-(1+v^2)+(1-v^2)} = \int \frac{2dv}{-2v^2}$$

$$= \frac{1}{v} = \frac{1}{\operatorname{tg}\frac{t}{2}}$$

$$\frac{1}{\operatorname{tg}(\frac{1}{2}\ln u)} = \ln|x| + C$$

$$\operatorname{cotg}(\tfrac{1}{2}\ln u) = \ln|x| + C$$

$$\operatorname{cotg}\left(\frac{1}{2}\ln\frac{y}{x}\right) = \ln Cx$$

33) Lineární diferenciální rovnice 1. řádu

$$y' = f(x)y + g(x)\ ,\ y(x_0) = y_0$$

$$y(x) = \exp\left(\int_{x_0}^{x} f(t)dt\right)\left[y_0 + \int_{x_0}^{x} g(t)\exp\left(-\int_{x_0}^{t} f(s)ds\right)dt\right]$$

homogenní:

$$y' = f(x)y + g(x) \;/\cdot\; e^{-\int f(t)dt}$$

$$y' \cdot e^{-\int f(t)dt} - y \cdot e^{-\int f(t)dt} f(x) = g(x)\, e^{-\int f(t)dt}$$

$$\left(y e^{-\int f(t)dt}\right)' = g(x)\, e^{-\int f(t)dt}$$

a) $y' = 6x - 2y$

I) homogenní

$$y' = -2y$$

$$\frac{dy}{y} = -2\,dx$$

$$\ln|y| = -2x + C$$

$$y = Ce^{-2x}$$

variace konstant

$$y(x) = C(x)\cdot e^{-2x}$$

$$y'(x) = C'(x)e^{-2x} - 2\cdot C(x)\cdot e^{-2x} = 6x - 2\cdot C(x)e^{-2x}$$

$$C' e^{-2x} = 6x$$

$$C' = 6x e^{+2x}$$

$$C = \int 6x e^{+2x}\,dx \left| \begin{matrix} u = 6x & u' = 6 \\ v' = e^{+2x} & v = +\frac{1}{2}e^{+2x} \end{matrix} \right| = +3xe^{+2x} - \int 3e^{+2x}\,dx =$$

$$= +3xe^{-2x} - \frac{3}{2}e^{+2x} + C$$

$$\underline{\underline{y(x) = 3x - \frac{3}{2} + C\cdot e^{-2x}}}$$

II) $y' = 6x - 2y$

$$y' + 2y = 6x \quad \Big| \cdot e^{\int 2\,dx} = e^{2x}$$

$$y'e^{2x} + 2ye^{2x} = 6xe^{2x}$$

$$(ye^{2x})' = 6xe^{2x}$$

$$ye^{2x} = \int 6xe^{2x}\,dx \left| \begin{matrix} u = 6x & u' = 6 \\ v' = e^{2x} & v = \frac{1}{2}e^{2x} \end{matrix} \right| = 3e^{2x} - \int 3e^{2x} = 3e^{2x} - \frac{3}{2}e^{2x} + C$$

$$\underline{\underline{y(x) = 3 - \frac{3}{2} + Ce^{-2x}}}$$

b) $y' = 4xy + (2x+1)e^{2x^2}$

$y' - 4xy = (2x+1)e^{2x^2} \quad / \cdot e^{-\int 4x dx} = e^{-2x^2}$

$y' e^{-2x^2} - 4x e^{-2x^2} y = 2x+1$

$\left(y e^{-2x^2}\right)' = 2x+1$

$y e^{-2x^2} = x^2 + x + C$

$$\underline{\underline{y(x) = (x^2+x+C)e^{2x^2}}}$$

c) $y' \cos x = (y + 2\cos x)\sin x$

$y' = y \, tg x$

$\frac{dy}{y} = tg \, dx$

$\ln|y| = -\ln|\cos x| + C$

$y = C \cdot \frac{1}{\cos x}$

$y' = C' \frac{1}{\cos x} + C \cdot \frac{+\sin x}{\cos^2 x} = C \cdot \frac{1}{\cos x} \cdot \frac{\sin}{\cos x} + 2 \cdot \sin x$

$C' = 2 \cdot \sin x \cdot \cos x$

$C = 2\int \sin x \cos x \, dx \left| \begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right| = 2\int u \, du = u^2 = \sin^2 x + C$

$$\underline{\underline{y(x) = \frac{\sin^2 x + C}{\cos x}}}$$

d) $y' - 4y = \cos x, \quad y(0) = 1$

$$y' - 4y = \cos x \quad \Big| \cdot e^{-\int 4\,dx}$$

$$y' e^{-4x} - 4 \cdot e^{-4x} \cdot y = e^{-4x} \cdot \cos x$$

$$\left(y e^{-4x}\right)' = e^{-4x} \cdot \cos x$$

$$\int e^{-4x} \cdot \cos x \, dx \quad \left| \begin{matrix} u = \cos x & u' = -\sin x \\ v' = e^{-4x} & v = -\frac{1}{4} e^{-4x} \end{matrix} \right| =$$

$$= -\frac{1}{4} \cos x \cdot e^{-4x} - \frac{1}{4} \int e^{-4x} \sin x \quad \left| \begin{matrix} u = \sin x & u' = \cos x \\ v' = e^{-4x} & v = -\frac{1}{4} e^{-4x} \end{matrix} \right|$$

$$= -\frac{1}{4} \cos x \cdot e^{-4x} + \frac{1}{16} \sin x \cdot e^{-4x} - \frac{1}{16} \int \cos x \cdot e^{-4x} \, dx$$

$$\Downarrow$$

$$\int e^{-4x} \cdot \cos x \, dx = \frac{16}{17} \cdot \left(-\frac{1}{4}\right) \cos x \cdot e^{-4x} + \frac{16}{17} \cdot \frac{1}{16} \sin x \, e^{-4x} + C$$

$$y = -\frac{4}{17} \cos x + \frac{1}{17} \sin x + C \cdot e^{4x}$$

$y(0)$: $\quad 1 = -\frac{4}{17} + C$

$$C = \frac{21}{17}$$

$$\underline{\underline{y(x) = \frac{1}{17} \sin x - \frac{4}{17} \cos x + \frac{21}{17} e^{-4x}}}$$

9) <u>Bernoulliova rovnice</u>

$$y' + f(x) y = y^n g(x) \quad , n \neq 0, n \neq 1, n \in \mathbb{R}$$

$$\frac{y'}{y^n} + \frac{f(x)}{y^{n-1}} = g(x) \quad , y \neq 0$$

↑ pro $r > 0$ je $y \equiv 0$ také řešení

$$z = \frac{1}{y^{n-1}} = y^{1-n}$$

$$z' = (1-n) y^{-n} \cdot y'$$

$$z' = (1-n) \left[g(x) - f(x) \cdot z \right]$$

↳ lineární dif. rovnice 1. řádu

a) $y' - xy = -y^3 e^{-x^2}$

$\frac{y'}{y^3} - \frac{x}{y^2} = -e^{-x^2}$, $y \neq 0$

$z = \frac{1}{y^2}$

$z' = -2 \cdot \frac{1}{y^3} \cdot y'$

$z' = -2\left(\cancel{\tfrac{x}{y}}\, xz - e^{-x^2}\right)$

$z' + 2xz = 2e^{-x^2} \quad / \cdot e^{\int 2x\,dx} = e^{x^2}$

$z'e^{x^2} + 2xze^{x^2} = 2$

$\left(ze^{x^2}\right)' = 2$

$ze^{x^2} = 2x + C$

$\frac{1}{y^2} = z = 2xe^{-x^2} + Ce^{-x^2}$

$\frac{1}{y^2} = e^{-x^2}(2x + C)$

$y^2 = \frac{e^{x^2}}{2x + C}$, $y = 0$

b) $3x^2 y' + xy = y^{-2}$

$y^2 y' + \frac{y^3}{3x} = \frac{1}{3x^2}$

$z = y^3$

$$z' = 3y^2 y'$$

$$z' = 3\cancel{y^2}\left(\frac{1}{3x^2} - \frac{z}{3x}\right)$$

$$z' = \frac{1}{x^2} - \frac{z}{x}$$

$$z' + \frac{z}{x} = \frac{1}{x^2} \quad \Big/ \cdot \quad e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x$$

$$(xz' + z) = \frac{1}{x}$$

$$(xz)' = \frac{1}{x}$$

$$xz = \ln|x| + C$$

$$z = \frac{\ln|x| + C}{x}$$

$$\underline{\underline{y^3 = \frac{\ln|x| + C}{x}}}$$

c) $y' = \frac{4}{x} y + x\sqrt{y}$

$$\frac{1}{\sqrt{y}} y' = \frac{4}{x}\sqrt{y} + x$$

$$z = \sqrt{y}$$

$$z' = \frac{1}{2\sqrt{y}} \cdot y' = \frac{1}{2}\left(\frac{4}{x} z + x\right)$$

$$z' = \frac{2z}{x} + \frac{1}{2}x$$

$$z' - \frac{2z}{x} = \frac{1}{2}x \quad \Big| \quad e^{-\int \frac{2}{x} dx} = e^{-2\ln|x|} = e^{-\ln x^2} = \frac{1}{x^2}$$

$$\left(z' \frac{1}{x^2} - \frac{2z}{x^3}\right) = \frac{1}{2x}$$

$$\left(z \cdot \frac{1}{x^2}\right)' = \frac{1}{2x}$$

$$\frac{1}{x^2} \cdot z = \frac{1}{2} \ln|x| + C$$

$$\frac{1}{x^2} \cdot \sqrt{y} = \ln\sqrt{|x|} + C$$

$$\sqrt{y} = x^2\left(\ln\sqrt{|x|} + C\right)$$

$$\underline{y = x^4\left(\ln\sqrt{|x|} + C\right)^2}, \quad \cancel{y = 0}$$

d) $y\,dy = \left(\frac{ay^2}{x^2} + \frac{b}{x^2}\right)dx, \quad a \neq 0$

$$y y' = \frac{ay^2}{x^2} + \frac{b}{x^2}$$

$$y y' - \frac{ay^2}{x^2} = \frac{b}{x^2}$$

$$z = y^2$$

$$z' = 2yy' = 2\left(\frac{az}{x^2} + \frac{b}{x^2}\right)$$

$$z' - \frac{2az}{x^2} = \frac{b}{x^2} \quad \Big| \cdot \quad e^{-\int \frac{2a}{x^2}dx} = e^{+2a \cdot \frac{1}{x}}$$

$$z' e^{\frac{2a}{x}} - \frac{2az}{x^2} \cdot e^{\frac{2a}{x}} = \frac{b}{x^2} \cdot e^{\frac{2a}{x}}$$

$$\left(z e^{\frac{2a}{x}}\right)' = \frac{b}{x^2} \cdot e^{\frac{2a}{x}}$$

$$z e^{\frac{2a}{x}} = -\frac{b}{2a} e^{\frac{2a}{x}} + C$$

$$z = -\frac{b}{2a} + C e^{-\frac{2a}{x}}$$

$$\underline{y^2 + \frac{b}{2a} = C e^{-\frac{2a}{x}}}$$

$$\int \frac{1}{x^2} \cdot e^{\frac{2a}{x}} dx = \left| \begin{matrix} u = \frac{1}{x} \\ du = -\frac{1}{x^2}dx \end{matrix} \right| =$$

$$= -\int e^{2au} du = -\frac{1}{2a} e^{2au} =$$

$$= -\frac{1}{2a} e^{\frac{2a}{x}}$$