9) Vyšetřete průběh funkce

$$f(x) = \frac{x^3}{x^2-1}$$

1) $D(f) = \mathbb{R} \setminus \{\pm 1\}$

2) funkce je spojitá v $D(f)$

3) $f(-x) = \frac{-x^3}{x^2-1} = -f(x)$ ... je lichá

4) $f(x) = 0 \iff x = 0$

$f(x)$: $-$ | -1 | $+$ | 0 | $-$ | 1 | $+$

5) $$f'(x) = \frac{3x^2(x^2-1) - x^3(2x)}{(x^2-1)^2} = \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2} = \frac{x^4 - 3x^2}{(x^2-1)^2}$$

$D(f') = \mathbb{R} \setminus \{\pm 1\}$,

$x^4 - 3x^2 = 0$

$x^2(x^2-3) = 0$

$x_{1,2} = 0$, $x_{3,4} = \pm\sqrt{3}$

6) $f(-\sqrt{3}) = -\frac{3}{2}\sqrt{3}$ $\qquad$ $f(\sqrt{3}) = \frac{3}{2}\sqrt{3}$

$f'(x)$: $+$ | $-\sqrt{3}$ | $-$ | -1 | $-$ | 0 | $-$ | 1 | $-$ | $\sqrt{3}$ | $+$

$f(x)$: $\nearrow$ $\searrow$ $\searrow$ $\searrow$ $\searrow$ $\nearrow$

lok. max $\qquad$ lok. min.

7) $$f''(x) = \frac{(4x^3-6x)(x^2-1)^2 - (x^4-3x^2)\cdot 2\cdot(x^2-1)\cdot 2x}{(x^2-1)^4} = \frac{(4x^3-6x)(x^2-1) - 4x(x^4-3x^2)}{(x^2-1)^3} = \frac{4x^5-4x^3-6x^3+6x-4x^5+12x^3}{(x^2-1)^3} =$$

$$= \frac{2x^3+6x}{(x^2-1)^3}$$ $\qquad$ $D(f'') = \mathbb{R} \setminus \{\pm 1\}$

8) $2x^3 + 6x = 0$

$x(2x^2+6) = 0$

$x_1 = 0$ $\qquad$ $x^2 = -3$

$f(0) = 0$

$f''(x)$: $-$ | -1 | $+$ | 0 | $-$ | 1 | $+$

$f(x)$: $\cap$ $\cup$ $\cap$ $\cup$

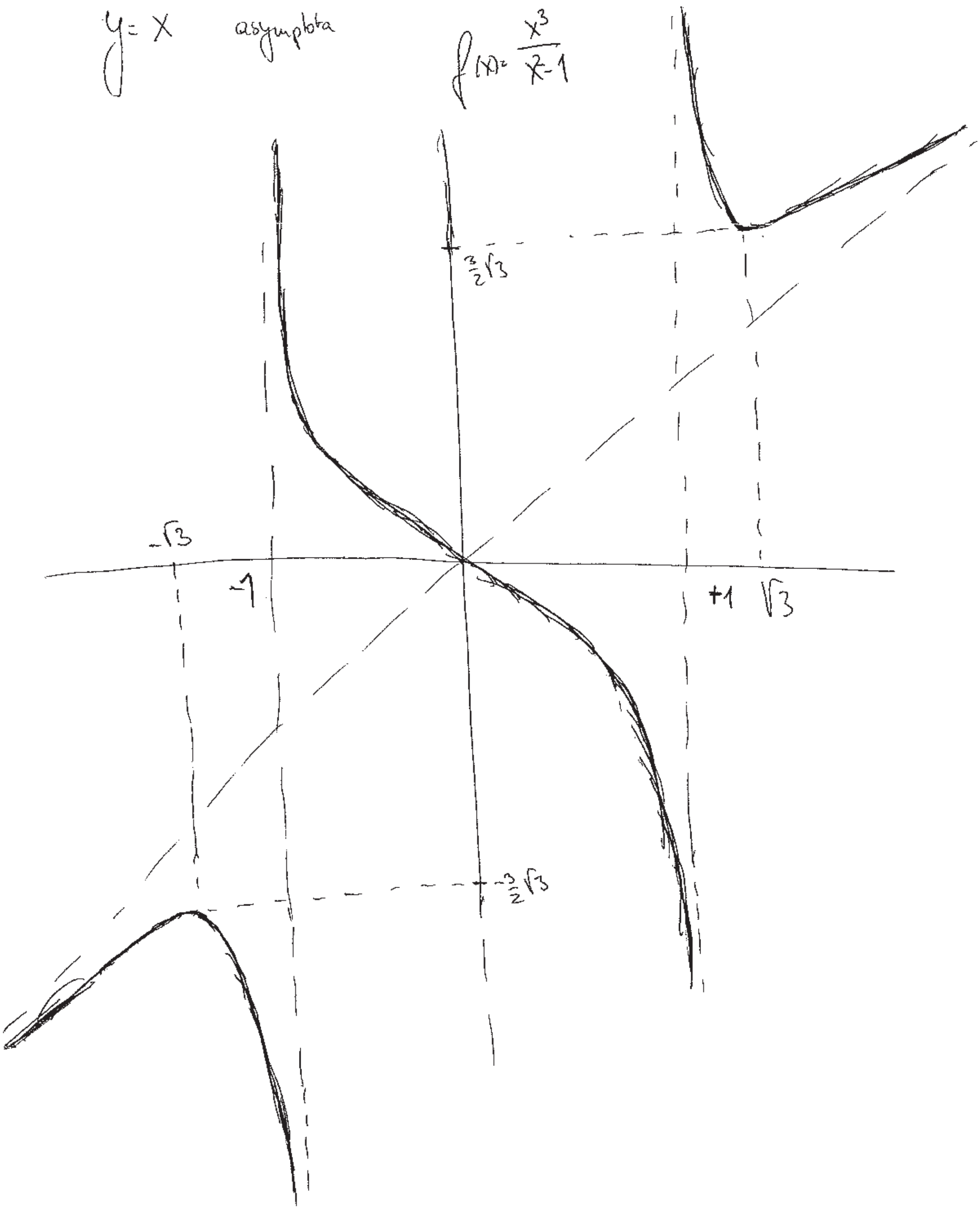
9) $\lim\limits_{x \to 1^+} \frac{x^3}{x^2-1} = \infty$

$\lim\limits_{x \to -1^+} \frac{x^3}{x^2-1} = +\infty$ asymptoty: $x_0 = 1$, $x_0 = -1$

10) $\lim\limits_{x \to \pm\infty} \frac{x^2}{x^2-1} = 1$

$\lim\limits_{x \to \pm\infty} \left(\frac{x^3}{x^2-1} - x \right) = \lim\limits_{x \to \pm\infty} \frac{x^3 - x^3 + x}{x^2-1} = \lim\limits_{x \to \pm\infty} \frac{x}{x^2-1} = 0$

$y = x$ asymptota



$\cancel{A}$) Průběh funkce

$$f(x) = x - \operatorname{arctg} x$$

1) $D(f) = \mathbb{R}$

2) je spojitá

3) $f(-x) = -x - (-\operatorname{arctg} x) = -x + \operatorname{arctg} x = -f(x)$ ... je lichá

4) $x = \operatorname{arctg} x$ jistě pro $x = 0$
zbytek vyřešíme později

~~$\tan x = x$~~

$f(x)$: $-$ | $+$ (bod 0)

5) $f'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{1+x^2-1}{1+x^2} = \dfrac{x^2}{1+x^2}$ $D(f') = \mathbb{R}$

6) $x^2 = 0$
$x = 0$

$f(0) = 0$

$f'(x)$: $+$ | $+$ (bod 0)
$f(x)$: $\nearrow$ $\nearrow$

($\Rightarrow$ funkce je stále rostoucí, proto nemůže mít více než jeden průsečík s osou x)

7) $f''(x) = \dfrac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x + 2x^3 - 2x^3}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2}$ $D(f'') = \mathbb{R}$

8) $f''(x) = 0 \iff 2x = 0$
$x = 0$

$f''(x)$: $-$ | $+$ (bod 0)
$f(x)$: $\cap$ $\cup$
infl. bod

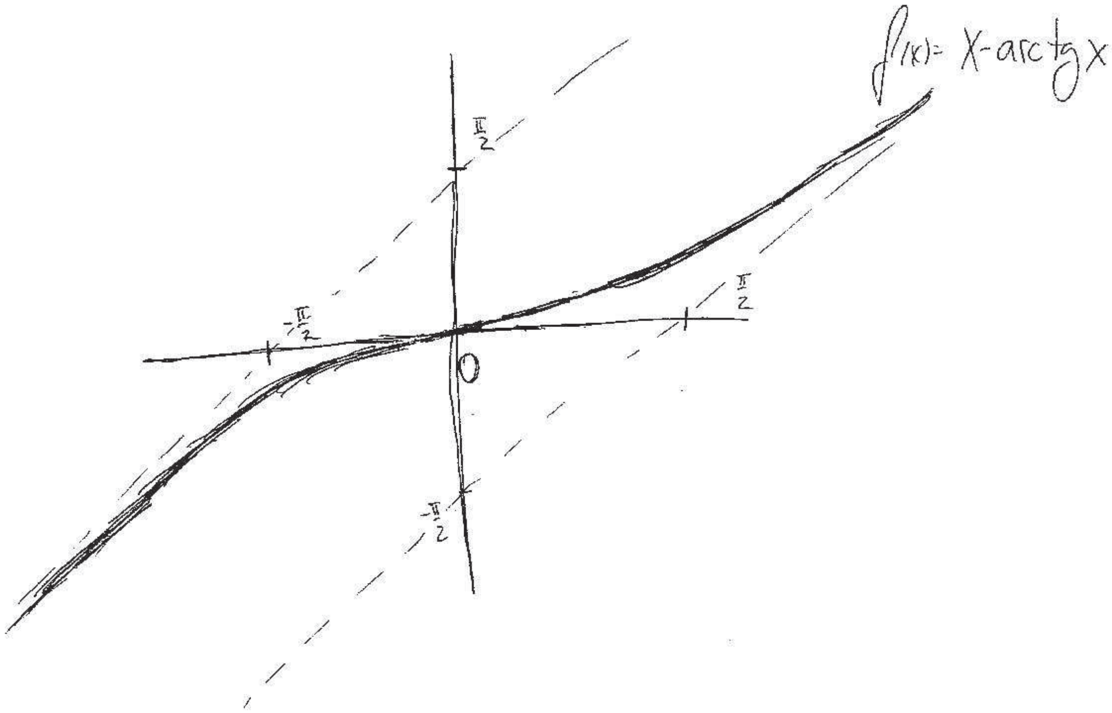
směrnice tečny
$f'(0) = 0 \Rightarrow$ tečna rovnoběžná s osou x

9) asymptoty bez směrnice – nejsou

10) $\lim\limits_{x\to\pm\infty} \left(\dfrac{x - \operatorname{arctg} x}{x}\right) \overset{L'H}{=} \lim\limits_{x\to\pm\infty} \left(\dfrac{1 - \frac{1}{1+x^2}}{1}\right) = 1$

$\lim\limits_{x\to\pm\infty} (x - \operatorname{arctg} x - x) = \lim\limits_{x\to\pm\infty} \operatorname{arctg} x = \pm\dfrac{\pi}{2}$

$y = x - \dfrac{\pi}{2}$, $x \to +\infty$
$y = x + \dfrac{\pi}{2}$ $x \to -\infty$



c) Průběh funkce

$$f(x) = \sqrt[3]{2x^2 - x^3}$$

1) $D(f) = \mathbb{R}$, 2) spojitá všude

3) $f(-x) = \sqrt[3]{2x^2 + x^3}$ $\quad \cancel{S}, \cancel{L}$

4) $f(x) = 0$

$$2x^2 - x^3 = 0$$
$$x^2(2 - x) = 0$$
$$x = 0, \; x = 2$$

+		+		−
	0		2	

5) $f'(x) = \frac{1}{3} \cdot (2x^2 - x^3)^{-2/3} \cdot (4x - 3x^2) = \dfrac{4x - 3x^2}{3\sqrt[3]{(2x^2 - x^3)^2}}$ $\quad D(f') = \mathbb{R} \setminus \{0, 2\}$

$$4x - 3x^2 = 0$$
$$x(4 - 3x) = 0$$
$$x = 0, \quad x = \frac{4}{3}$$

6)

$f(0) = 0$ $\quad$ lok. max

$f'(x)$:	−	0	+	$\frac{4}{3}$	−	2	−
$f(x)$:	↘		↗		↘		↘

lok. max. $\quad$ $f\left(\frac{4}{3}\right) = \frac{2}{3}\sqrt[3]{4}$

7) $f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)(2x^2-x^3)^{-4/3} \cdot (4x-3x^2)^2 + \frac{1}{3}(2x^2-x^3)^{-2/3} \cdot (4-6x) =$

$$= -\frac{8}{9\cdot(2-x)\sqrt[3]{(2x^2-x^3)^2}}$$

$D(f'') = \mathbb{R}\setminus\{0, 2\}$

8) f'': − (0) − (2) +

f: $\cap$ $\cap$ $\cup$

infl. bod

$f(2) = 0$

směrnice tečny v bodě 2 je $f'(2) \Rightarrow$ neexist.

proto

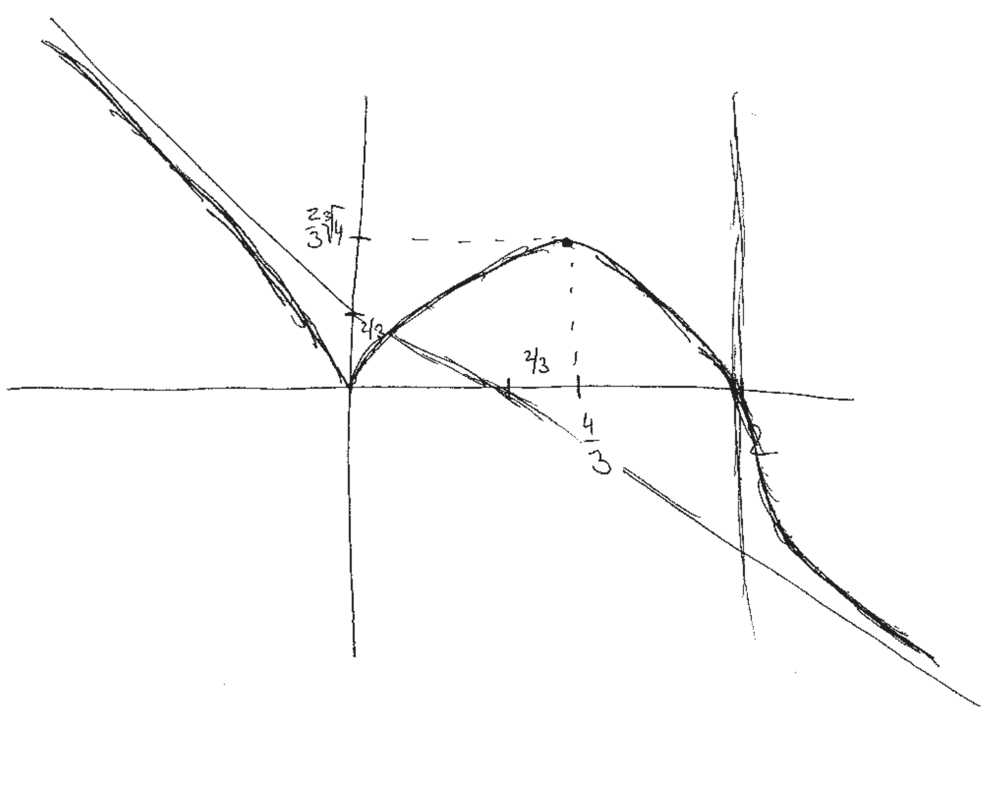
$\lim_{x\to 2} f'(x) = -\infty \Rightarrow$ tečna je rovnoběžná s osou y

9) asympt. bez směrnice neexist.

10) $\lim_{x\to\pm\infty} \frac{\sqrt[3]{2x^2-x^3}}{x} = \lim_{x\to\pm\infty} \sqrt[3]{\frac{2x^2-x^3}{x^3}} = -1$

$\lim_{x\to\pm\infty} \left(\sqrt[3]{2x^2-x^3} + x\right) = \frac{2}{3}$

$y = -x + \frac{2}{3}$



d) Průběh funkce

$$y = \frac{\ln x}{x}$$

1) $D(f) = (0, \infty)$ $\qquad \lim\limits_{x \to 0^+} \frac{\ln x}{x} = -\infty$

2) spojitá v def. oboru

3) ∞ $\nexists$, $\nexists$

4) $f(x) = 0$ pro $x = 1$

$f(x)$: 0 | $-$ | 1 | $+$

5) $f'(x) = \dfrac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \dfrac{1 - \ln x}{x^2}$ $\qquad D(f) = (0, \infty)$

6) $f'(x) = 0$

$1 - \ln x = 0$

$\ln x = 1$

$x = e$

$f(e) = 1/e$

$f'(x)$: 0 | $+$ | e | $-$

$f(x)$: $\nearrow$ $\searrow$

lok. max

7) $f''(x) = \dfrac{-\frac{1}{x} \cdot x^2 - (1 - \ln x) 2x}{x^4} = \cancel{\dfrac{-x - x^2 + x^2 \ln x}{x^4}} = \cancel{\dfrac{-(1+x) + x \ln x}{x^3}}$

$= \dfrac{-x - 2x + 2x \ln x}{x^4} = \dfrac{-3 + 2 \ln x}{x^3}$

$D(f) = (0, \infty)$

8) $f''(x) = 0$

$2 \ln x = 3$

$\ln x = 3/2$

$x = e^{3/2}$

infl. bod

$f''(x)$: 0 | $-$ | $e^{3/2}$ | $+$

$f(x)$: $\cap$ $\cup$

$f'(e^{3/2}) = \dfrac{-1/2}{e^{6/2}} = -\dfrac{1}{\ldots}$

$f(e^{3/2}) = \dfrac{3/2}{e^{3/2}}$

9) A.b.s. „nejsou"

10) $\lim\limits_{x\to\infty} \dfrac{\frac{\ln x}{x}}{x} = \lim\limits_{x\to\infty} \dfrac{\ln x}{x^2} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{2x} = 0$

$\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{1} = 0$

$y=0$