

31) Diferenciální rovnice se separovanými proměnnými

$$y' = f(x)g(y)$$

$$\frac{dy}{dx} = f(x)g(y) \quad g(y) \neq 0$$

$$\int \frac{dy}{g(y)} = \int f(x) dx$$

$$\Downarrow \\ G(y) = F(x) + c, \quad c \in \mathbb{R}$$

a) $2y - x^3 y' = 0$

$$2y = x^3 y'$$

$$2 \frac{1}{x^3} = \frac{1}{y} y', \quad y \neq 0$$

$$2 \frac{dx}{x^3} = \frac{1}{y} dy \quad | \int$$

$$2 \cdot \frac{x^{-2}}{-2} + c = \ln |y|, \quad c \in \mathbb{R}$$

$$-\frac{1}{x^2} + c = \ln |y|$$

$$e^{-\frac{1}{x^2} + c} = |y|, \quad c \in \mathbb{R}$$

$$K \cdot e^{-\frac{1}{x^2}} = |y|, \quad K \in \mathbb{R}^+$$

$$\underline{\underline{C \cdot e^{-\frac{1}{x^2}} = y, \quad C \in \mathbb{R}}}$$

pro $c=0$ obdržíme

proto řešení je pouze
 $y = C \cdot e^{-\frac{1}{x^2}}, \quad C \in \mathbb{R}$

$$b) (x+1)dy + xydx = 0$$

$$(x+1)dy = -xydx \quad | y \neq 0$$

$$\frac{1}{y} dy = -\frac{x}{x+1} dx \quad | \int$$

$$\ln|y| = -\int \frac{x}{x+1} dx = -\int \left(1 - \frac{1}{x+1}\right) dx = -(x - \ln|x+1|) + C$$

$$|y| = e^{-x + \ln|x+1|} \cdot K, \quad K \in \mathbb{R}^+$$

$$\underline{y = (x+1)e^{-x}, \quad C \in \mathbb{R}}$$

$$c) y - y^2 + xy' = 0$$

$$xy' = y^2 - y, \quad y \neq 0, y \neq 1$$

$$\frac{dy}{y^2 - y} = \frac{dx}{x}$$

$$\frac{dy}{y(y-1)} = \frac{dx}{x} \quad | \int$$

$$-\ln|y| + \ln|y-1| = \int \frac{dx}{x} = \ln|x| + C$$

$$e^{-\ln|y| + \ln|y-1|} = e^{\ln|x|} \cdot K, \quad K \in \mathbb{R}^+$$

$$e^{\ln\left|\frac{y-1}{y}\right|} = e^{\ln|x|} \cdot K$$

$$\frac{y-1}{y} = Cx, \quad C \in \mathbb{R}$$

$$\underline{y = (1 - Cx)^{-1}, \quad C \in \mathbb{R}, \quad y \neq 0}$$

$$d) \quad y' \cos^3 x = (1 + \cos^3 x) \cdot \sqrt{1-y^2}$$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{1 + \cos^3 x}{\cos^3 x} dx \quad | y \neq \pm 1$$

$$\frac{dy}{\sqrt{1-y^2}} = \left(\frac{1}{\cos^3 x} + 1 \right) dx \quad | \int$$

$$\arcsin y = \int \frac{1}{\cos^3 x} + x + C, \quad C \in \mathbb{R}$$

$$\underline{y = \sin(\int \frac{1}{\cos^3 x} + x + C)}, \quad C \in \mathbb{R}, \quad \underline{y = \pm 1}$$

$$e) \quad e^{-s} \left(1 + \frac{ds}{dt} \right) = 1$$

$$1 + \frac{ds}{dt} = e^s$$

$$\frac{ds}{dt} = e^s - 1$$

$$\frac{ds}{e^s - 1} = dt, \quad e^s \neq 1$$

$$\ln \left| \frac{e^s - 1}{e^s} \right| = t + C$$

$$\left| \frac{e^s - 1}{e^s} \right| = e^t \cdot k, \quad k \in \mathbb{R}^+$$

$$\frac{e^s - 1}{e^s} = C e^t, \quad C \in \mathbb{R}$$

$$1 - \frac{1}{e^s} = C e^t$$

$$\underline{\underline{1 - C e^t = e^{-s}}}, \quad C \in \mathbb{R}$$

$$\int \frac{ds}{e^s - 1} \quad \left| \begin{array}{l} u = e^s - 1 \\ du = e^s ds \\ ds = \frac{du}{u+1} \end{array} \right| = \int \frac{1}{u} \cdot \frac{du}{u+1} =$$

$$= \int \frac{1}{u} - \frac{1}{u+1} du = \ln|u| - \ln|u+1| =$$

$$= \ln \left| \frac{u}{u+1} \right| = \ln \left| \frac{e^s - 1}{e^s} \right|$$

$$f) y' + \operatorname{tg} x - y^2 = 1 - 2y$$

$$y' + \operatorname{tg} x = 1 - 2y + y^2$$

$$\frac{y'}{1 - 2y + y^2} = \frac{1}{\operatorname{tg} x} \quad | \quad (1 - 2y + y^2) = (1 - y)^2 \neq 0 \Rightarrow y \neq 1$$

$$\frac{dy}{(1-y)^2} = \frac{dx}{\operatorname{tg} x}$$

$$\int \frac{1}{(1-y)^2} dy \quad \left| \begin{array}{l} u = 1-y \\ du = -dy \end{array} \right| = - \int \frac{1}{u^2} du =$$

$$= + \frac{1}{u} = \frac{1}{1-y}$$

$$\frac{1}{1-y} = \ln |\sin x| + C$$

$$1 = (1-y) \left[\ln |\sin x| + C \right], C \in \mathbb{R}, \quad \underline{y=1}$$

$$g) y' = 6x + 2y + 3 \quad \Rightarrow \text{substitute} \quad z = 6x + 2y + 3$$

$$\frac{dz}{dx} = z$$

$$\frac{dz}{dx} \cdot \frac{dz}{dy} = z \cdot \frac{dz}{dy}$$

$$\frac{dz}{dx} = z \cdot 2$$

$$\frac{dz}{z} = 2 dx, \quad z \neq 0$$

$$\ln |z| = 2x + C$$

$$z = C \cdot e^{2x}, \quad C \in \mathbb{R}$$

$$6x + 2y + 3 = C \cdot e^{2x}$$

$$\underline{y = C \cdot e^{2x} - 3(x+1)}, \quad C \in \mathbb{R}$$

$$h) 2(1+e^x)yy' = e^x, y(0)=0$$

$$2y dy = \frac{e^x}{1+e^x} dx$$

$$y^2 = \ln|e^x+1| + C$$

$$\underline{e^{y^2} = k \cdot (e^x+1)}, k \in \mathbb{R}^+$$

$$y(0)=0:$$

$$1 = k \cdot (1+1)$$

$$1 = 2k$$

$$\underline{k = \frac{1}{2}}$$

$$\underline{\underline{e^{y^2} = \frac{1}{2}(e^x+1)}}$$

$$i) \sin y \cos x dy = \cos y \sin x dx, y(0) = \frac{\pi}{4}$$

$$\int \sin y dy = \int \cos x dx$$

$$-\ln|\cos y| = -\ln|\cos x| + C$$

$$|\cos y| = k \cdot \cos x, k \in \mathbb{R}^+$$

$$\cos y = C \cdot \cos x, C \in \mathbb{R}$$

$$y(0): \cos \frac{\pi}{4} = C \cdot \cos 0$$

$$\frac{\sqrt{2}}{2} = C$$

$$\cos y = \frac{\sqrt{2}}{2} \cos x \Rightarrow \frac{2}{\sqrt{2}} \cos y = \cos x \Rightarrow \underline{\underline{\sqrt{2} \cos y = \cos x}}$$

$$j) (x^2+1)(y^2-1) + xy y' = 0, \quad y(1) = \sqrt{2}$$

$$(x^2+1)(y^2-1) = -xy \frac{dy}{dx}$$

$$-\frac{x^2+1}{x} dx = \frac{y}{y^2-1} dy$$

$$-(x + \frac{1}{x}) dx = \frac{y}{(y+1)(y-1)} dy$$

$$-\frac{x^2}{2} - \ln|x| + C = \frac{1}{2} \ln|y^2-1|$$

$$-x^2 - 2\ln|x| + L = \ln|y^2-1|, \quad L \in \mathbb{R}$$

$$e^{-x^2-2\ln|x|} \cdot K = |y^2-1|, \quad K \in \mathbb{R}^+$$

$$e^{-x^2} \cdot x^{-2} \cdot C = |y^2-1|, \quad C \in \mathbb{R}$$

$$y = \sqrt{e^{-x^2} \cdot x^{-2} \cdot C + 1}$$

$$y(1): \sqrt{2} = \sqrt{e^{-1} \cdot 1 \cdot C + 1} \Rightarrow C = e^{2}$$

$$y(x) = \sqrt{e^{1-x^2} \cdot x^{-2} + 1}$$

$$\int \frac{y}{(y+1)(y-1)} = \int \frac{1/2}{y+1} + \frac{1/2}{y-1} dy$$

$$= \frac{1}{2} \ln|y+1| + \frac{1}{2} \ln|y-1| =$$

$$= \frac{1}{2} \ln|y^2-1|$$

$$2) y' = \frac{x-y+1}{x-y}$$

$$z = x - y$$

$$\begin{aligned} y &= x - z \\ y' &= 1 - z' \end{aligned}$$

$$1 - z' = \frac{z+1}{z}$$

$$1 - z' = 1 + \frac{1}{z}$$

$$z' = -\frac{1}{z}$$

$$z \, dz = -1 \, dx$$

$$\frac{z^2}{2} = -x + C$$

$$z^2 + 2x + C = 0$$

$$\underline{\underline{(x-y)^2 + 2x + C = 0}}$$

32) Homogenní diferenciální rovnice

funkce homogenní k-tého stupně

$$f(tx, ty) = t^k f(x, y)$$

$$f(x, y) = x^2 + y^2 - xy \leftarrow \text{homogenní 2. stupně}$$

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \leftarrow \text{homogenní 0. tého stupně}$$

homogenní DR lze psát ve tvaru

$$y' = g\left(\frac{y}{x}\right)$$

řešme substitucí

$$u = \frac{y}{x}$$

$$ux = y$$

$$y' = u + u'x$$

potom dostaneme

$$u + u'x = g(u)$$

máme-li DR

$$P(x,y)dx + Q(x,y)dy = 0$$

ta je homogenní $\Leftrightarrow P, Q$ jsou homogenní funkce téhož stupně

a) $xy' = \sqrt{x^2 - y^2} + y$

$$y' = \frac{\sqrt{x^2 - y^2}}{x} + \frac{y}{x}$$

$$y' = \sqrt{\frac{x^2 - y^2}{x^2}} + \frac{y}{x}$$

$$y' = \sqrt{1 - \left(\frac{y}{x}\right)^2} + \frac{y}{x}, \quad u = \frac{y}{x}, \quad y' = u + u'x$$

$$u + u'x = \sqrt{1 - u^2} + u$$

$$\frac{du}{\sqrt{1 - u^2}} = \frac{dx}{x}, \quad 1 - u^2 \neq 0 \Rightarrow u^2 \neq 1 \Rightarrow u \neq \pm 1$$

$$\arcsin u = \ln|x| + C = \ln|kx|$$

$$\text{máme } |\ln|kx|| \leq \frac{\pi}{2}$$

$$\arcsin \frac{y}{x} = \ln|kx|$$

$$\underline{y = x \cdot \sin \ln|kx|, \quad k \in \mathbb{R}}$$

$$\underline{y = \pm x}$$

$$b) (y^2 - x^2) dx - 2xy dy = 0$$

$$xy \Rightarrow (tx) \cdot (ty) = t^2 xy \leftarrow \text{1. stupen'}$$

$$y^2 - x^2 \Rightarrow (ty)^2 - (tx)^2 = t^2 (y^2 - x^2) \leftarrow \text{1. stupen'}$$

$$(y^2 - x^2) dx = 2xy dy$$

$$\frac{y^2 - x^2}{2xy} = \frac{dy}{dx}$$

$$\frac{1}{2} \cdot \left(\frac{y}{x} - \frac{x}{y} \right) = \frac{dy}{dx}, \quad u = \frac{y}{x}, \quad xy \neq 0$$

$$\frac{1}{2} \left(u - \frac{1}{u} \right) = u + u'x$$

$$\frac{1}{2}u - \frac{1}{2} \frac{1}{u} = u + u'x$$

$$-\frac{1}{2}u - \frac{1}{2} \frac{1}{u} = u'x$$

$$\frac{1}{x} dx = \frac{du}{-\frac{1}{2}u - \frac{1}{2u}}$$

$$-\frac{1}{x} dx = \frac{u}{\frac{1}{2}u^2 + \frac{1}{2}} du$$

$$\cancel{\frac{1}{x}} -\frac{1}{x} dx = \frac{2u}{u^2 + 1} du$$

$$-\ln|x| + C = \ln|u^2 + 1|$$

$$-\ln|kx| = \ln|u^2 + 1|$$

$$\left| \frac{1}{kx} \right| = u^2 + 1$$

$$\pm C \cdot \frac{1}{x} = \frac{y^2}{x^2} + 1$$

$$\pm C^* = y^2 + x^2, \quad C \in \mathbb{R}$$

$$c) \quad xy' + y \ln x = y \ln y$$

$$xy' = y(\ln y - \ln x)$$

$$y' = \frac{y}{x} \cdot \ln \frac{y}{x}, \quad \frac{y}{x} = u, \quad y' = u + u'x$$

$$u + u'x = u \cdot \ln u$$

$$u'x = u(-1 + \ln u)$$

$$\frac{du}{u(\ln u - 1)} = \frac{dx}{x}, \quad u \neq 0, \quad 1 \neq \ln u$$

$$\ln |\ln u - 1| = \ln |x| + C$$

$$|\ln u - 1| = |kx|$$

$$\ln u - 1 = Cx$$

$$\ln u = Cx + 1$$

$$u = e^{Cx+1}$$

$$\underline{y = x \cdot e^{Cx+1}}, \quad C \in \mathbb{R}$$

$$\int \frac{du}{u(-1 + \ln u)} \quad \left| \begin{array}{l} t = -1 + \ln u \\ dt = +\frac{1}{u} du \end{array} \right|$$

$$= \int \frac{+dt}{t} = +\ln |t|$$

$$d) \quad (xy' - y) \cos \frac{y}{x} = x$$

$$\left(\frac{y'}{x} - \frac{y}{x^2} \right) \cos \frac{y}{x} = 1, \quad \frac{y}{x} = u, \quad y' = u + u'x$$

$$(u + u'x - u) \cos u = 1$$

$$\cos u \, du = \frac{1}{x} \, dx$$

$$\sin u = \ln |x| + C$$

$$\sin \frac{y}{x} = \ln |x| + C$$

$$\underline{X = C \cdot e^{\sin \frac{y}{x}}}, \quad C \in \mathbb{R}$$

$$e) xy' = y \cos \ln \frac{y}{x}, \quad \frac{y}{x} > 0$$

$$y' = \frac{y}{x} \cos \ln \frac{y}{x}, \quad \frac{y}{x} = u, \quad y' = u + u'x$$

$$u + u'x = u \cdot \cos \ln u \quad \text{uvio}$$

$$u'x = u(-1 + \cos \ln u)$$

$$\frac{u'}{u(-1 + \cos \ln u)} = \frac{1}{x} dx$$

$$\int \frac{du}{u(-1 + \cos \ln u)} \quad \left| \begin{array}{l} t = \ln u \\ dt = \frac{1}{u} du \end{array} \right| =$$

$$= \int \frac{dt}{-1 + \cos t} = \left| \begin{array}{l} \operatorname{tg} \frac{t}{2} = v \\ dt = \frac{2}{1+v^2} dv \\ \cos t = \frac{1-v^2}{1+v^2} \end{array} \right|$$

$$= \int \frac{\frac{2}{1+v^2} dv}{-1 + \frac{1-v^2}{1+v^2}} = \int \frac{2 dv}{-(1+v^2) + (1+v^2)} = \int \frac{2 dv}{-2v^2}$$

$$= \frac{1}{v} = \frac{1}{\operatorname{tg} \frac{t}{2}}$$

$$\frac{1}{\operatorname{tg}(\ln u)} = \ln |x| + C$$

$$\operatorname{cotg}(\ln u) = \ln |x| + C$$

$$\operatorname{cotg} \left(\ln \frac{y}{x} \right) = \ln Cx$$

33) Lineární diferenciální rovnice 1. řádu

$$y' = f(x)y + g(x), \quad y(x_0) = y_0$$

$$y(x) = \exp \left(\int_{x_0}^x f(t) dt \right) \left[y_0 + \int_{x_0}^x g(t) \exp \left(- \int_{x_0}^t f(s) ds \right) dt \right]$$

homogenní:

$$y' = f(x)y + g(x) \quad / \cdot e^{-\int f(x) dx}$$

$$y' e^{-\int f(x) dx} - y \cdot e^{-\int f(x) dx} = g(x) e^{-\int f(x) dx}$$

$$(y e^{-\int f(x) dx})' = g(x) e^{-\int f(x) dx}$$

$$a) y' = 6x - 2y$$

I) homogenisiert

$$y' = -2y$$

$$\frac{dy}{y} = -2 dx$$

$$\ln|y| = -2x + C$$

$$y = C e^{-2x}$$

Variation der Konstanten

$$y(x) = C(x) \cdot e^{-2x}$$

$$y'(x) = C'(x) e^{-2x} - 2 \cdot C(x) \cdot e^{-2x} = 6x - 2 \cdot C(x) e^{-2x}$$

$$C' e^{-2x} = 6x$$

$$C' = 6x e^{+2x}$$

$$C = \int 6x e^{+2x} dx \quad \left| \begin{array}{l} u=6x \\ v=e^{+2x} \end{array} \right.$$

$$\left. \begin{array}{l} u=6 \\ v=\frac{1}{2} e^{+2x} \end{array} \right| = +3x e^{+2x} - \int 3 e^{+2x} dx =$$

$$= +3x e^{-2x} - \frac{3}{2} e^{-2x} + C$$

$$\underline{y(x) = 3x - \frac{3}{2} + C e^{-2x}}$$

$$II) y' = 6x - 2y$$

$$y' + 2y = 6x \quad | \cdot e^{\int 2 dx} = e^{2x}$$

$$y' e^{2x} + 2y e^{2x} = 6x e^{2x}$$

$$(y e^{2x})' = 6x e^{2x}$$

$$y e^{2x} = \int 6x e^{2x} dx \quad \left| \begin{array}{l} u=6x \\ v=e^{2x} \end{array} \right.$$

$$\left. \begin{array}{l} u=6 \\ v=\frac{1}{2} e^{2x} \end{array} \right| = 3e^{2x} - \int 3e^{2x} = 3e^{2x} - \frac{3}{2} e^{2x} + C$$

$$\underline{y(x) = 3 - \frac{3}{2} + C e^{-2x}}$$

$$b) y' = 4xy + (2x+1)e^{2x^2}$$

$$y' - 4xy = (2x+1)e^{2x^2} \quad / \cdot e^{-\int 4x dx} = e^{-2x^2}$$

$$y'e^{-2x^2} - 4xe^{-2x^2}y = 2x+1$$

$$(ye^{-2x^2})' = 2x+1$$

$$ye^{-2x^2} = x^2 + x + C$$

$$\underline{\underline{y(x) = (x^2 + x + C)e^{2x^2}}}$$

$$c) y' \cos x = (y + 2 \cos x) \sin x$$

$$y' = y \tan x$$

$$\frac{dy}{y} = \tan x dx$$

$$\ln|y| = -\ln|\cos x| + C$$

$$y = C \cdot \frac{1}{\cos x}$$

$$y' = C' \frac{1}{\cos x} + C \cdot \frac{\sin x}{\cos^2 x} = C' \frac{1}{\cos x} + \frac{\sin x}{\cos x} + 2 \cdot \sin x$$

$$C' = 2 \cdot \sin x \cdot \cos x$$

$$C = 2 \int \sin x \cos x dx \quad \left| \begin{array}{l} u = \sin x \\ du = \cos x dx \end{array} \right. = 2 \int u du = u^2 = \sin^2 x + C$$

$$\underline{\underline{y(x) = \frac{\sin^2 x + C}{\cos x}}}$$

$$d) \quad y' - 4y = \cos x, \quad y(0) = 1$$

$$y' - 4y = \cos x \quad | \cdot e^{-\int 4 dx}$$

$$y e^{-4x} - 4 \cdot e^{-4x} \cdot y = e^{-4x} \cdot \cos x$$

$$(y e^{-4x})' = e^{-4x} \cdot \cos x$$

$$y = -\frac{4}{17} \cos x + \frac{1}{17} \sin x + C e^{-4x}$$

$$y(0): \quad 1 = -\frac{4}{17} + C$$

$$C = \frac{21}{17}$$

$$\underline{y(x) = \frac{1}{17} \sin x - \frac{4}{17} \cos x + \frac{21}{17} e^{-4x}}$$

$$\int e^{-4x} \cdot \cos x \, dx \quad \left| \begin{array}{l} u = \cos x \quad u' = -\sin x \\ v = e^{-4x} \quad v' = -\frac{1}{4} e^{-4x} \end{array} \right. =$$

$$= -\frac{1}{4} \cos x \cdot e^{-4x} + \frac{1}{4} \int e^{-4x} \sin x \, dx \quad \left| \begin{array}{l} u = \sin x \quad u' = \cos x \\ v = e^{-4x} \quad v' = -\frac{1}{4} e^{-4x} \end{array} \right. =$$

$$= -\frac{1}{4} \cos x \cdot e^{-4x} + \frac{1}{16} \sin x \cdot e^{-4x} - \frac{1}{16} \int \cos x \cdot e^{-4x} \, dx$$

⇓

$$\int e^{-4x} \cdot \cos x \, dx = \frac{16}{17} \cdot \left(-\frac{1}{4}\right) \cos x \cdot e^{-4x} + \frac{16}{17} \cdot \frac{1}{16} \sin x \cdot e^{-4x} + C$$

g) Bernoulliova rovnice

$$y' + f(x)y = y^n g(x), \quad n \neq 0, n \neq 1, n \in \mathbb{R}$$

$$\frac{y'}{y^n} + \frac{f(x)}{y^{n-1}} = g(x), \quad y \neq 0$$

↑ pro $n > 0$ je $y = 0$ také řešením

$$z = \frac{1}{y^{n-1}} = y^{1-n}$$

$$z' = (1-n) y^{-n} \cdot y'$$

$$z' = (1-n) [g(x) - f(x) \cdot z]$$

↳ lineární dif. rovnice 1. řádu

$$a) y' - xy = -y^3 e^{-x^2}$$

$$\frac{y'}{y^3} - \frac{x}{y^2} = -e^{-x^2}, \quad y \neq 0$$

$$z = \frac{1}{y^2}$$

$$z' = -2 \cdot \frac{1}{y^3} \cdot y'$$

$$z' = -2(xz - e^{-x^2})$$

$$z' + 2xz = 2e^{-x^2} \quad | \cdot e^{\int 2x dx} = e^{x^2}$$

$$z' e^{x^2} + 2xz e^{x^2} = 2$$

$$(z e^{x^2})' = 2$$

$$z e^{x^2} = 2x + C$$

$$\frac{1}{y^2} = z = 2x e^{-x^2} + C e^{-x^2}$$

$$\frac{1}{y^2} = e^{-x^2} (2x + C)$$

$$y^2 = \frac{e^{x^2}}{2x + C}$$

$$y = \pm \sqrt{\frac{e^{x^2}}{2x + C}}, \quad y \neq 0$$

$$b) 3x^2 y' + xy = y^{-2}$$

$$y^2 y' + \frac{y^3}{3x} = \frac{1}{3x^2}$$

$$z = y^3$$

$$z' = 3y^2 y'$$

$$z' = 3y^2 \left(\frac{1}{3x^2} - \frac{z}{3x} \right)$$

$$z' = \frac{1}{x^2} - \frac{z}{x}$$

$$z' + \frac{z}{x} = \frac{1}{x^2} \quad / \cdot e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x$$

$$(xz' + z) = \frac{1}{x}$$

$$(xz)' = \frac{1}{x}$$

$$xz = \ln|x| + C$$

$$z = \frac{\ln|x| + C}{x}$$

$$\underline{\underline{y^3 = \frac{\ln|x| + C}{x}}}$$

$$e) \quad y' = \frac{4}{x}y + x\sqrt{y}$$

$$\frac{1}{\sqrt{y}} y' = \frac{4}{x}\sqrt{y} + x$$

$$z = \sqrt{y}$$

$$z' = \frac{1}{2\sqrt{y}} \cdot y' = \frac{1}{2} \left(\frac{4}{x}z + x \right)$$

$$z' = \frac{2z}{x} + \frac{1}{2}x$$

$$z' - \frac{2z}{x} = \frac{1}{2}x \quad / \cdot e^{-\int \frac{2}{x} dx} = e^{-2\ln|x|} = e^{-\ln x^2} = \frac{1}{x^2}$$

$$\left(z \cdot \frac{1}{x^2} - \frac{2z}{x^3} \right) = \frac{1}{2x}$$

$$\left(z \cdot \frac{1}{x^2}\right)' = \frac{1}{2x}$$

$$\frac{1}{x^2} \cdot z = \frac{1}{2} \ln|x| + C$$

$$\frac{1}{x^2} \cdot \sqrt{y} = \ln\sqrt{|x|} + C$$

$$\sqrt{y} = x^2 (\ln\sqrt{|x|} + C)$$

$$\underline{y = x^4 (\ln\sqrt{|x|} + C)^2} \quad y=0$$

d) $y dy = \left(\frac{ay^2}{x^2} + \frac{b}{x^2}\right) dx, a \neq 0$

$$y y' = \frac{ay^2}{x^2} + \frac{b}{x^2}$$

$$y y' - \frac{ay^2}{x^2} = \frac{b}{x^2}$$

$$z = y^2$$

$$z' = 2y y' = 2 \left(\frac{az}{x^2} + \frac{b}{x^2}\right)$$

$$z' - \frac{2az}{x^2} = \frac{b}{x^2} \quad | \cdot e^{-\int \frac{2a}{x^2} dx + 2a \cdot \frac{1}{x}} = e$$

$$z' e^{\frac{2a}{x}} - \frac{2az}{x^2} e^{\frac{2a}{x}} = \frac{b}{x^2} e^{\frac{2a}{x}}$$

$$\left(z e^{\frac{2a}{x}}\right)' = \frac{b}{x^2} e^{\frac{2a}{x}}$$

$$z e^{\frac{2a}{x}} = -\frac{b}{2a} e^{\frac{2a}{x}} + C$$

$$z = -\frac{b}{2a} + C e^{-\frac{2a}{x}}$$

$$\underline{y^2 + \frac{b}{2a} = C e^{-\frac{2a}{x}}}$$

$$\int \frac{1}{x^2} e^{\frac{2a}{x}} dx \quad \left| u = \frac{1}{x} \right. \\ \left. du = -\frac{1}{x^2} dx \right| =$$

$$= -\int e^{2au} du = -\frac{1}{2a} e^{2au} =$$

$$= -\frac{1}{2a} e^{\frac{2a}{x}}$$