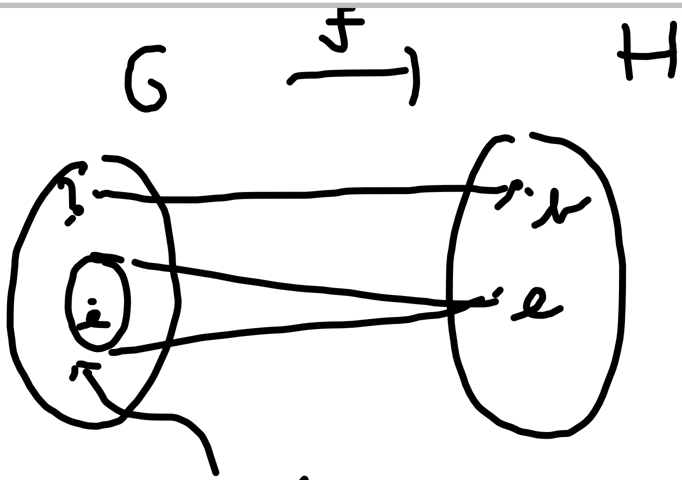
(6)



$f^{-1}(e) = \mathrm{Ker}\, f$

$x, y \in G$

$f(x) = f(y)$

$\Updownarrow$

$f(x) \cdot \big(f(y)\big)^{-1} = e$

$\Updownarrow$

$f(x \cdot y^{-1}) = e$

$\Updownarrow$

$x \cdot y^{-1} \in \mathrm{Ker}\, f$

$\mathrm{Ker}\, f = \{e\} \Rightarrow x \cdot y^{-1} = e \Rightarrow x = y$

$f: G \to H$

$\operatorname{Ker} f \subset G$ je podgrupa

$\operatorname{Im} f \subset H$ je podgrupa

$\parallel$

$f(G)$

př.

(1) $\mathrm{sgn}: \Sigma_n \to \{-1, 1\}$

$(\{-1,1\}, \cdot) \cong (\mathbb{Z}_2, +)$

(2) POZOR!

$\mathrm{sgn}\ \square \not= \Sigma_4$

$\mathrm{sgn}\ \square \subset \Sigma_4$ podgrupa

(Cayley)

(3) $z \mapsto e^{z}$

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$$

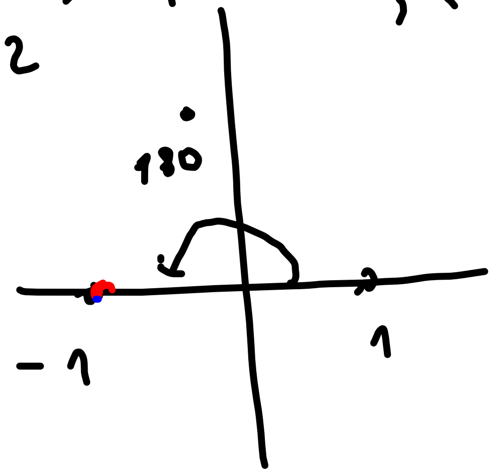
$$\mathrm{Ker}(\exp) = \{ 2k\pi i : k \in \mathbb{Z} \} \cong \mathbb{Z}$$

$$\left(e^{2k\pi i} = \cos 2k\pi + i \sin 2k\pi \right)$$
$$= 1$$

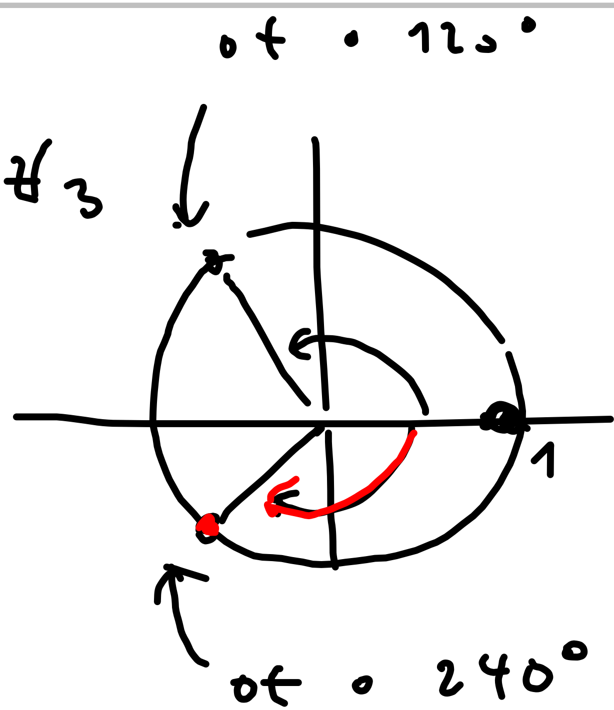
Pozn. $|G| = 4 \Rightarrow G \cong \mathbb{Z}_4$

nebo $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

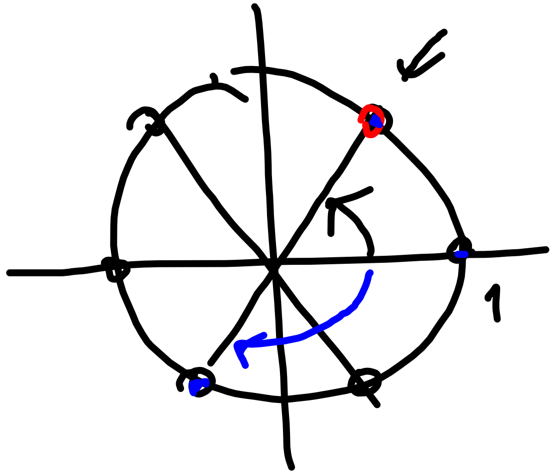
$\mathbb{Z}_2 \cong \{\overset{\text{id}}{1}, -1\}$, stř. sym.



$\mathbb{Z}_3$



$\mathbb{Z}_6$



$$\boxed{\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3}$$

$H \subset G$ podgr.

$\rightsquigarrow$ relace na G

$b \in aH \iff b \sim a$

$b = a \cdot h$ pro nějaké H

$\overset{?}{\iff} a^{-1} \cdot b = h \in H$

$\iff b^{-1} \cdot a = h^{-1} \in H$

(1) $aH = Ha \iff aHa^{-1} \subseteq H$

$a \cdot h_1 = h_2 \cdot a \iff a h_1 a^{-1} = h_2$

...

G komut $\rightsquigarrow$ $a h_1 a^{-1} =$
$a a^{-1} h_1 = h_1$

(2) $H = eH = He$

jako množiny

$\forall a \in G: \quad H \cong aH$

$h \mapsto ah$ je bijekce

$H < G \rightsquigarrow G/H$

$p: G \to G/H$

$a \mapsto aH$ je surjektivní zobrazení

PŘÍKLAD: $G = (\mathbb{Z}, +)$ $H = m\mathbb{Z}$

G komut. $\Rightarrow G/H = H\backslash G$

a $\boxed{G/H \cong \mathbb{Z}_m}$

$k \in \mathbb{Z} \rightsquigarrow kH = \{k + m \cdot l : l \in \mathbb{Z}\}$

1. $G =$ sjednocení tříd rozkladu.
2. Lagrange ...
3. řád $a \in G$ = řád $\langle a \rangle = \{\underset{e}{a^0}, a, a^2, \dots\}$
4. $a \in G$: $a^{|G|} = e$

$k =$ řád $a \Rightarrow a^k = e$

$k \mid n \Rightarrow n = k \cdot l \dots$

$a^n = (a^k)^l = e^l = e$

5. n prvoč. $\Rightarrow$ jediní dělitelé jsou

1 a n → $\forall a$ je řád 1 / n

$e \leftarrow 1$

$\cdot/\cdot$ $\quad \{\underset{\overset{\|}{e}}{a^0}, a, a^2, \ldots, \underset{\overset{\|}{e}}{a^n}\} = \underset{\overset{\|}{G}}{\{a^0, a, \ldots, a^{n-1}\}}$

$H \subset G \rightsquigarrow$ relaci $\sim$

$\rightsquigarrow$ rozklad

$\rightsquigarrow$ G/H

Pokud H je normální

$\Rightarrow$ $\cdot$ na G indukuje

$\cdot$ na G/H

$\Rightarrow$ $(G/H, \cdot)$ je opět GRUPA

(faktorová)

$G = \mathbb{Z}$ $\quad H = m\mathbb{Z}$

$G/H = \mathbb{Z}_m$

$p : G \to G/H$

surj. zobr.

$\rightsquigarrow$ $\forall$ normální podgr. je jádrem homo.

Tedy jádro homom. je normální

$f: G \to H$

$\operatorname{Ker} f \subset G$ normální?

$\forall a \in G,\ \underline{h \in \operatorname{Ker} f} \Rightarrow a^{-1} h a \in \operatorname{Ker} f$?

$\Uparrow$

$$f(a^{-1} h a) = e$$

$$\|$$

$$f(a^{-1}) f(h) f(a)$$

$$\|$$

$$f(a^{-1}) \cdot e \cdot f(a)$$

$$\|$$

$$e$$

L:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & K \\ \underbrace{G/\mathrm{Ker}\, f} \arrow[ur, dotted, "\tilde{f}"'] & \end{tikzcd}$$

$$\tilde{f} \circ p = f$$

dobře def
a injekt.

$$G/\mathrm{Ker}\, f \cong \mathrm{Im}\, f$$

Př.

$$\mathbb{C}^* = (\mathbb{C} - \{0\}, \cdot)$$

$$f : \mathbb{C}^* \to \mathbb{C}^*$$

$$z \mapsto z^k$$

$$\operatorname{Ker} f = \{ z \in \mathbb{C} : z^k = 1 \}$$

$k2$

$\mathbb{Z}_k$

$$\operatorname{Im} f = \mathbb{C}^*$$

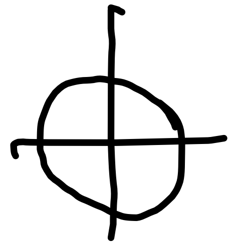
$$\mathbb{C}^* / \mathbb{Z}_k \cong \mathbb{C}^*$$

Př. $f : \mathbb{R}_{+} \to \mathbb{C} - \{0\}$

$$x \mapsto e^{ix}$$

$$\operatorname{Ker} f = \{x = 2k\pi \,;\ k \in \mathbb{Z}\} \cong \mathbb{Z}$$

$$\operatorname{Im} f = \{z \in \mathbb{C} : |z| = 1\} =: C_1$$



$$\mathbb{R}/\operatorname{Ker} f \cong \operatorname{Im} f$$

$$\boxed{\mathbb{R}/\mathbb{Z} \cong C_1}$$