Introduction to Modal and Temporal Logic

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History: Logic of Necessity and Possibility

Classical logic is truth-functional: truth value of larger formula determined by truth value(s) of its subformula(e) via truth tables for $\wedge$, $\vee$, $\neg$, and $\rightarrow$.

Lewis 1920s: How to capture a non-truth-functional notion of "A Necessarily Implies B"? $(A \prec B)$

Take $A \prec B$ to mean "it is impossible for A to be true and B to be false"

Write $\mathbf{P}A$ for "A is possible" then:

$\neg\mathbf{P}A$ is "A is impossible"

$\neg\mathbf{P}\neg A$ is "not-A is impossible"

$\mathbf{N}A := \neg\mathbf{P}\neg A$ "A is necessary"

$$A \prec B := \mathrm{N}(A \rightarrow B) = \neg\mathbf{P}\neg(A \rightarrow B) = \neg\mathbf{P}\neg(\neg A \vee B) = \neg\mathbf{P}(A \wedge \neg B)$$

Modal Logic: "possibly true" and "necessarily true" are modes of truth

Preliminaries

Directed Graph $\langle V, E\rangle$: where

$V = \{v_0, v_1, \cdots\}$ is a set of vertices

$E = \{(s_1, t_1), (s_2, t_2), \cdots\}$ is a set of edges from source vertex $s_i \in V$ to target vertex $t_i \in V$ for $i = 1, 2, \cdots$.

Cross Product: $V \times V$ stands for $\{(v, w) \mid v \in V, w \in V\}$ the set of all ordered pairs (v, w) where v and w are from V.

Directed Graph $\langle V, E\rangle$: where $V = \{v_0, v_1, \cdots\}$ is a set of vertices and $E \subseteq V \times V$ is a binary relation over V.

Iff: means if and only if.

Logic = Syntax and (Semantics or Calculus)

Syntax: formation rules for building formulae $\varphi, \psi, \cdots$ for our logical language

Assumptions: a (usually) finite collection Γ of formulae

Semantics: φ is a logical consequence of Γ $(\Gamma \models \varphi)$

Calculi: φ is derivable (purely syntactically) from Γ $(\Gamma \vdash \varphi)$

Soundness: If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

Completeness: If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$

Consistency: Both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ should not hold for any φ

Decidability: Is there an algorithm to tell whether or not $\Gamma \models \varphi$?

Complexity: Time/space required by algorithm for deciding whether $\Gamma \models \varphi$?

Syntax of Modal Logic

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$ $\quad (Atm)$

Formulae: $\varphi ::= p \mid \neg\varphi \mid \langle\rangle\varphi \mid []\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi$ $\quad (Fml)$

Examples: $[]p_0 \to p_2$ $\quad []p_3 \to [][]p_1$ $\quad [](p_1 \to p_2) \to (([]p_1) \to ([]p_2))$

Variables: p, q, r stand for atomic formulae while φ, ψ possibly with subscripts stand for arbitrary formulae (including atomic ones)

Schema/Shapes: $[]\varphi \to \varphi$ $\quad []\varphi \to [][]\varphi$ $\quad [](\varphi \to \psi) \to ([]\varphi \to []\psi)$

Schema Instances: Uniformly replace the formula variables with formulae

Examples: $[]p_0 \to p_0$ is an instance of $[]\varphi \to \varphi$ but $[]p_0 \to p_2$ is not

Formula Length: number of logical symbols, excluding parentheses, where length(p_0) = length(p_1) = $\cdots$ = 1

Example: length($[]p_0 \to p_2$) = 4

Kripke Semantics for Logical Consequence

Motivation: Give an intuitive meaning to syntactic symbols.

Motivation: Give the meaning of "φ is true"

Motivation: Define a meaning of "φ is a logical consequence of Γ" ($\Gamma \models \varphi$)

Goal: Prove some interesting properties of logical consequence.

Kripke Semantics for Logical Consequence

Kripke Frame: directed graph $\langle W, R\rangle$ where W is a non-empty set of points/worlds/vertices and $R \subseteq W \times W$ is a binary relation over W

Valuation: on a Kripke frame $\langle W, R\rangle$ is a map $\vartheta : W \times Atm \mapsto \{\mathbf{t}, \mathbf{f}\}$ telling us the truth value ($\mathbf{t}$ or else $\mathbf{f}$) of every atomic formula at every point in W

Kripke Model: $\langle W, R, \vartheta\rangle$ where ϑ is a valuation on a Kripke frame $\langle W, R\rangle$

Example: If $W = \{w_0, w_1, w_2\}$ and $R = \{(w_0, w_1), (w_0, w_2)\}$ and $\vartheta(w_1, p_3) = \mathbf{t}$ then $\langle W, R, \vartheta\rangle$ is a Kripke model as pictured below:

$$\begin{tikzcd} & w_1 \\ w_0 \arrow[ur, "R"] \arrow[dr, "R"'] & \\ & w_2 \end{tikzcd}$$

$$\begin{aligned}
\vartheta(w_0, p) &= \mathbf{f} \text{ for all } p \in Atm \\
\vartheta(w_1, p) &= \mathbf{f} \text{ for all } p \neq p_3 \in Atm \\
\vartheta(w_2, p) &= \mathbf{f} \text{ for all } p \in Atm \\
\vartheta(w_0, \langle\rangle p_1) &= ? \\
\vartheta(w_0, []p_1) &= ?
\end{aligned}$$

Kripke Semantics for Logical Consequence

Given some model $\langle W, R, \vartheta\rangle$ and some $w \in W$, we compute the truth value of a non-atomic formula by recursion on its shape:

$$\vartheta(w, \neg\varphi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(w,\varphi) = \mathrm{f} \\ \mathrm{f} & \text{otherwise} \end{cases}$$

$$\vartheta(w, \varphi\wedge\psi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(w,\varphi) = \mathrm{t} \text{ and } \vartheta(w,\psi) = \mathrm{t} \\ \mathrm{f} & \text{otherwise} \end{cases}$$

$$\vartheta(w, \varphi\vee\psi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(w,\varphi) = \mathrm{t} \text{ or } \vartheta(w,\psi) = \mathrm{t} \\ \mathrm{f} & \text{otherwise} \end{cases}$$

$$\vartheta(w, \varphi \to \psi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(w,\varphi) = \mathrm{f} \text{ or } \vartheta(w,\psi) = \mathrm{t} \\ \mathrm{f} & \text{otherwise} \end{cases} \qquad (\neg\varphi \vee \psi)$$

Intuition: classical connectives behave as usual at a world (truth functional)

Kripke Semantics for Logical Consequence

Given some model $\langle W, R, \vartheta\rangle$ and some $w \in W$, we compute the truth value of a non-atomic formula by recursion on its shape:

$$\vartheta(w, \langle\rangle\varphi) = \begin{cases} \mathrm{t} & \vartheta(v,\varphi) = \mathrm{t} \text{ for some } v \in W \text{ with } wRv \\ \mathrm{f} & \text{otherwise} \end{cases}$$

$$\vartheta(w, []\varphi) = \begin{cases} \mathrm{t} & \vartheta(v,\varphi) = \mathrm{t} \text{ for every } v \in W \text{ with } wRv \\ \mathrm{f} & \text{otherwise} \end{cases}$$

Example: If $W = \{w_0, w_1, w_2\}$ and $R = \{(w_0, w_1), (w_0, w_2)\}$ and $\vartheta(w_1, p_3) = \mathrm{t}$ then $\langle W, R, \vartheta\rangle$ is a Kripke model as pictured below:

$$\begin{tikzcd} & w_1 \\ w_0 \arrow[ur, "R"] \arrow[dr, "R"'] & \\ & w_2 \end{tikzcd}$$

$$\begin{aligned} \vartheta(w_0, \langle\rangle p_3) &= \mathrm{t} \\ \vartheta(w_0, []p_3) &= \mathrm{f} \\ \vartheta(w_1, []p_1) &= \mathrm{t} \\ \vartheta(w_1, []\neg p_1) &= \mathrm{t} \\ \vartheta(w_0, \langle\rangle[]p_1) &= \mathrm{t} \end{aligned}$$

Intuition: truth of modalities depends on underlying R (not truth functional)

Semantics: Examples

Let $\mathcal{M} = \langle W, R, \vartheta \rangle$ be any Kripke model, and $w \in W$.

Example: If $\vartheta(w, []\varphi) = \mathrm{t}$ then $\vartheta(w, \langle\rangle\neg\varphi) = \mathrm{f}$

Example: If $\vartheta(w, \langle\rangle\neg\varphi) = \mathrm{f}$ then $\vartheta(w, \neg\langle\rangle\neg\varphi) = \mathrm{t}$ $\qquad []\varphi \rightarrow \neg\langle\rangle\neg\varphi$

Example: If $\vartheta(w, \langle\rangle\varphi) = \mathrm{t}$ then $\vartheta(w, []\neg\varphi) = \mathrm{f}$

Example: If $\vartheta(w, []\neg\varphi) = \mathrm{f}$ then $\vartheta(w, \neg[]\neg\varphi) = \mathrm{t}$ $\qquad \langle\rangle\varphi \rightarrow \neg[]\neg\varphi$

Exercise: Show that all these implications are reversible.

Example: $\vartheta(w, []\varphi) = \mathrm{t}$ if and only if $\vartheta(w, \neg\langle\rangle\neg\varphi) = \mathrm{t}$

Example: $\vartheta(w, \langle\rangle\varphi) = \mathrm{t}$ if and only if $\vartheta(w, \neg[]\neg\varphi) = \mathrm{t}$

Classical (Two-Valued) Nature of Kripke Semantics

Lemma 1 *For any Kripke model $\langle W, R, \vartheta \rangle$, any $w \in W$ and any formula φ, either $\vartheta(w, \varphi) = \mathbf{t}$ or else $\vartheta(w, \varphi) = \mathbf{f}$.*

Proof: Pick any Kripke model $\langle W, R, \vartheta \rangle$, any $w \in W$, and any formula φ. Proceed by induction on the length l of φ.

Base Case $l = 1$: If φ is an atomic formula p, either $\vartheta(w, p) = \mathbf{t}$ or $\vartheta(w, p) = \mathbf{f}$ by definition of ϑ. So the lemma holds for all atomic formulae.

Ind. Hyp. : Lemma holds for all formulae of length less than some $n > 0$.

Induction Step: If φ is of length n, then consider the shape of φ.

$\varphi = \langle\rangle\psi$: If w has no R-successors, then $\vartheta(w, \langle\rangle\psi) = \mathbf{f}$, and $\vartheta(w, \langle\rangle\psi) = \mathbf{t}$ is impossible by its definition. Else pick any $v \in W$ with wRv. By IH, either $\vartheta(v, \psi) = \mathbf{t}$ or else $\vartheta(v, \psi) = \mathbf{f}$ since ψ is smaller than φ. Either all R-successors of w make ψ false, or else at least one of them makes ψ true. Hence, either $\vartheta(w, \langle\rangle\psi) = \mathbf{f}$ or else $\vartheta(w, \langle\rangle\psi) = \mathbf{t}$.

Semantic Forcing Relation $\Vdash$ and its negation $\not\Vdash$

Let $\mathcal{K}$ be the class of all Kripke models, and $\mathcal{M} = \langle W, R, \vartheta \rangle$ a Kripke model

Let $\mathfrak{K}$ be the class of all Kripke frames and let $\mathfrak{F}$ be a Kripke frame

Let Γ be a set of formulae, and φ be a formula

Forces	We say	We write	When	$\bullet \not\Vdash \varphi$
in a world	w forces φ	$w \Vdash \varphi$	$\vartheta(w, \varphi) = \mathbf{t}$	$\vartheta(w, \varphi) = \mathbf{f}$
in a model	$\mathcal{M}$ forces φ	$\mathcal{M} \Vdash \varphi$	$\forall w \in W . w \Vdash \varphi$	$\exists w \in W . w \not\Vdash \varphi$
in a frame	$\mathfrak{F}$ forces φ	$\mathfrak{F} \Vdash \varphi$	$\forall \vartheta . \langle \mathfrak{F}, \vartheta \rangle \Vdash \varphi$	$\exists \vartheta . \langle \mathfrak{F}, \vartheta \rangle \not\Vdash \varphi$

Classicality: either $\bullet \Vdash \varphi$ or else $\bullet \not\Vdash \varphi$ holds for $\bullet \in \{w, \mathcal{M}, \mathfrak{F}\}$

Exercise: Work out the negation of each fully e.g. $\mathcal{M} \not\Vdash \varphi$ is $\exists w \in W . w \Vdash \neg\varphi$

Either $w \Vdash \varphi$ or else $w \Vdash \neg\varphi$ holds (Lemma 1)

But this does not apply to all: e.g. either $\mathcal{M} \Vdash \varphi$ or else $\mathcal{M} \Vdash \neg\varphi$ is rarely true.

$W \Vdash \varphi$ meaning “every frame built out of given W forces φ” is not interesting

Various Consequence Relations

Let $\mathcal{K}$ be the class of all Kripke models, and $\mathcal{M} = \langle W, R, \vartheta \rangle$ a Kripke model

Let $\mathfrak{K}$ be the class of all Kripke frames and let $\mathfrak{F}$ be a Kripke frame

Let Γ be a set of formulae, and φ be a formula

Forces	We say	We write	When	$\bullet \not\Vdash \varphi$
in a world	w forces φ	$w \Vdash \varphi$	$\vartheta(w, \varphi) = \mathbf{t}$	$\vartheta(w, \varphi) = \mathbf{f}$
in a model	$\mathcal{M}$ forces φ	$\mathcal{M} \Vdash \varphi$	$\forall w \in W. w \Vdash \varphi$	$\exists w \in W. w \not\Vdash \varphi$
in a frame	$\mathfrak{F}$ forces φ	$\mathfrak{F} \Vdash \varphi$	$\forall \vartheta. \langle \mathfrak{F}, \vartheta \rangle \Vdash \varphi$	$\exists \vartheta. \langle \mathfrak{F}, \vartheta \rangle \not\Vdash \varphi$

Let $\bullet \Vdash \Gamma$ stand for $\forall \psi \in \Gamma. \bullet \Vdash \psi$ $\quad$ ($\bullet \in \{w, \mathcal{M}, \mathfrak{F}\}$)

World: every world that forces Γ also forces φ $\quad \forall w \in W. w \Vdash \Gamma \Rightarrow w \Vdash \varphi$

Model: every model that forces Γ also forces φ $\quad \forall \mathcal{M} \in \mathcal{K}. \mathcal{M} \Vdash \Gamma \Rightarrow \mathcal{M} \Vdash \varphi$

Frame: every frame that forces Γ also forces φ $\quad \forall \mathfrak{F} \in \mathfrak{K}. \mathfrak{F} \Vdash \Gamma \Rightarrow \mathfrak{F} \Vdash \varphi$

Various Consequence Relations

Let $\mathcal{K}$ be the class of all Kripke models, and $\mathcal{M} = \langle W, R, \vartheta \rangle$ a Kripke model

Let $\mathfrak{K}$ be the class of all Kripke frames and let $\mathfrak{F}$ be a Kripke frame.

Let Γ be a set of formulae, and φ be a formula

Forces	We say	We write	When	$\bullet \nVdash \varphi$
in a world	w forces φ	$w \Vdash \varphi$	$\vartheta(w, \varphi) = \mathbf{t}$	$\vartheta(w, \varphi) = \mathbf{f}$
in a model	$\mathcal{M}$ forces φ	$\mathcal{M} \Vdash \varphi$	$\forall w \in W . w \Vdash \varphi$	$\exists w \in W . w \nVdash \varphi$
in a frame	$\mathfrak{F}$ forces φ	$\mathfrak{F} \Vdash \varphi$	$\forall \vartheta . \langle \mathfrak{F}, \vartheta \rangle \Vdash \varphi$	$\exists \vartheta . \langle \mathfrak{F}, \vartheta \rangle \nVdash \varphi$

Let $\bullet \Vdash \Gamma$ stand for $\forall \psi \in \Gamma . \bullet \Vdash \psi$ $\quad$ $(\bullet \in \{w, \mathcal{M}, \mathfrak{F}\})$

World: $\forall w \in W . w \Vdash \Gamma \Rightarrow w \Vdash \varphi$ iff $\forall w \in W . w \Vdash \bigwedge \Gamma \to \varphi$ iff $\mathcal{M} \Vdash \bigwedge \Gamma \to \varphi$

Model: $\forall \mathcal{M} \in \mathcal{K} . \mathcal{M} \Vdash \Gamma \Rightarrow \mathcal{M} \Vdash \varphi$ $\quad$ is the one we study

Frame: $\forall \mathfrak{F} \in \mathfrak{K} . \mathfrak{F} \Vdash \Gamma \Rightarrow \mathfrak{F} \Vdash \varphi$ $\quad$ usually undecidable

Logical Consequence, Validity and Satisfiability

Logical Consequence: $\Gamma \models \varphi$ iff $\forall \mathcal{M} \in \mathcal{K}. \mathcal{M} \Vdash \Gamma \Rightarrow \mathcal{M} \Vdash \varphi$

Validity: φ is $\mathcal{K}$-valid iff $\emptyset \models \varphi$

Satisfiability: φ is $\mathcal{K}$-satisfiable iff $\exists \mathcal{M} = \langle W, R, \vartheta \rangle \in \mathcal{K}, \exists w \in W, w \Vdash \varphi$

Example: $\{p_0\} \models []p_0$. If every world in a model makes p_0 true, then every world in that model must make $[]p_0$ true.

For a contradiction, assume $\{p_0\} \not\models []p_0$.

i.e. exists $\mathcal{M} = \langle W, R, \vartheta \rangle \in \mathcal{K}. \mathcal{M} \Vdash p_0$ and $\mathcal{M} \not\Vdash []p_0$.

i.e. exists $w_0 \in W$ and $w_0 \not\Vdash []p_0$

i.e. exists $w_0 \in W$ and $w_1 \in W$ with $w_0 R w_1$ and $w_1 \not\Vdash p_0$

i.e. But $\mathcal{M} \Vdash p_0$ means $\forall w \in W. w \Vdash p_0$, hence $w_1 \Vdash p_0$ (contradiction)

Logical Consequence: Examples

Example 1 *All instances of* $\varphi \rightarrow (\psi \rightarrow \varphi)$ *are* $\mathcal{K}$*-valid.*

For a contradiction, assume some instance $\varphi_1 \rightarrow (\psi_1 \rightarrow \varphi_1)$ not $\mathcal{K}$-valid.

i.e. exists model $\mathcal{M} = \langle W, R, \vartheta \rangle$ and $w \in W$ with $w \nVdash \varphi_1 \rightarrow (\psi_1 \rightarrow \varphi_1)$.

i.e. $w \Vdash \varphi_1$ and $w \nVdash \psi_1 \rightarrow \varphi_1$.

i.e. $w \Vdash \varphi_1$ and $w \Vdash \psi_1$ and $w \nVdash \varphi_1$. (contradiction)

Exercise 1 *All instances of* $\neg\neg\varphi \rightarrow \varphi$ *are* $\mathcal{K}$*-valid.*

Exercise 2 *All instances of* $(\varphi \rightarrow (\psi \rightarrow \xi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \xi))$ *are* $\mathcal{K}$*-valid.*

Logical Consequence: Examples

Example 2 *All instances of* $[](\varphi \rightarrow \psi) \rightarrow ([]\varphi \rightarrow []\psi)$ *are* $\mathcal{K}$*-valid.*

For a contradiction, assume there is some instance $[](\varphi_1 \rightarrow \psi_1) \rightarrow ([]\varphi_1 \rightarrow []\psi_1)$ which is not $\mathcal{K}$-valid.

Therefore, there is some model $\mathcal{M} = \langle W, R, \vartheta \rangle$ and some $w \in W$ such that $w \not\Vdash [](\varphi_1 \rightarrow \psi_1) \rightarrow ([]\varphi_1 \rightarrow []\psi_1)$.

i.e. $\vartheta(w, [](\varphi_1 \rightarrow \psi_1) \rightarrow ([]\varphi_1 \rightarrow []\psi_1)) = \mathbf{f}$

i.e. $w \Vdash [](\varphi_1 \rightarrow \psi_1)$ and $w \not\Vdash ([]\varphi_1 \rightarrow []\psi_1)$

i.e. $w \Vdash [](\varphi_1 \rightarrow \psi_1)$ and $w \Vdash []\varphi_1$ and $w \not\Vdash []\psi_1$

i.e. $w \Vdash [](\varphi_1 \rightarrow \psi_1)$ and $w \Vdash []\varphi_1$ and $v \in W$ with wRv and $v \not\Vdash \psi_1$

i.e. $v \Vdash \varphi_1 \rightarrow \psi_1$ and $v \Vdash \varphi_1$ and $v \not\Vdash \psi_1$

i.e. $v \Vdash \psi_1$ and $v \not\Vdash \psi_1$ (contradiction)

Logical Consequence: Examples

Example 3 *If* $\varphi \in \Gamma$ *then* $\Gamma \models \varphi$ *(by definition of* $\models$*)*

Example 4 *If* $\Gamma \models \varphi$ *then* $\Gamma \models []\varphi$

For a contradiction, assume $\Gamma \models \varphi$ and $\Gamma \not\models []\varphi$.

i.e. exists $\mathcal{M} = \langle W, R, \vartheta \rangle \Vdash \Gamma$ and $w \in W$ with $w \Vdash \neg[]\varphi$.

i.e. exists $\mathcal{M} = \langle W, R, \vartheta \rangle \Vdash \Gamma$ and $w \in W$ with $w \Vdash \langle\rangle\neg\varphi$.

i.e. exists $\mathcal{M} = \langle W, R, \vartheta \rangle \Vdash \Gamma$ and $w \in W$ with wRv and $v \Vdash \neg\varphi$.

But $\Gamma \models \varphi$ means $\forall \mathcal{M} \in \mathcal{K}.(\mathcal{M} \Vdash \Gamma \Rightarrow \mathcal{M} \Vdash \varphi)$, hence $v \Vdash \varphi$. Contradiction.

Exercise 3 *If* $\Gamma \models \varphi$ *and* $\Gamma \models \varphi \rightarrow \psi$ *then* $\Gamma \models \psi$

Logical Implication as Logical Consequence

Lemma 2 *For any* w *in any model* $\langle W, R, \vartheta \rangle$, *if* $w \Vdash \{\varphi, \varphi \to \psi\}$ *then* $w \Vdash \psi$

Lemma 3 *For any model* $\mathcal{M}$, *if* $\mathcal{M} \Vdash \{\varphi, \varphi \to \psi\}$ *then* $\mathcal{M} \Vdash \psi$

Lemma 4 *If* $\Gamma \models \varphi \to \psi$ *then* $\Gamma, \varphi \models \psi$ *(writing* Γ, φ *for* $\Gamma \cup \{\varphi\}$*)*

Proof: Suppose $\Gamma \models \varphi \to \psi$. Suppose $\mathcal{M} \Vdash \Gamma, \varphi$. Must show $\mathcal{M} \Vdash \psi$. But $\mathcal{M} \Vdash \Gamma$ implies $\mathcal{M} \Vdash \varphi \to \psi$, so $\mathcal{M} \Vdash \{\varphi, \varphi \to \psi\}$. Lemma 3 gives $\mathcal{M} \Vdash \psi$.

Remark: Converse of Lemma 4 fails! e.g. We know $p_0 \models []p_0$. But $\emptyset \models p_0 \to []p_0$ is falsified in a model where $w \Vdash p_0$ with wRv and $v \Vdash \neg p_0$.

Lemma 5 *If* $\Gamma, \varphi \models \psi$ *then there exists an* n *such that*

$$\Gamma \models ([]^0\varphi \wedge []^1\varphi \wedge []^2\varphi \wedge \cdots \wedge []^n\varphi) \to \psi$$

where $[]^0\varphi = \varphi$ *and* $[]^n\varphi = [][]^{n-1}\varphi$ *(See Kracht for details)*

e.g. $p_0 \models []p_0$ implies $\emptyset \models (p_0 \wedge []p_0) \to []p_0$ so $n = 1$ for this example

Summary: Logic = Syntax and Semantics

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$ $\qquad (Atm)$

Formulae: $\varphi ::= p \mid \neg\varphi \mid \langle\rangle\varphi \mid []\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$ $\qquad (Fml)$

Kripke Frame: directed graph $\langle W, R\rangle$ where W is a non-empty set of points/worlds/vertices and $R \subseteq W \times W$ is a binary relation over W

Valuation on a Kripke frame $\langle W, R\rangle$ is a map $\vartheta : W \times Atm \mapsto \{\mathbf{t}, \mathbf{f}\}$ telling us the truth value ($\mathbf{t}$ or $\mathbf{f}$) of every atomic formula at every point in W

Kripke Model: $\langle W, R, \vartheta\rangle$ where ϑ is a valuation on a Kripke frame $\langle W, R\rangle$

Logical consequence: $\Gamma \models \varphi$ iff $\forall \mathcal{M} \in \mathcal{K}. \mathcal{M} \Vdash \Gamma \Rightarrow \mathcal{M} \Vdash \varphi$

Having defined $\Gamma \models \varphi$, we can consider a logic to be a set of formulae:

$$\mathbb{K} = \{\varphi \mid \emptyset \models \varphi\} = \{\varphi \mid \forall \mathcal{M} \in \mathcal{K}. \mathcal{M} \Vdash \varphi\} = \{\varphi \mid \forall \mathfrak{F} \in \mathfrak{K}. \mathfrak{F} \Vdash \varphi\}$$

Lecture 2: Hilbert Calculi

Motivation: Define a notion of deducibility "φ is deducible from Γ"

Requirement: Purely syntax manipulation, no semantic concepts allowed.

Judgment: $\Gamma \vdash \varphi$ where Γ is a finite set of assumptions (formulae)

Read $\Gamma \vdash \varphi$ as "φ is derivable from assumptions Γ"

Soundness: If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

If φ is derivable from Γ then φ is a logical consequence of Γ

Completeness: If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$

If φ is a logical consequence of Γ then φ is derivable from Γ

Goal: Deducibility captures logical consequence via syntax manipulation.

Hilbert Calculi: Derivation and Derivability

Assumptions: finite set of formulae accepted as derivable in one step (instantiation forbidden)

Axiom Schemata: Formula shapes, all of whose instances are accepted unquestionably as derivable in one step (listed shortly)

Rules of Inference: allow us to extend derivations into longer derivations

Judgment: $\Gamma \vdash \varphi$ where Γ is a finite set of assumptions (formulae)

Rules: (Name) $\dfrac{\text{Judgment}_1 \ldots \text{Judgment}_n}{\text{Judgment}}$ (Condition) $\dfrac{\text{premisses}}{\text{conclusion}}$

Read as: if premisses hold and condition holds then conclusion holds

Rule Instances: Uniformly replace formula variables and set variables in judgements with formulae and formula sets

Hilbert Derivability for Modal Logics

Assumptions: finite set of formulae accepted as derivable in one step (instantiation forbidden)

(Id) $\dfrac{}{\Gamma \vdash \varphi}\ \varphi \in \Gamma$ e.g. (Id) $\dfrac{}{\{p_0\} \vdash p_0}$

Axiom Schemata: Formula shapes, all of whose instances are accepted unquestionably as derivable in one step (listed shortly)

(Ax) $\dfrac{}{\Gamma \vdash \varphi}\ \varphi$ is an instance of an axiom schema

Rules of Inference: allow us to extend derivations into longer derivations

Modus Ponens (MP) $\dfrac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi \to \psi}{\Gamma \vdash \psi}$

Necessitation (Nec) $\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash [\,]\varphi}$

Hilbert Derivability for Modal Logics

(Id) $\dfrac{}{\Gamma \vdash \varphi}\ \varphi \in \Gamma$ (Ax) $\dfrac{}{\Gamma \vdash \varphi}\ \varphi$ is an instance of an axiom schema

(MP) $\dfrac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi \rightarrow \psi}{\Gamma \vdash \psi}$ (Nec) $\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash []\varphi}$

Rule Instances: Uniformly replace formula and set variables with formulae and formula sets

Derivation of φ_0 from assumptions Γ_0: is a finite tree of judgments with:

1. a root node $\Gamma_0 \vdash \varphi_0$
2. only (Ax) judgment instances and (Id) instances as leaves (sic!)
3. and such that all parent judgments are obtained from their child judgments by instantiating a rule of inference

Hilbert Calculus for Modal Logic $\mathbb{K}$

Axiom Schemata:

PC: $\varphi \longrightarrow (\psi \longrightarrow \varphi)$

$\neg\neg\varphi \longrightarrow \varphi$

$(\varphi \longrightarrow (\psi \longrightarrow \xi)) \longrightarrow ((\varphi \longrightarrow \psi) \longrightarrow (\varphi \longrightarrow \xi))$

K: $[](\varphi \longrightarrow \psi) \longrightarrow ([]\varphi \longrightarrow []\psi)$

How used: Create the leaves of a derivation via:

$$\text{(Ax)} \quad \frac{}{\Gamma \vdash \varphi} \; \varphi \text{ is an instance of an axiom schema}$$

$\varphi \wedge \psi \; := \neg(\varphi \longrightarrow \neg\psi)$

$\varphi \vee \psi \; := (\neg\varphi \longrightarrow \psi)$

$\varphi \leftrightarrow \psi \; := (\varphi \longrightarrow \psi) \wedge (\psi \longrightarrow \varphi)$

Hilbert Derivations: Examples

Let $\Gamma_0 = \{p_0, p_0 \to p_1\}$ and $\varphi_0 = []p_1$. Usually omit braces.

Below is a derivation of $[]p_1$ from $\{p_0, p_0 \to p_1\}$.

$$\dfrac{\dfrac{\dfrac{}{p_0, p_0 \to p_1 \vdash p_0}\,(\mathrm{Id}) \qquad \dfrac{}{p_0, p_0 \to p_1 \vdash p_0 \to p_1}\,(\mathrm{Id})}{p_0, p_0 \to p_1 \vdash p_1}\,(\mathrm{MP})}{p_0, p_0 \to p_1 \vdash []p_1}\,(\mathrm{Nec})$$

A derivation of φ_0 from assumptions Γ_0 is a finite tree of judgments with:

1. a root node $\Gamma_0 \vdash \varphi_0$
2. only (Ax) judgment instances and (Id) instances as leaves
3. and such that all parent judgments are obtained from their child judgments by instantiating a rule of inference

Hilbert Derivations: Examples

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$$\dfrac{\dfrac{}{p_0, p_0 \to p_1 \vdash p_0}\text{(Id)} \quad \dfrac{}{p_0, p_0 \to p_1 \vdash p_0 \to p_1}\text{(Id)}}{\dfrac{p_0, p_0 \to p_1 \vdash p_1}{p_0, p_0 \to p_1 \vdash []p_1}\text{(Nec)}}\text{(MP)}$$

(Nec) $\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash []\varphi}$ $\qquad \Gamma := \{p_0, p_0 \to p_1\} \qquad \varphi := p_1$

Hilbert Derivations: Examples

Let $\Gamma_0 = \{p_0, p_0 \to p_1\}$ and $\varphi_0 = []p_1$. Usually omit braces.

Below is a derivation of $[]p_1$ from $\{p_0, p_0 \to p_1\}$.

$$
\dfrac{\dfrac{\dfrac{}{p_0, p_0 \to p_1 \vdash p_0}\,(\mathrm{Id}) \qquad \dfrac{}{p_0, p_0 \to p_1 \vdash p_0 \to p_1}\,(\mathrm{Id})}{p_0, p_0 \to p_1 \vdash p_1}\,(\mathrm{MP})}{p_0, p_0 \to p_1 \vdash []p_1}\,(\mathrm{Nec})
$$

$$
(\mathrm{MP})\ \dfrac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi \to \psi}{\Gamma \vdash \psi} \qquad \Gamma := \{p_0, p_0 \to p_1\} \qquad \varphi := p_0 \qquad \psi := p_1
$$

Hilbert Derivations: Examples

Let $\Gamma_0 = \{p_0, p_0 \to p_1\}$ and $\varphi_0 = []p_1$. Usually omit braces.

Below is a derivation of $[]p_1$ from $\{p_0, p_0 \to p_1\}$.

$$\dfrac{\dfrac{\dfrac{}{p_0, p_0 \to p_1 \vdash p_0}\,(\mathrm{Id}) \qquad \dfrac{}{p_0, p_0 \to p_1 \vdash p_0 \to p_1}\,(\mathrm{Id})}{p_0, p_0 \to p_1 \vdash p_1}\,(\mathrm{MP})}{p_0, p_0 \to p_1 \vdash []p_1}\,(\mathrm{Nec})$$

(Id) $\dfrac{}{\Gamma \vdash \varphi}\,\varphi \in \Gamma$

$\Gamma := \{p_0, p_0 \to p_1\}$

$\varphi := p_0$

(Id) $\dfrac{}{\Gamma \vdash \varphi}\,\varphi \in \Gamma$

$\Gamma := \{p_0, p_0 \to p_1\}$

$\varphi := p_0 \to p_1$

Hilbert Derivations: Examples

Let $\Gamma = \{p_0, p_0 \rightarrow p_1\}$. Another derivation of $[]p_1$ from $\{p_0, p_0 \rightarrow p_1\}$:

$$\dfrac{\dfrac{\dfrac{}{p_0, p_0 \rightarrow p_1 \vdash p_0 \rightarrow p_1}\,(\mathrm{Id})}{p_0, p_0 \rightarrow p_1 \vdash [](p_0 \rightarrow p_1)}\,(\mathrm{Nec}) \qquad \dfrac{}{p_0, p_0 \rightarrow p_1 \vdash [](p_0 \rightarrow p_1) \rightarrow ([]p_0 \rightarrow []p_1)}\,(\mathrm{Ax})}{p_0, p_0 \rightarrow p_1 \vdash []p_0 \rightarrow []p_1}\,(\mathrm{MP})$$

1

$$\dfrac{\dfrac{\dfrac{}{p_0, p_0 \rightarrow p_1 \vdash p_0}\,(\mathrm{Id})}{p_0, p_0 \rightarrow p_1 \vdash []p_0}\,(\mathrm{Nec}) \qquad \overset{1}{p_0, p_0 \rightarrow p_1 \vdash []p_0 \rightarrow []p_1}}{p_0, p_0 \rightarrow p_1 \vdash []p_1}\,(\mathrm{MP})$$

K: $[](\varphi \rightarrow \psi) \rightarrow ([]\varphi \rightarrow []\psi)$ $\qquad$ $\varphi := p_0$ $\qquad$ $\psi := p_1$

Summary: Logic = Syntax and Calculus

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$ $\qquad (Atm)$

Formulae: $\varphi ::= p \mid \neg\varphi \mid \langle\rangle\varphi \mid []\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$ $\qquad (Fml)$

Hilbert Calculus $\mathbf{K}$: $[](\varphi \rightarrow \psi) \rightarrow ([]\varphi \rightarrow []\psi)$ only modal axiom

(Id) $\dfrac{}{\Gamma \vdash \varphi}\ \varphi \in \Gamma$ (Ax) $\dfrac{}{\Gamma \vdash \varphi}$ φ is an instance of an axiom schema

(MP) $\dfrac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi \rightarrow \psi}{\Gamma \vdash \psi}$ (Nec) $\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash []\varphi}$

$\Gamma \vdash \varphi$ iff there is a derivation of φ from Γ in $\mathbf{K}$.

Having defined $\Gamma \vdash \varphi$, we can consider a logic to be a set of formulae:

$$\mathbf{K} \quad = \quad \{\varphi \mid \emptyset \vdash \varphi\}$$

φ is a theorem of $\mathbf{K}$ iff $\varphi \in \mathbf{K}$ i.e. if it is deducible from the empty set

A modal logic is called “normal” if it extends $\mathbf{K}$ with extra modal axioms.

Soundness: all derivations are semantically correct

Theorem: if $\Gamma \vdash \psi$ then $\Gamma \models \psi$ ($\Gamma \models \psi$ means $\forall M \in \mathcal{K}. M \Vdash \Gamma \Rightarrow M \Vdash \psi$)

Proof: By induction on the length l of the derivation of $\Gamma \vdash \psi$

$l = 0$**:** So $\Gamma \vdash \psi$ because $\psi \in \Gamma$. But $\mathcal{M} \Vdash \Gamma$ implies $\mathcal{M} \Vdash \psi$ for all $\psi \in \Gamma$.

$l = 0$**:** So $\Gamma \vdash \psi$ because ψ is an axiom schema instance. By Eg 1, Ex 1, Ex 2, Eg 2, we know $\emptyset \models \psi$ for every axiom schema instance ψ, hence $\Gamma \models \psi$.

Ind. Hyp. : Theorem holds for all derivations of length less than some $k > 0$.

Ind. Step: Suppose $\Gamma \vdash \psi$ has a derivation of length k. Bottom-most rule?

MP: So both $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$ are shorter than k. By IH $\Gamma \models \varphi \to \psi$ and $\Gamma \models \varphi$. But if $w \Vdash \varphi \to \psi$ and $w \Vdash \varphi$ then $w \Vdash \psi$, hence $\Gamma \models \psi$

Nec: Then we know that $\Gamma \vdash \psi$ has length shorter than k. By IH we know $\Gamma \models \psi$. But if $\Gamma \models \psi$ then $\Gamma \models []\psi$ by Eg 4.

Completeness: all semantic consequences are derivable

Theorem: if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$

Proof Method: Prove contrapositive, if $\Gamma \nvdash \varphi$ then $\Gamma \not\models \varphi$

Proof Plan: Assume $\Gamma \nvdash \varphi$. Show there is a $\mathcal{K}-$model $\mathcal{M}_c = \langle W_c, R_c, \vartheta_c \rangle$ such that $\mathcal{M}_c \Vdash \Gamma$ and $\mathcal{M}_c \nVdash \varphi$ (i.e. $\exists w \in W_c . w \Vdash \neg\varphi$)

Technique: is known as the canonical model construction

Local Consequence: Write $X \vdash_l \varphi$ iff there exists a finite subset $\{\psi_1, \psi_2, \cdots, \psi_n\} \subseteq X$ such that $\emptyset \vdash (\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n) \rightarrow \varphi$

Exercise: if $X \vdash_l \varphi$ then $X \vdash \varphi$ by (MP) on $X \vdash \bigwedge(\psi_i)$ and $X \vdash \bigwedge(\psi_i) \rightarrow \varphi$

Set X is Maximal: if $\forall\psi.\psi \in X$ or $\neg\psi \in X$

Set X is Consistent: if both $X \vdash_l \psi$ and $X \vdash_l \neg\psi$ never hold, for any ψ

Set X is Maximal-Consistent: if it is maximal and consistent.

Lindenbaum's Construction of Maximal-Consistent Sets

Lemma 6 *Every consistent* Γ *is extendable into a maximal-consistent* $X^* \supset \Gamma$.

Proof: Choose an enumeration $\varphi_1, \varphi_2, \varphi_3, \cdots$ of the set of all formulae.

Stage 0: Let $X_0 := \Gamma$

Stage $n > 0$: $X_n := \begin{cases} X_{n-1} \cup \{\varphi_n\} & \text{if } X_{n-1} \vdash_l \varphi_n \\ X_{n-1} \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases}$

Stage ω: $X^* := \bigcup_{n=0}^{\omega} X_n$

Question: Every Stage is deterministic so why is X^* not unique ? (choice)

Not Effective: Relies on classicality: either $X_{n-1} \vdash_l \varphi_n$ or $X_{n-1} \not\vdash_l \varphi_n$ is true, but does not say how we decide the question.

Exercise: Why is having both $X_{n-1} \vdash_l \varphi_n$ and $X_{n-1} \vdash_l \neg\varphi_n$ impossible ?

Lindenbaum's Construction of Maximal-Consistent Sets

Lemma 7 *Every consistent Γ is extendable into a maximal-consistent $X^* \supset \Gamma$.*

Proof: Choose an enumeration $\varphi_1, \varphi_2, \varphi_3, \cdots$ of the set of all formulae.

Stage 0: Let $X_0 := \Gamma$

Stage $n > 0$: $X_n := \begin{cases} X_{n-1} \cup \{\varphi_n\} & \text{if } X_{n-1} \vdash_l \varphi_n \\ X_{n-1} \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases}$

Stage ω: $X^* := \bigcup_{n=0}^{\omega} X_n$

Chain of consistent sets: $X_0 \subset X_1 \subset \cdots$

Maximality: Clearly, for all φ either $\varphi \in X^*$ or else $\neg\varphi \in X^*$

X^* is consistent: Suppose for a contradiction that X^* is inconsistent. Thus $X^* \vdash_l \psi$ and $X^* \vdash_l \neg\psi$ for some ψ. Hence $\psi \in X_i$ and $\neg\psi \in X_j$ for some i and j. Let $k := max\{i, j\}$. Then $X_k \vdash_l \psi$ by (Id) and $X_k \vdash_l \neg\psi$ by (Id). Contradiction since X_k is consistent.

The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

W_c $:= \{X^* \mid X^*$ is a maximal-consistent extension of $\Gamma\} \neq \emptyset$

$w\ R_c\ v$ iff $\{\varphi \mid []\varphi \in w\} \subseteq v$

$$\vartheta_c(w,p) := \begin{cases} \mathbf{t} & \text{if } p \in w \\ \mathbf{f} & \text{otherwise} \end{cases}$$

Claim: wR_cv iff $\{\langle\rangle\varphi \mid \varphi \in v\} \subseteq w$

Proof left to right: Suppose wR_cv and $\{\langle\rangle\varphi \mid \varphi \in v\} \not\subseteq w$. Hence, there is some $\varphi \in v$ such that $\langle\rangle\varphi \notin w$. By maximality, $\neg\langle\rangle\varphi \in w$. By consistency, $[]\neg\varphi \in w$. By definition of wR_cv, we must have $\neg\varphi \in v$. Contradiction.

Proof right to left: Suppose $\{\langle\rangle\varphi \mid \varphi \in v\} \subseteq w$ and not wR_cv. Hence, there is some $[]\varphi \in w$ such that $\varphi \notin v$. By maximality, $\neg\varphi \in v$. By supposition, $\langle\rangle\neg\varphi \in w$. By consistency, $\neg[]\varphi \in w$. Contradiction.

The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

W_c $:= \{X^* \mid X^*$ is a maximal-consistent extension of $\Gamma\} \neq \emptyset$

$w\ R_c\ v$ iff $\{\varphi \mid []\varphi \in w\} \subseteq v$

$$\vartheta_c(w,p) := \begin{cases} \mathrm{t} & \text{if } p \in w \\ \mathrm{f} & \text{otherwise} \end{cases}$$

Lemma 8 *For every formula φ and every formula ψ and every $w \in W_c$:*

$\neg$:	$\neg\varphi \in w$	*iff*	$\varphi \notin w$	*i.e.* $\neg\varphi \notin w$ *iff* $\varphi \in w$
$\wedge$:	$\varphi \wedge \psi \in w$	*iff*	$\varphi \in w$ and $\psi \in w$	
$\vee$:	$\varphi \vee \psi \in w$	*iff*	$\varphi \in w$ or $\psi \in w$	
$\longrightarrow$:	$\varphi \longrightarrow \psi \in w$	*iff*	$\varphi \notin w$ or $\psi \in w$	
$[]$:	$[]\varphi \in w$	*iff*	$\forall v \in w.wR_cv \Rightarrow \varphi \in v$	
$\langle\rangle$:	$\langle\rangle\varphi \in w$	*iff*	$\exists v \in w.wR_cv\ \&\ \varphi \in v$	

The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

W_c $:= \{X^* \mid X^*$ is a maximal-consistent extension of $\Gamma\} \neq \emptyset$

$w\ R_c\ v$ iff $\{\varphi \mid []\varphi \in w\} \subseteq v$

$$\vartheta_c(w,p) := \begin{cases} \mathbf{t} & \text{if } p \in w \\ \mathbf{f} & \text{otherwise} \end{cases}$$

Claim: $\varphi \wedge \psi \in w$ iff $\varphi \in w$ and $\psi \in w$

Proof right to left : Suppose $\varphi \wedge \psi \in w$ and $\varphi \notin w$. Then $\neg\varphi \in w$.

Note $(\varphi \wedge \psi) \to \varphi \in w$ since $\emptyset \vdash_l (\varphi \wedge \psi) \to \varphi$ by $\mathbf{PC}$ (exercise)

Exists k with $X_k \vdash_l \neg\varphi$, and $X_k \vdash_l \varphi \wedge \psi$, and $X_k \vdash_l (\varphi \wedge \psi) \to \varphi$, by (Id).

Then $X_k \vdash_l \varphi$ by (MP) Contradiction.

Proof left to right: Suppose $\varphi \in w$ and $\psi \in w$ and $\varphi \wedge \psi \notin w$.

i.e. $(\varphi \to \neg\psi) \in w$ since $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$

i.e. exists k such that $X_k \vdash_l \varphi$ and $X_k \vdash_l \varphi \to \neg\psi$ and $X_k \vdash_l \psi$ by (id)

Then $X_k \vdash_l \neg\psi$ by (MP) Contradiction

The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

W_c $:= \{X^* \mid X^*$ is a maximal-consistent extension of $\Gamma\} \neq \emptyset$

$w\ R_c\ v$ iff $\{\psi \mid []\psi \in w\} \subseteq v$ $\qquad$ $\vartheta_c(w,p) := \begin{cases} \mathbf{t} & \text{if } p \in w \\ \mathbf{f} & \text{otherwise} \end{cases}$

Claim: $[]\varphi \in w$ iff $\forall v \in W_c.(wR_cv \Rightarrow \varphi \in v)$

Proof left to right: Suppose $[]\varphi \in w$ and $\forall v \in W_c.wR_cv \not\Rightarrow \varphi \in v$

i.e. $[]\varphi \in w$ and $\exists v \in W_c.wR_cv \;\&\; \varphi \notin v$

i.e. $[]\varphi \in w$ and $\exists v \in W_c.\varphi \in v \;\&\; \varphi \notin v$ $\qquad$ Contradiction.

The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

W_c $:= \{X^* \mid X^*$ is a maximal-consistent extension of $\Gamma\} \neq \emptyset$

$w\ R_c\ v$ iff $\{\psi \mid []\psi \in w\} \subseteq v$ $\qquad \vartheta_c(w,p) := \begin{cases} \mathrm{t} & \text{if } p \in w \\ \mathrm{f} & \text{otherwise} \end{cases}$

Claim: $[]\varphi \in w$ iff $\forall v \in W_c.(wR_cv \Rightarrow \varphi \in v)$

Proof right to left: Suppose $\forall v \in W_c.(wR_cv \Rightarrow \varphi \in v)$. Must show $[]\varphi \in w$.

i.e. $\forall v \in W_c.(\{\psi \mid []\psi \in w\} \subseteq v \Rightarrow \varphi \in v)$ $\qquad$ Let $\Psi := \bigwedge\{\psi \mid []\psi \in w\}$

i.e. $\forall v \in W_c.(\Psi \in v \Rightarrow \varphi \in v)$ i.e. $\forall v \in W_c.\Psi \rightarrow \varphi \in v$ by Lemma 8($\rightarrow$).

i.e. $\Gamma \vdash_l \Psi \rightarrow \varphi$ $\qquad$ (else can choose $\varphi_0 = \Psi \rightarrow \varphi$ for some v)

i.e. $\Gamma \vdash_l [](\Psi \rightarrow \varphi)$ by (Nec)

Note $\Gamma \vdash_l [](\Psi \rightarrow \varphi) \rightarrow ([]\Psi \rightarrow []\varphi)$ by (Ax)

Hence $\Gamma \vdash_l ([]\Psi \rightarrow []\varphi)$ by (MP) $\qquad$ Hence $([]\Psi \rightarrow []\varphi) \in w$.

Note, $\emptyset \vdash_l (([]\psi_0) \wedge ([]\psi_1)) \rightarrow [](\psi_0 \wedge \psi_1)$ $\qquad$ (exercise)

Hence $\{[]\Psi, ([]\Psi \rightarrow []\varphi)\} \subset w$. $\qquad$ Hence $[]\varphi \in w$ by (MP).

Truth Lemma

Lemma 9 *For every* φ *and every* $w \in W_c$*:* $\vartheta_c(w, \varphi) = \mathbf{t}$ *iff* $\varphi \in w$*.*

Proof: Pick any φ, any $w \in W$. Proceed by induction on length l of φ.

$l = 0$**:** So $\varphi = p$ is atomic. Then, $\vartheta_c(w, p) = \mathbf{t}$ iff $p \in w$ by definition of ϑ_c.

Ind. Hyp. : Lemma holds for all formulae with length l less than some $n > 0$

Ind. Step: Assume $l = n$ and proceed by cases on main connective

$\varphi = []\psi$**:** We have $\vartheta_c(w, []\psi) = \mathbf{t}$

iff $\forall v \in W_c.(wR_cv \Rightarrow \vartheta_c(v, \psi) = \mathbf{t}$ (by defn of valuations ϑ)

iff $\forall v \in W_c.(wR_cv \Rightarrow \psi \in v)$ (by IH)

iff $[]\psi \in w$ by Lemma 8([]).

Exercise: complete the proof

Completeness Proof

Corollary 1 $\langle W_c, R_c, \vartheta_c \rangle \Vdash \Gamma$

Proof: Since Γ is in every maximal-consistent set extending it, we must have $\Gamma \subset w$ for all $w \in W_c$. By Lemma 9, $w \Vdash \Gamma$, hence $\langle W_c, R_c, \vartheta_c \rangle \Vdash \Gamma$

Proof of Completeness: if $\Gamma \nvdash \varphi$ then $\Gamma \not\models \varphi$

Suppose $\Gamma \nvdash \varphi$. Hence $\Gamma \nvdash_l \varphi$. Construct the canonical model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$. Consider any ordering of formulae where φ is the first formula and let the associated maximal-consistent extension of Γ be X^*. Since $\Gamma \nvdash_l \varphi$ we must have $\neg\varphi \in X^*$. The set X^* appears as some world $w_0 \in W_c$ (say). Hence there exists at least one world where $\neg\varphi \in w_0$. By Lemma 9 $w_0 \Vdash \neg\varphi$ i.e. $\mathcal{M}_\Gamma \nVdash \varphi$. By Corollary 1, we know $\mathcal{M}_\Gamma \Vdash \Gamma$. Since the canonical model is a Kripke model, we have $\Gamma \not\models \varphi$. (i.e. not $\forall \mathcal{M} \in \mathcal{K}. \mathcal{M} \Vdash \Gamma \Rightarrow \mathcal{M} \Vdash \varphi$)

Completeness: By contraposition, if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Notes

$\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$ relies on the canonical frame $\langle W_c, R_c \rangle$ being a Kripke frame by its definition. (i.e. $\langle W_c, R_c \rangle \in \mathfrak{K}$)

Later we shall see that the canonical model is not always sound for $\vdash$: that is we can have φ where $\Gamma \vdash \varphi$ and $\mathcal{M}_\Gamma \not\Vdash \varphi$ (incomplete logics)

Beware: some books (e.g. Goldblatt) use the notation $\Gamma \vdash \varphi$ for our $\Gamma \vdash_l \varphi$ because then the deduction theorem holds: $\Gamma, \varphi \vdash_l \psi$ iff $\Gamma \vdash_l \varphi \to \psi$

Exercise: Prove it.

For us, the syntactic counterparts of Lemma 4 and Lemma 5 are:

Lemma 10 $\Gamma \vdash \varphi \to \psi$ *implies* $\Gamma, \varphi \vdash \psi$

Lemma 11 $\Gamma, \varphi \vdash \psi$ *implies* $\exists n. \Gamma \vdash []^0\varphi \wedge \cdots \wedge []^n\varphi \to \psi$

Lecture 3: Logic = Syntax and (Semantics or Calculus)

$\Gamma \models \varphi$: semantic consequence in class of Kripke models $\mathcal{K}$

$\Gamma \vdash \varphi$: deducibility in Hilbert calculus $\mathbf{K}$

Soundness: if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

Completeness: if $\Gamma \nvdash \varphi$ then $\mathcal{M}_\Gamma \not\models \varphi$ and $\mathcal{M}_\Gamma \in \mathcal{K}$.

$\mathbb{K}$	$=$	$\{\varphi \mid \emptyset \models \varphi\}$	the validities of Kripke frames $\mathfrak{K}$
$\mathbf{K}$	$=$	$\{\varphi \mid \emptyset \vdash \varphi\}$	the theorems of Hilbert calculus $\mathbf{K}$

Theorem 1 $\mathbf{K} = \mathbb{K}$

The presence of R makes modal logics non-truth-functional.

But Kripke models put no conditions on R.

So what happens if we put conditions on R ?

Valid Shapes and Frame Conditions

A binary relation R is reflexive if $\forall w \in W.wRw$.

A frame $\langle W, R\rangle$ or model $\langle W, R, \vartheta\rangle$ is reflexive if R is reflexive.

The shape $[]\varphi \rightarrow \varphi$ is called T.

A frame $\langle W, R\rangle$ validates a shape iff it forces all instances of that shape.

i.e. for all instances ψ of the shape and all valuations ϑ we have $\langle W, R, \vartheta\rangle \Vdash \psi$

Lemma 12 *A frame $\langle W, R\rangle$ validates T iff R is reflexive.*

Intuition: the shape T captures or corresponds to reflexivity of R.

Valid Shapes and Frame Conditions

A relation R is reflexive if $\forall w \in W.wRw$. The shape $[]\varphi \to \varphi$ is called T.

Lemma 13 *[Correspondence] A frame $\langle W, R\rangle$ validates T iff R is reflexive.*

Proof(i): Assume R is reflexive and $\langle W, R\rangle \not\Vdash []\psi \to \psi$ for some $[]\psi \to \psi$.

Exists model $\langle W, R, \vartheta\rangle$ and $w_0 \in W$ with $w_0 \Vdash []\psi$ and $w_0 \not\Vdash \psi$.

$v \Vdash \psi$ for all v with $w_0 R v$ $\qquad$ $w_0 R w_0$ $\qquad$ Hence, $w_0 \Vdash \psi$. Contradiction

Proof(ii): Assume $\langle W, R\rangle$ forces all instances of $[]\varphi \to \varphi$, and R not reflexive.

Exists $w_0 \in W$ such that $w_0 R w_0$ does not hold.

For all $w \in W$, let $\vartheta(w, p_0) = \mathrm{t}$ iff $w_0 R w$. (we define ϑ)

$\vartheta(v, p_0) = \mathrm{t}$ for every v with $w_0 R v$, and $\vartheta(w_0, p_0) = \mathrm{f}$ since not $w_0 R w_0$.

$w_0 \Vdash []p_0$ and $w_0 \not\Vdash p_0$ hence $w_0 \not\Vdash []p_0 \to p_0$

But $[]p_0 \to p_0$ is an instance of T hence $w_0 \Vdash []p_0 \to p_0$. Contradiction.

Valid Shapes and Frame Conditions

A frame $\langle W, R\rangle$ is reflexive if $\forall w \in W. wRw$. The shape $[]\varphi \to \varphi$ is called T.

A frame $\langle W, R\rangle$ validates T iff R is reflexive.

This correspondence does not work for models!

A model $\langle W, R, \vartheta\rangle$ validates T iff R is reflexive is false!

Consider the reflexive model $\mathcal{M}$ where:
$W = \{w_0\}$ and $R = \{(w_0, w_0)\}$ and ϑ is arbitrary.

This model must validate T since $\langle W, R\rangle$ is reflexive.

Now consider the model $\mathcal{M}'$ where:
$W' = \{v_0, v_1\}$ $\quad$ $R' = \{(v_0, v_1), (v_1, v_0)\}$ $\quad$ ϑ' is:

$$\vartheta'(v_i, p) = \begin{cases} \mathbf{t} & \text{if} \quad \vartheta(w_0, p) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

Exercise: model $\mathcal{M}'$ also validates T. But $\mathcal{M}'$ is not reflexive!

Summary: The Logic of Reflexive Kripke Frames

Let $\mathfrak{KT}$ be the class of all reflexive Kripke frames.

Let $\mathcal{KT}$ be the class of all reflexive Kripke models.

Let $\mathbf{KT} = \mathbf{K} + []\varphi \to \varphi$ (shape T) as an extra modal axiom.

Define $\Gamma \models_{\mathcal{KT}} \varphi$ to mean $\forall \mathcal{M} \in \mathcal{KT}. \mathcal{M} \Vdash \Gamma \Rightarrow \mathcal{M} \Vdash \varphi$.

Define $\Gamma \vdash_{KT} \varphi$ to mean there is a derivation of φ from Γ in $\mathbf{KT}$.

Soundness: if $\Gamma \vdash_{KT} \varphi$ then $\Gamma \models_{\mathcal{KT}} \varphi$

Proof: all instances of T are valid in reflexive frames.

Completeness: if $\Gamma \nvdash_{KT} \varphi$ then $\mathcal{M}_\Gamma \not\models_{\mathcal{KT}} \varphi$ and $\mathcal{M}_\Gamma \in \mathcal{KT}$

Proof: if $\mathcal{M}_\Gamma$ validates (all instances of) T then $\mathcal{M}_\Gamma$ is reflexive. (sic!)

i.e. T-instance $[]\psi_1 \to \psi_1 \in w$ iff $[]\psi_1 \in w \Rightarrow \psi_1 \in w$ by Lemma 8($\to$).

$\forall w, v \in W. w \ R_c \ v$ iff $\{\psi \mid []\psi \in w\} \subseteq v$ implies wR_cw

More Axiom and Frame Correspondences

Name	Axiom	Frame Class	Condition
T	$[]\varphi \to \varphi$	Reflexive	$\forall w \in W. wRw$
D	$[]\varphi \to \langle\rangle\varphi$	Serial	$\forall w \in W \exists v \in W. wRv$
4	$[]\varphi \to [][]\varphi$	Transitive	$\forall u, v, w \in W. uRv \& vRw \Rightarrow uRw$
5	$\langle\rangle[]\varphi \to []\varphi$	Euclidean	$\forall u, v, w \in W. uRv \& uRw \Rightarrow vRw$
B	$\varphi \to []\langle\rangle\varphi$	Symmetric	$\forall u, v \in W. uRv \Rightarrow vRu$
Alt_1	$\langle\rangle\varphi \to []\varphi$	Weakly-Functional	$\forall u, v, w \in W. uRv \& uRw \Rightarrow v = w$
2	$\langle\rangle[]\varphi \to []\langle\rangle\varphi$	Weakly-Directed	$\forall u, v, w \in W. uRv \& uRw \Rightarrow$ $\exists x \in W. vRx \& wRx$
3	$\langle\rangle\varphi \wedge \langle\rangle\psi \to$ $\langle\rangle(\varphi \wedge \langle\rangle\psi)$ $\vee\langle\rangle(\langle\rangle\varphi \wedge \psi)$ $\vee\langle\rangle(\varphi \wedge \psi)$	Weakly-Linear	$\forall u, v, w \in W. uRv \& uRw \Rightarrow$ vRw or wRv or $w = v$

Let $\mathbf{KA_1A_2 \cdots A_n} = \mathbf{K} + \mathsf{A}_1 + \mathsf{A}_2 + \cdots + \mathsf{A}_n$. (any A_is from above)

Theorem 2 $\Gamma \vdash_{\mathbf{KA_1A_2 \cdots A_n}} \varphi$ *iff* $\Gamma \models_{\mathcal{K}\mathcal{A}_1\mathcal{A}_2 \cdots \mathcal{A}_n} \varphi$

Correspondence, Canonicity and Completeness

Normal modal logic $\mathbf{L}$ is determined by class of Kripke frames $\mathfrak{C}$ if: $\forall\varphi.\mathfrak{C} \Vdash \varphi \iff \vdash_L \varphi$. Normal modal logic $\mathbf{L}$ is complete if determined by some class of Kripke frames. A normal modal logic is canonical if it is determined by its canonical frame.

A Sahlqvist formula is a formula with a particular shape (too complicated to define here but see Blackburn, de Rijke and Venema)

Theorem 3 *Every Sahlqvist formula φ corresponds to some first-order condition on frames, which is effectively computable from φ.*

Theorem 4 *If each axiom A_i is a Sahlqvist formula, then the Hilbert logic $\mathbf{KA_1A_2\cdots A_n}$ is canonical, and is determined by a class of frames which is first-order definable.*

Theorem 5 *Given a collection of Sahlqvist axioms $A_1, \cdots, A_k$, the logic $\mathbf{KA_1A_2\cdots A_k}$ is complete wrt the class of frames determined by $A_1 \cdots A_k$.*

Not All First-Order Conditions Are Captured By Shapes

Theorem 6 (Chagrov) *It is undecidable whether an arbitrary modal formula has a first-order correspondent.*

Question: Are there conditions on R not captured by any shape ?

Yes: the following conditions cannot be captured by any shape:

Irreflexivity: $\forall w \in W.$ not wRw

Anti-Symmetry: $\forall u, v \in W. uRv \& vRu \Rightarrow u = v$

Asymmetry: $\forall u, v \in W. uRv \Rightarrow$ not (vRu)

See Goldblatt for details.

Second-Order Aspects of Modal Logics

All of these conditions are first-order definable so it looked like modal logic was just a fragment of first-order logic ...

An R-chain is a sequence of distinct worlds $w_0 R w_1 R w_2 \cdots$.

Name	Shape	R Condition
G	$[]([]\varphi \to \varphi) \to []\varphi$	transitive and no infinite R-chains
Grz	$[]([](\varphi \to []\varphi) \to \varphi) \to []\varphi$	reflexive, transitive and no infinite R-chains

The condition "no infinite R-chains" is not first-order definable since "finiteness" is not first-order definable. It requires second-order logic, so propositional modal logic is a fragment of quantified second-order logic.

The logic $\mathbf{KG}$ has an interesting interpretation where $[]\varphi$ can be read as "φ is provable in Peano Arithmetic".

These logics are not Sahlqvist.

Shapes Not Captured By Any Kripke Frame Class

Consider logic $\mathbf{KH}$ where H is the axiom schema $[]([]\varphi \leftrightarrow \varphi) \rightarrow []\varphi$.

Theorem 7 (Boolos and Sambin) *The logic $\mathbf{KH}$ is not determined by any class of Kripke frames.*

G Boolos and G Sambin. *An Incomplete System of Modal Logic*, Journal of Philosophical Logic, 14:351-358, 1985.

Incompleteness first found in modal logic by S K Thomason in 1972. Beware, there is also a R H Thomason in modal logic literature.

Can regain a general frame correspondence by using general frames instead of Kripke frames: see Kracht.

Kracht shows how to compute modal Sahlqvist formulae from first-order formulae.

SCAN Algorithm of Dov Gabbay and Hans Juergen Ohlbach automatically computes first-order equivalents via the web.

Sub-Normal Mono-Modal Logics

Hilbert Calculus $\mathbf{S}$ = $\mathbf{PC}$ plus modal axioms (not $\mathbf{K}$)

(Id) $\dfrac{}{\Gamma \vdash_S \varphi}\ \varphi \in \Gamma$ (Ax) $\dfrac{}{\Gamma \vdash_S \varphi}\ \varphi$ is an instance of an axiom schema

(MP) $\dfrac{\Gamma \vdash_S \varphi \quad \Gamma \vdash_S \varphi \rightarrow \psi}{\Gamma \vdash_S \psi}$ (Mon) $\dfrac{\Gamma \vdash_S \varphi \rightarrow \psi}{\Gamma \vdash_S []\varphi \rightarrow []\psi}$ no rule (Nec)

$\Gamma \vdash_S \varphi$: iff there is a derivation of φ from Γ in $\mathbf{S}$.

Such modal logics are called "sub-normal".

$\Gamma \models_S \varphi$: needs Kripke models $\langle W, Q, R, \vartheta \rangle$ where: W is a set of "normal" worlds and ϑ behaves as usual, and Q is a set of "queer" or "non-normal" worlds where $\vartheta(w_q, \langle\rangle\varphi) = \mathbf{t}$ for all φ and all $w_q \in Q$ by definition. Then (Nec) fails since $\mathcal{M} \Vdash \varphi \not\Rightarrow \mathcal{M} \Vdash []\varphi$ i.e. every non-normal world makes $[]\varphi$ false.

Applications in logics for agents: $\models \varphi \Rightarrow \models []\varphi$ says that "if φ is valid, then φ is known", but agents may not be omniscient, hence want to go "sub-normal".

Regaining Expressive Power Via Nominals

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$ (Atm)

Nominals: $i ::= i_0 \mid i_1 \mid i_2 \mid \cdots$ (Nom)

Formulae: $\varphi ::= p \mid i \mid \neg\varphi \mid \langle\rangle\varphi \mid [\,]\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$ (Fml)

Valuation: for every i, $\vartheta(w, i) = \mathfrak{t}$ at only one world

Intuition: i is the name of w

Expressive Power:

Irreflexivity: $\forall w \in W.$ not wRw $i \rightarrow \neg\langle\rangle i$

Anti-Symmetry: $\forall u, v \in W. uRv \& vRu \Rightarrow u = v$ $i \rightarrow [\,](\langle\rangle i \rightarrow i)$

Asymmetry: $\forall u, v \in W. uRv \Rightarrow$ not (vRu) $i \rightarrow \neg\langle\rangle\langle\rangle i$

And many more see: Blackburn P. Nominal Tense Logics, Notre Dame Journal Of Formal Logic, 14:56-83, 1993.

Lecture 4: Tableaux Calculi and Decidability

Motivation: Finding derivations in Hilbert Calculi is cumbersome:

$\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \varphi \rightarrow \psi$ fails! $\quad \Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash ([]^0\varphi \wedge []^1\varphi \cdots []^n\varphi) \rightarrow \psi$

$$\frac{\dfrac{?}{\vdash \xi} \quad \dfrac{?}{\vdash \xi \rightarrow (\varphi \rightarrow \psi)}}{\vdash \varphi \rightarrow \psi}\ \text{(MP)} \qquad\qquad \frac{\dfrac{?}{\vdash \varphi}}{\vdash []\varphi}\ \text{(Nec)}$$

Resolution: one rule suffices for classical first-order logic, but not so for modal resolution

Decidability: questions can be answered via refinements of canonical models called filtrations, but there are better ways ...

For filtrations see Goldblatt.

Negated Normal Form

NNF: A formula is in negation normal form iff all occurrences of $\neg$ appear in front of atomic formulae only, and there are no occurrences of $\rightarrow$.

Lemma 14 *Every formula φ can be rewritten into a formula φ' such that φ' is in negation normal form, the length of φ' is at most polynomially longer than the length of φ, and $\emptyset \models \varphi \leftrightarrow \varphi'$.*

Proof: Repeatedly distribute negation over subformulae using the following valid principles:

$\models (\varphi_1 \rightarrow \psi_1) \leftrightarrow (\neg\varphi_1 \vee \psi_1)$ $\qquad$ $\models \neg(\varphi_1 \rightarrow \psi_1) \leftrightarrow (\varphi_1 \wedge \neg\psi_1)$

$\models \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$ $\qquad$ $\models \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$ $\qquad$ $\models \neg\neg\varphi \leftrightarrow \varphi$

$\models \neg\langle\rangle\varphi \leftrightarrow []\neg\varphi$ $\qquad$ $\models \neg[]\varphi \leftrightarrow \langle\rangle\neg\varphi$

Examples: NNF

Example:

$$\neg([](p_0 \to p_1) \to ([]p_0 \to []p_1))$$
$$[](p_0 \to p_1) \wedge \neg([]p_0 \to []p_1)$$
$$[](p_0 \to p_1) \wedge ([]p_0 \wedge \neg[]p_1)$$
$$[](\neg p_0 \vee p_1) \wedge ([]p_0 \wedge \langle\rangle\neg p_1)$$

Example:

$$\neg([]p_0 \to p_0)$$
$$([]p_0) \wedge (\neg p_0)$$

$$\neg([]p_0 \to [][]p_0)$$
$$([]p_0) \wedge (\neg[][]p_0)$$
$$([]p_0) \wedge (\langle\rangle\neg[]p_0)$$
$$([]p_0) \wedge (\langle\rangle\langle\rangle\neg p_0)$$

Tableau Calculi for Normal Modal Logics

Static Rules: (id) $\dfrac{p; \neg p; X}{\times}$ $\quad$ ($\wedge$) $\dfrac{\varphi \wedge \psi; X}{\varphi; \psi; X}$ $\quad$ ($\vee$) $\dfrac{\varphi \vee \psi; X}{\varphi; X \mid \psi; X}$

Transitional Rule: ($\langle\rangle\mathbf{K}$) $\dfrac{\langle\rangle\varphi; []X; Z}{\varphi; X} \; \forall\psi.[]\psi \notin Z$ $\qquad$ $[]X = \{[]\psi \mid \psi \in X\}$

X, Y, Z are possibly empty multisets of formulae and
$\varphi; X$ stands for $\{\varphi\}$ multiset-union X so number of occurences matter

Rules: (Name) $\dfrac{\text{MSet}}{\text{MSet}_1 \mid \ldots \mid \text{MSet}_n}$ $\qquad$ $\dfrac{\text{if numerator is } \mathcal{K}\text{-satisfiable}}{\text{then some denominator is } \mathcal{K}\text{-satisfiable}}$

A $\mathbf{K}$-tableau for Y is an inverted tree of nodes with:

1. a root node $\texttt{nnf}\ Y$
2. and such that all children nodes are obtained from their parent node by instantiating a rule of inference

A $\mathbf{K}$-tableau is closed (derivation) if all leaves are (id) instances, else it is open.

Examples of $\mathbf{K}$-Tableau

(id) $\dfrac{p; \neg p; X}{\times}$ $(\wedge)$ $\dfrac{\varphi \wedge \psi; X}{\varphi; \psi; X}$ $(\vee)$ $\dfrac{\varphi \vee \psi; X}{\varphi; X \mid \psi; X}$ $(\langle\rangle\mathbf{K})$ $\dfrac{\langle\rangle\varphi; []X; Z}{\varphi; X}$ $\forall\psi.[]\psi \notin Z$

$$\neg([](p_0 \to p_1) \to ([]p_0 \to []p_1))$$
$$\text{- - - - - - - - - - - - - - - - - - - (nnf)}$$
$$\dfrac{[](\neg p_0 \vee p_1) \wedge ([]p_0 \wedge \langle\rangle\neg p_1)}{[](\neg p_0 \vee p_1); ([]p_0 \wedge \langle\rangle\neg p_1)}\ (\wedge)$$
$$\dfrac{[](\neg p_0 \vee p_1); ([]p_0 \wedge \langle\rangle\neg p_1)}{[](\neg p_0 \vee p_1); []p_0; \langle\rangle\neg p_1}\ (\wedge)$$
$$\dfrac{[](\neg p_0 \vee p_1); []p_0; \langle\rangle\neg p_1}{\neg p_0 \vee p_1; p_0; \neg p_1}\ (\langle\rangle\mathbf{K})$$
$$\dfrac{\neg p_0 \vee p_1; p_0; \neg p_1}{\neg p_0; p_0; \neg p_1 \quad \mid \quad p_1; p_0; \neg p_1}\ (\vee)$$
$$\times \qquad\qquad\qquad \times$$

There is a closed $\mathbf{K}$-tableau for $\neg([](p_0 \to p_1) \to ([]p_0 \to []p_1))$

Examples of Tableau

(id) $\dfrac{p;\neg p; X}{\times}$ ($\wedge$) $\dfrac{\varphi \wedge \psi; X}{\varphi; \psi; X}$ ($\vee$) $\dfrac{\varphi \vee \psi; X}{\varphi; X \mid \psi; X}$ ($\langle\rangle\mathbf{K}$) $\dfrac{\langle\rangle\varphi; []X; Z}{\varphi; X}\ \forall\psi.[]\psi \notin Z$

$$\begin{array}{ll}
\neg([]p_0 \to p_0) & \\
\hdashline
([]p_0) \wedge \neg p_0 & \texttt{nnf} \\
\hline
([]p_0); \neg p_0 & (\wedge)
\end{array}$$

$$\begin{array}{ll}
\neg([]p_0 \to [][]p_0) & \\
\hdashline
([]p_0) \wedge (\langle\rangle\langle\rangle\neg p_0) & \texttt{nnf} \\
\hline
[]p_0; \langle\rangle\langle\rangle\neg p_0 & (\wedge) \\
\hline
p_0; \langle\rangle\neg p_0 & (\langle\rangle\mathbf{K}) \\
\hline
\neg p_0 & (\langle\rangle\mathbf{K})
\end{array}$$

There is no closed $\mathbf{K}$-tableau for $\neg([]p_0 \to p_0)$

There is no closed $\mathbf{K}$-tableau for $\neg([]p_0 \to [][]p_0)$

How can we be sure, we only looked at one $\mathbf{K}$-tableau for each ?

Some Proof Theory

(id) $\dfrac{p; \neg p; X}{\times}$ ($\wedge$) $\dfrac{\varphi \wedge \psi; X}{\varphi; \psi; X}$ ($\vee$) $\dfrac{\varphi \vee \psi; X}{\varphi; X \mid \psi; X}$ ($\langle\rangle\mathbf{K}$) $\dfrac{\langle\rangle\varphi; []X; Z}{\varphi; X}$ $\forall\psi.[]\psi \notin Z$

Weakening: **Lemma 15** *If $\varphi; X$ has a closed $\mathbf{K}$-tableau then so does $\varphi; X; Y$ for all multisets Y* *(adding junk does not destroy closure)*

Inversion $\wedge$: **Lemma 16** *If $\varphi \wedge \psi; X$ has a closed $\mathbf{K}$-tableau then so does $\varphi; \psi; X$* *(applying $(\wedge)$ cannot destroy closure)*

Inversion $\vee$: **Lemma 17** *If $\varphi \vee \psi; X$ has a closed $\mathbf{K}$-tableau then so do $\varphi; X$ and $\psi; X$* *(applying $(\vee)$ cannot destroy closure)*

Inversion fails for $(\langle\rangle\mathbf{K})$: $\dfrac{\langle\rangle(p \vee \neg p); (q \wedge \neg q)}{p \vee \neg p}$ $\dfrac{\longleftarrow \text{ has closed } \mathbf{K}\text{-tableau}}{\longleftarrow \text{ has no closed } \mathbf{K}\text{-tableau}}$

Contraction: **Lemma 18** *$\varphi; X$ has a closed $\mathbf{K}$-tableau iff $\varphi; \varphi; X$ has a closed $\mathbf{K}$-tableau.* *Can treat multisets as sets and vice-versa!*

Soundness of Modal Tableaux W.R.T. $\mathcal{K}$-satisfiability

A multiset of formulae Y is $\mathcal{K}$-satisfiable iff there is some Kripke model $\langle W, R, \vartheta \rangle$ and some $w \in W$ with $w \Vdash Y$ i.e. $\forall \varphi \in Y . w \Vdash \varphi$.

Lemma 19 (id) *The multiset* $p ; \neg p ; X$ *is never* $\mathcal{K}$*-satisfiable.*

Lemma 20 ($\wedge$) *If* $\varphi \wedge \psi ; X$ *is* $\mathcal{K}$*-satisfiable then* $\varphi ; \psi ; X$ *is* $\mathcal{K}$*-satisfiable.*

Lemma 21 ($\vee$) *If* $\varphi \vee \psi ; X$ *is* $\mathcal{K}$*-satisfiable then* $\varphi ; X$ *is* $\mathcal{K}$*-satisfiable or* $\psi ; X$ *is* $\mathcal{K}$*-satisfiable.*

Lemma 22 ($\langle\rangle$) *If* $\langle\rangle\varphi ; []X ; Z$ *is* $\mathcal{K}$*-satisfiable then* $\varphi ; X$ *is* $\mathcal{K}$*-satisfiable.*

Proof: Suppose $\langle\rangle\varphi ; []X ; Z$ is $\mathcal{K}$-satisfiable.

i.e. exists Kripke model $\langle W, R, \vartheta \rangle$ and some $w \in W$ with $w \Vdash \langle\rangle\varphi ; []X ; Z$

i.e. exists Kripke model $\langle W, R, \vartheta \rangle$ and some $v \in W$ with wRv and $v \Vdash \varphi$

i.e. $v \Vdash \varphi$ and $v \Vdash X$ i.e. $v \Vdash \varphi ; X$

i.e. $(\varphi ; X)$ is $\mathcal{K}$-satisfiable. (transitional)

Soundness of Modal Tableaux

Theorem 8 *If there is a closed* $\mathbf{K}$*-tableau for* Y *then* Y *is not* $\mathcal{K}$*-satisfiable.*

Proof: Suppose there is a closed $\mathbf{K}$-tableau for $\mathtt{nnf}\ Y$. Proceed by induction on length of $\mathbf{K}$-tableau, recall that $\models (\bigwedge Y) \leftrightarrow (\bigwedge \mathtt{nnf}\ Y)$.

$l = 0$**:** So $\mathtt{nnf}\ Y$ is an instance of (id). But $p; \neg p; X$ is never $\mathcal{K}$-satisfiable.

Ind. Hyp. : Theorem holds for all derivations of length less than some $k > 0$.

Ind. Step: Then $\mathtt{nnf}\ Y$ has a closed $\mathbf{K}$-tableau of length k. Top-most rule?

($\langle\rangle\mathbf{K}$): So the top-most rule application is an instance of the $(\langle\rangle\mathbf{K})$-rule.

$\varphi; X$ has closed $\mathbf{K}$-tableau By IH. $\varphi; X$ is not $\mathcal{K}$-satisfiable.

Lemma 22: if $\langle\rangle\varphi; []X; Z$ is $\mathcal{K}$-satisfiable then $\varphi; X$ is $\mathcal{K}$-satisfiable.

Hence $Y = (\langle\rangle\varphi; []X; Z)$ cannot be $\mathcal{K}$-satisfiable.

Corollary 2 *If* $\{\neg\varphi\}$ *has a closed* $\mathbf{K}$*-tableau then* $\emptyset \models \varphi$

Downward Saturated Or Hintikka Sets

A set Y is downward-saturated or an Hintikka set iff:

$$\begin{array}{lll}
\neg: & \neg\neg\varphi \in Y & \Rightarrow \quad \varphi \in Y \\
\wedge: & \varphi \wedge \psi \in Y & \Rightarrow \quad \varphi \in Y \text{ and } \psi \in Y \\
\vee: & \varphi \vee \psi \in Y & \Rightarrow \quad \varphi \in Y \text{ or } \psi \in Y \\
\rightarrow: & \varphi \rightarrow \psi \in Y & \Rightarrow \quad \varphi \notin Y \text{ or } \psi \in Y
\end{array}$$

Downward-saturated set is consistent if it does not contain $\{\varphi, \neg\varphi\}$, for any φ.

Don't need maximality: it is not demanded that $\forall\varphi.\varphi \in Y$ or $\neg\varphi \in Y$. (Hintikka)

Model Graphs

A $\mathbf{K}$-model-graph for set Y is a pair $\langle W, \lhd \rangle$ where W is a non-empty set of downward-saturated and consistent sets, some $w_0 \in W$ contains Y, and $\lhd$ is a binary relation over W such that for all w:

$\langle\rangle$: $\langle\rangle\varphi \in w \Rightarrow (\exists v \in W. w \lhd v \; \& \; \varphi \in v)$

$[]$: $[]\varphi \in w \Rightarrow (\forall v \in W. w \lhd v \Rightarrow \varphi \in v)$.

Lemma 23 (Hintikka) *If there is a $\mathbf{K}$-model-graph $\langle W, \lhd \rangle$ for set Y then Y is $\mathcal{K}$-satisfiable.*

Proof: Let $\langle W, R, \vartheta \rangle$ be the model where $R = \lhd$ and $\vartheta(w, p) = \mathbf{t}$ iff $p \in w$. By induction on the length of a formula φ, show that $\vartheta(w, \varphi) = \mathbf{t}$ iff $\varphi \in w$. Since $Y \subseteq w_0$ we have $w_0 \Vdash Y$.

Creating Downward-Saturated and Consistent Sets

Lemma 24 *If every $\mathbf{K}$-tableau for Y is open, then Y can be extended into a downward-saturated and consistent Y^* so every $\mathbf{K}$-tableau for Y^* is also open.*

Proof: Suppose no $\mathbf{K}$-tableau for Y closes. Now consider the following systematically constructed $\mathbf{K}$-tableau.

Stage 0: Let $w_0 = Y$.

Stage 1: Apply static rules giving finite open branch of nodes $w_0, w_1, \cdots, w_k$.
Let Y^* be the multiset-union of $w_0, \cdots, w_k$.

Claim: Y^* is downward-saturated (obvious) and consistent, and $Y \subseteq Y^*$.

By Contraction Lemma 18, we know $\varphi; X$ has (no) closed $\mathbf{K}$-tableau iff $\varphi; \varphi; X$ has (no) closed $\mathbf{K}$-tableau. (adding copies cannot affect closure)

Tableau for Y^* cannot close since construction of Y^* just adds back the principal formulae of each static rule application. can treat Y^* as a set!

Completeness and Decidability

Lemma 25 *If no $\mathbf{K}$-tableau for Y is closed, there is a $\mathbf{K}$-model-graph for Y.*

Proof: Suppose no $\mathbf{K}$-tableau for Y closes. Now consider the following systematic procedure

Stage 0: Let $w = Y$.

Stage 1: Apply static rules giving downward-saturated and consistent node w^* (Lemma 24)

Stage 2: Let $\langle\rangle\varphi_1, \langle\rangle\varphi_1, \cdots \langle\rangle\varphi_n$ be all the $\langle\rangle$-formulae in the current node.

So the current node looks like: $\langle\rangle\varphi_i; []X; Z_i$ for each $i = 1 \cdots n$.

For each $i = 1 \cdots n$ apply: $(\langle\rangle)\ \dfrac{\langle\rangle\varphi_i; []X; Z_i}{\varphi_i; X}$ $\qquad \dfrac{\longleftarrow w^*}{\longleftarrow v_i}$

Repeat Stages 1 and 2 on each node $v_i = (\varphi_i; X)$, and so on ad infinitum.

Each $(\langle\rangle)$-rule application reduces maximal-modal degree, giving termination.

Let W be set of all $*$-nodes, let $w^* \lhd v_i^*$ $\qquad \langle W, \lhd\rangle$ is a $\mathbf{K}$-model-graph for Y.

Decidability and Analytic Superformula Property

Subformula property: the nodes (sets) of a $\mathbf{K}$-tableau for Y (i.e. $\mathtt{nnf}\ Y$) only contain formulae from $\mathtt{nnf}\ Y$.

Subformula property will hold if all rules simply break down formulae or copy formulae across.

Analytic superformula property: the nodes (sets) of a $\mathbf{L}$-tableau for Y (i.e. $\mathtt{nnf}\ Y$) only contain formulae from a finite set Y' computable from $\mathtt{nnf}\ Y$ (but possibly larger than $\mathtt{nnf}\ Y$).

Analytic superformula property will hold if all rules that build up formulae cannot be applied ad infinitum.

The main skill in tableau calculi is to invent rules with the subformula property or the analytic superformula property!

Completeness W.R.T. $\mathcal{K}$-Satisfiability

Theorem 9 *If there is no closed $\mathbf{K}$-tableau for Y then Y is $\mathcal{K}$-satisfiable.*

Proof: Suppose every $\mathbf{K}$-tableau for Y is open.

Use Lemma 25 to construct a $\mathbf{K}$-model-graph $\langle W, \lhd \rangle$ for Y.

For all $w \in W$, let $\vartheta(w, p) = \mathbf{t}$ iff $p \in w$.

Then $\langle W, \lhd, \vartheta \rangle$ contains a world w_0 with $w_0 \models Y$ by Hintikka's Lemma 23.

Corollary 3 *If there is no closed $\mathbf{K}$-tableau for $\{\neg\varphi\}$ then $\not\models \varphi$.*

Corollary 4 *There is a closed $\mathbf{K}$-tableau for Y iff Y is not $\mathcal{K}$-satisfiable.*

Corollary 5 *There is a closed $\mathbf{K}$-tableau for $\{\neg\varphi\}$ iff φ is $\mathcal{K}$-valid.*

What About Logical Consequence: a concrete example

Write $\Gamma \vdash^{\mathcal{T}} \varphi$: iff there is a closed $\mathbf{K}$-tableau for $(\Gamma; \neg\varphi)$ i.e. $\mathtt{nnf}\ (\Gamma; \neg\varphi)$

Want Completeness: $\Gamma \not\vdash^{\mathcal{T}} \varphi \Rightarrow \exists \mathcal{M}. \mathcal{M} \Vdash \Gamma \ \& \ \mathcal{M} \not\Vdash \varphi$

Consider: $\Gamma := \{p_0\}$ and $\varphi := []p_1$.

Then $\mathtt{nnf}\ (\Gamma; \neg\varphi)$ has only one (open) $\mathbf{K}$-tableau:

$$\cfrac{\cfrac{\dfrac{(\Gamma; \neg\varphi)}{(p_0; \neg[]p_1)}}{(p_0; \langle\rangle\neg p_1)}\,(\mathtt{nnf}\ )}{\neg p_1}\,(\langle\rangle)$$

$w_0 = \{p_0, \langle\rangle\neg p_1\}$ $w_1 = \{\neg p_1\}$ $w_0 R w_1$

Problem: although $w_0 \Vdash \Gamma$, we don't have $w_1 \Vdash \Gamma$. So $\mathcal{M} \not\Vdash \varphi$ but $\mathcal{M} \not\Vdash \Gamma$.

If only we could make w_1 force Γ too ...

Regaining Completeness WRT Logical Consequence

Change ($\langle\rangle$) rule from ($\langle\rangle$) $\dfrac{\langle\rangle\varphi;\,[]X;\,Z}{\varphi;\,X}\ \forall\psi.[]\psi\notin Z$ to:

Transitional Rule: ($\langle\rangle\Gamma$) $\dfrac{\langle\rangle\varphi;\,[]X;\,Z}{\varphi;\,X;\,\mathtt{nnf}\ \Gamma}\ \forall\psi.[]\psi\notin Z$ (R-successor forces Γ)

Semantic reading:

$$\frac{\text{if numerator is } \mathbf{L}\text{-satisfiable in a model that forces } \Gamma}{\text{then some denominator is } \mathbf{L}\text{-satisfiable in a model that forces } \Gamma}$$ (new)

Stage 2: For each $i = 1 \cdots n$ apply: ($\langle\rangle\Gamma$) $\dfrac{\langle\rangle\varphi_i;\,[]X;\,Z_i}{\varphi_i;\,X;\,\mathtt{nnf}\ \Gamma}$ $\dfrac{\longleftarrow w^*}{\longleftarrow v_i \supseteq \mathtt{nnf}\ \Gamma}$

By completeness: $\Gamma \not\vdash^{\mathcal{T}} \varphi$: iff $(\exists\mathcal{M}.\exists w.\mathcal{M} \Vdash \Gamma\ \& w \Vdash (\Gamma; \neg\varphi))$

iff $(\exists\mathcal{M}.\mathcal{M} \Vdash \Gamma\ \&\ \mathcal{M} \not\Vdash \varphi)$ iff $\Gamma \not\models \varphi$

But there is a slight problem ... (TINSTAAFL)

Regaining Decidability

Problem: $\mathbb{K}$-tableau can now loop for ever: $\Gamma := \{\langle\rangle p_0\}$, and $\varphi := p_1$:

$$
\cfrac{\cfrac{\cfrac{\cfrac{(\Gamma; \neg\varphi)}{(\langle\rangle p_0; \neg p_1)}\,(\mathtt{nnf})}{(p_0; \langle\rangle p_0)}\,(\langle\rangle\Gamma)}{(p_0; \langle\rangle p_0)}\,(\langle\rangle\Gamma)}{\ldots}\,(\langle\rangle\Gamma)
$$

Solution: if we ever see a repeated node, just add a $\lhd$-edge back to previous copy on path from current node to root.

Other Normal Modal Logics

$\mathbf{KT}$: Static Rules: (id), $(\wedge)$, $(\vee)$, plus (T) $\dfrac{[]\varphi; X}{\varphi; ([]\varphi)^*; X}$ $[]\varphi$ unstarred

Transitional Rule: $(\langle\rangle\Gamma)$ $\dfrac{\langle\rangle\varphi; []X^*; Z}{\varphi; X; \mathtt{nnf}\ \Gamma}$ $\forall\psi.[]\psi \notin Z$ (unstar all []-formulae)

$\mathbf{K4}$: Static Rules: (id), $(\wedge)$, $(\vee)$

Transitional Rule: $(\langle\rangle\Gamma 4)$ $\dfrac{\langle\rangle\varphi; []X; Z}{\varphi; X; []X; \mathtt{nnf}\ \Gamma}$ $\forall\psi.[]\psi \notin Z$

$\mathbf{KT4}$: Static Rules: (id), $(\wedge)$, $(\vee)$, (T)

Transitional Rule: $(\langle\rangle\Gamma T4)$ $\dfrac{\langle\rangle\varphi; []X^*; Z}{\varphi; []X; \mathtt{nnf}\ \Gamma}$ $\forall\psi.[]\psi \notin Z$ (unstar all []-formulae)

Examples of $\mathbf{KT}$-Tableau

$\mathbf{KT}$: Static Rules: (id), $(\wedge)$, $(\vee)$, plus (T) $\dfrac{[]\varphi; X}{\varphi; ([]\varphi)^*; X}$ $[]\varphi$ unstarred

Transitional Rule: $(\langle\rangle\Gamma)$ $\dfrac{\langle\rangle\varphi; []X^*; Z}{\varphi; X; \mathtt{nnf}\ \Gamma}$ $\forall\psi.[]\psi \notin Z$ (unstar all []-formulae)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg([]p_0 \to p_0)}{([]p_0) \wedge \neg p_0}\ \mathtt{nnf}}{([]p_0); \neg p_0}\ (\wedge)}{p_0, ([]p_0)*; \neg p_0}\ (T)}{\times}
$$

There is a closed KT-tableau for $\neg([]p_0 \to p_0)$ i.e. $\emptyset \vdash^{\mathcal{T}}_{KT} []p_0 \to p_0$

Starring stops infinite sequence of T-rule applications.

Examples of $K4$-Tableau

$\mathbf{K4}$: Static Rules: (id), $(\wedge)$, $(\vee)$

Transitional Rule: $(\langle\rangle\Gamma 4)$ $\dfrac{\langle\rangle\varphi;\ []X;\ Z}{\varphi;\ X;\ []X;\ \mathtt{nnf}\ \Gamma}\ \forall\psi.[]\psi \notin Z$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg([]p_0 \to [][]p_0)}{([]p_0) \wedge (\langle\rangle\langle\rangle\neg p_0)}\ \mathtt{nnf}}{[]p_0;\ \langle\rangle\langle\rangle\neg p_0}\ (\wedge)}{p_0;\ []p_0;\ \langle\rangle\neg p_0}\ (\langle\rangle\Gamma 4)}{p_0;\ []p_0;\ \neg p_0}\ (\langle\rangle\Gamma 4)}{\times}$$

$$\cfrac{\cfrac{\cfrac{\langle\rangle p_0;\ []\langle\rangle p_0}{p_0;\ \langle\rangle p_0;\ []\langle\rangle p_0}\ (\langle\rangle\Gamma 4)}{p_0;\ \langle\rangle p_0;\ []\langle\rangle p_0}\ (\langle\rangle\Gamma 4)}{\cdots}$$

There is closed $K4$-tableau for $\neg([]p_0 \to [][]p_0)$ i.e. $\emptyset \vdash^{\mathcal{T}}_{K4} []p_0 \to [][]p_0$

Need loop check: $K4$-tableau for $(\langle\rangle p_0;\ []\langle\rangle p_0)$ has infinite branch.

Follow The Procedure ...

Prove Weakening.

Prove Inversion for all Static Rules.

Check if Transitional Rule has Inversion (unlikely).

Prove Soundness: If there is a closed $\mathbf{KL}$-tableau for Y then Y is not $\mathcal{KL}$-satisfiable.

Define appropriate notion of $\mathbf{L}$-model-graph.

Prove Hintikka's Lemma: If there is an $\mathbf{L}$-model-graph for Y then Y is $\mathcal{KL}$-satisfiable.

Prove Completeness: If there is no closed $\mathbf{KL}$-tableau for Y then Y is $\mathcal{KL}$-satisfiable.

Add changes to transitional rule(s) for handling $\Gamma \vdash_L^{\tau} \varphi$

Prove termination (by analytic superformula property and tracking of loops).

Soundness for Rule ($\langle\rangle T4$)

Example: ($\langle\rangle T4$) $\dfrac{\langle\rangle\varphi; []X^*; Z}{\varphi; []X}\ \forall\psi.[]\psi \notin Z$

All depends upon:

Lemma : if $\langle\rangle\varphi; []X; Z$ is $\mathcal{KT}4$-satisfiable then $\varphi; X$ is $\mathcal{KT}4$-satisfiable.

Proof: Suppose $\langle\rangle\varphi; []X; Z$ is is $\mathcal{KT}4$-satisfiable.

i.e. exists transitive Kripke model $\langle W, R, \vartheta\rangle$ and some $w \in W$ with
$w \Vdash \langle\rangle\varphi; []X; Z$

i.e. exists transitive Kripke model $\langle W, R, \vartheta\rangle$ and some $v \in W$ with wRv and
$v \Vdash (\varphi; X; []X)$ $\qquad$ ($[]X \to [][]X$)

i.e. exists transitive Kripke model $\langle W, R, \vartheta\rangle$ and some $v \in W$ with wRv and
$v \Vdash (\varphi; []X)$ $\qquad$ can regain X by T rule

Tableaux Versus Hilbert Calculi

Algorithm: Systematic procedure gives algorithm for finding (closed) tableaux.

Decidability: easier than in Hilbert Calculi.

Modularity: Must invent new rules for new axioms. Reuse completeness proof based upon systematic procedure with tweaks. Rules require careful design to regain decidability e.g. starring, looping, dynamic looping etc.

Automated Deduction: Logics WorkBench `http://www.lwb.unibe.ch` has implementation of tableau theorem provers for many fixed logics e.g. $\mathbf{K}$, $\mathbf{KT}$, $\mathbf{K4}$, $\mathbf{KT4}$, ...

Automated Deduction: The Tableaux WorkBench `http://arp.anu.edu.au/~abate/twb` provides a way to implement tableau theorem provers for any tableau calculus that fits its syntax e.g. $\mathbf{KD45}$, $\mathbf{KtS4}$, $\mathbf{Int}$, $\mathbf{IntS4}$, ...

Lecture 5: Tense and Temporal Logics

Tense Logics: interpret $[]\varphi$ as “φ is true always in the future”.

W represents moments of time

R captures the flow of time

Temporal Logics: similar, but use a more expressive binary modality $\varphi\mathcal{U}\psi$ to capture “φ is true at all time points from now until ψ becomes true”.

Shall look at Syntax, Semantics, Hilbert and Tableau Calculi.

Tense Logics: Syntax and Semantics

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$

Formulae: $\varphi ::= p \mid \neg\varphi \mid \langle F\rangle\varphi \mid [F]\varphi \mid \langle P\rangle\varphi \mid [P]\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$

Boolean connectives interpreted as for modal logic.

Given some Kripke model $\langle W, R, \vartheta\rangle$ and some $w \in W$, we compute the truth value of a non-atomic formula by recursion on its shape:

$$\vartheta(w, \langle F\rangle\varphi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at some } v \in W \text{ with } wRv \\ \mathrm{f} & \text{otherwise} \end{cases}$$

$$\vartheta(w, [F]\varphi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at every } v \in W \text{ with } wRv \\ \mathrm{f} & \text{otherwise} \end{cases}$$

$$\vartheta(w, \langle P\rangle\varphi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at some } v \in W \text{ with } vRw \\ \mathrm{f} & \text{otherwise} \end{cases}$$

$$\vartheta(w, [P]\varphi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at every } v \in W \text{ with } vRw \\ \mathrm{f} & \text{otherwise} \end{cases}$$

Tense Logics: Syntax and Semantics

$$
\begin{aligned}
\vartheta(w, \langle F\rangle\varphi) &= \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at some } v \in W \text{ with } wRv \\ \mathrm{f} & \text{otherwise} \end{cases} \\
\vartheta(w, [F]\varphi) &= \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at every } v \in W \text{ with } wRv \\ \mathrm{f} & \text{otherwise} \end{cases} \\
\vartheta(w, \langle P\rangle\varphi) &= \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at some } v \in W \text{ with } vRw \\ \mathrm{f} & \text{otherwise} \end{cases} \\
\vartheta(w, [P]\varphi) &= \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at every } v \in W \text{ with } vRw \\ \mathrm{f} & \text{otherwise} \end{cases}
\end{aligned}
$$

Example: If $W = \{w_0, w_1, w_2\}$ and $R = \{(w_0, w_1), (w_0, w_2)\}$ and $\vartheta(w_1, p_3) = \mathrm{t}$ then $\langle W, R, \vartheta\rangle$ is a Kripke model as pictured below:

$$
\begin{tikzcd}
 & w_1 \\
w_0 \arrow[ur, "R"] \arrow[dr, "R"'] & \\
 & w_2
\end{tikzcd}
$$

$$
\begin{aligned}
\vartheta(w_0, \langle F\rangle p_3) &= \mathrm{t} \\
\vartheta(w_2, \langle P\rangle\langle F\rangle p_3) &= \mathrm{t} \\
\vartheta(w_0, [P]p_1) &= \mathrm{t}
\end{aligned}
$$

Hilbert Calculus for Modal Logic $\mathrm{K_t}$

Axiom Schemata: Axioms for $\mathbf{PC}$ plus:

K$[F]$**:** $[F](\varphi \to \psi) \to ([F]\varphi \to [F]\psi)$

K$[P]$**:** $[P](\varphi \to \psi) \to ([P]\varphi \to [P]\psi)$

FP: $\varphi \to [F]\langle P\rangle\varphi$ PF: $\varphi \to [P]\langle F\rangle\varphi$

Rules of Inference: (Ax) $\dfrac{}{\Gamma \vdash \varphi}$ φ is an instance of an axiom schema

(Id) $\dfrac{}{\Gamma \vdash_{K_t} \varphi}$ $\varphi \in \Gamma$ (MP) $\dfrac{\Gamma \vdash_{K_t} \varphi \quad \Gamma \vdash_{K_t} \varphi \to \psi}{\Gamma \vdash_{K_t} \psi}$

(Nec$[F]$) $\dfrac{\Gamma \vdash_{K_t} \varphi}{\Gamma \vdash_{K_t} [F]\varphi}$ (Nec$[P]$) $\dfrac{\Gamma \vdash_{K_t} \varphi}{\Gamma \vdash_{K_t} [P]\varphi}$

Soundness, Completeness, Correspondence etc. : Let $\mathcal{K}_t = \mathcal{K}$ be class of all Kripke Tense frames $\Gamma \vdash_{K_t A_1, A_2, \ldots, A_n} \varphi$ iff $\Gamma \models_{K_t A_1, A_2, \ldots, A_n} \varphi$

Different Models of Time

Arbitrary Time: $\mathbf{K_t}$

Reflexive Time: $\varphi \rightarrow \langle F \rangle \varphi$

Transitive Time: $\langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$

Dense Time: $\langle F \rangle \varphi \rightarrow \langle F \rangle \langle F \rangle \varphi$

Never Ending Time: $[F]\varphi \rightarrow \langle F \rangle \varphi$

Backward Linear: $\langle F \rangle \langle P \rangle \varphi \rightarrow \langle P \rangle \varphi \vee \varphi \vee \langle F \rangle \varphi$

Forward Linear: $\langle P \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi \vee \varphi \vee \langle P \rangle \varphi$

Tableau Calculi also exist but require even more complex loop detection often called “dynamic blocking”.

Discrete $\langle \mathbb{Z}, < \rangle$, Rational $\langle \mathbb{Q}, < \rangle$, Real $\langle \mathbb{R}, < \rangle$ linear and non-reflexive models of time also possible: see Goldblatt.

Tableau-like calculi exist: see Mosaic Method

PLTL: Propositional Linear Temporal Logic

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$

Formulae: $\varphi ::= p \mid \neg\varphi \mid \bigoplus\varphi \mid [F]\varphi \mid \langle F\rangle\varphi \mid \varphi\mathcal{U}\psi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$

Boolean connectives interpreted as for modal logic.

Linear Time Kripke Model: $\langle S, \sigma, R, \vartheta\rangle$

S**:** non-empty set of states

σ**:** $\mathbb{N} \rightarrow S$ enumerates S as sequence $\sigma_0, \sigma_1, \cdots$ with repetitions when S finite

ϑ**:** $S \times Atm \mapsto \{\mathbf{t}, \mathbf{f}\}$

R**:** is a binary relation over S

Condition: $R = \sigma^*$ (R is the reflexive and transitive closure of σ)

Semantics of PLTL

$$\vartheta(s_i, \bigoplus\varphi) = \begin{cases} \mathbf{t} & \text{if } \vartheta(s_{i+1}, \varphi) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$\vartheta(s_i, \langle F\rangle\varphi) = \begin{cases} \mathbf{t} & \text{if } \vartheta(s_j, \varphi) = \mathbf{t} \text{ for some } j \geq i \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$\vartheta(s_i, [F]\varphi) = \begin{cases} \mathbf{t} & \text{if } \vartheta(s_j, \varphi) = \mathbf{t} \text{ for all } j \geq i \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$\vartheta(s_i, \varphi\mathcal{U}\psi) = \begin{cases} \mathbf{t} & \text{if } \exists k \geq i.\vartheta(s_k, \psi) = \mathbf{t} \,\&\, \forall j.i \leq j < k \Rightarrow \vartheta(s_j, \varphi) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$\begin{array}{cccccc} s_i & s_{i+1} & \cdots & s_j & \cdots & s_k \\ \\ p\mathcal{U}q & p, \neg q & \cdots & p, \neg q & \cdots & q \end{array}$$

Note: when $k \neq i$, the state s_k is the first state after s_i where q is true.

Semantics of PLTL

$$\vartheta(s_i, \bigoplus\varphi) = \begin{cases} \mathbf{t} & \text{if } \vartheta(s_{i+1}, \varphi) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$\vartheta(s_i, \langle F\rangle\varphi) = \begin{cases} \mathbf{t} & \text{if } \vartheta(s_j, \varphi) = \mathbf{t} \text{ for some } j \geq i \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$\vartheta(s_i, [F]\varphi) = \begin{cases} \mathbf{t} & \text{if } \vartheta(s_j, \varphi) = \mathbf{t} \text{ for all } j \geq i \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$\vartheta(s_i, \varphi\mathcal{U}\psi) = \begin{cases} \mathbf{t} & \text{if } \exists k \geq i.\vartheta(s_k, \psi) = \mathbf{t} \ \&\ \forall j.i \leq j < k \Rightarrow \vartheta(s_j, \varphi) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

s_i	s_{i+1}	$\cdots$	s_j	$\cdots$	s_k	
$\neg(p\mathcal{U}q), \neg q$	$\neg q$	$\cdots$	$\neg q$	$\cdots$	$\neg q$	q is always false, or
$\neg(p\mathcal{U}q)$	$\neg q$	$\cdots$	$\neg p, \neg q$	$\cdots$	q	p false before q true

Note: when $k \neq i$, the state s_k is the first state after s_i where q is true. And p is false in some s_j before state s_k.

Hilbert Calculus for PLTL

Axiom Schemata: axioms for $\mathbf{PC}$ plus

K$[F]$: $[F](\varphi \to \psi) \to ([F]\varphi \to [F]\psi)$

K$\bigoplus$: $\bigoplus(\varphi \to \psi) \to (\bigoplus\varphi \to \bigoplus\psi)$

Fun: $\bigoplus\neg\varphi \leftrightarrow \neg\bigoplus\varphi$

Mix: $[F]\varphi \to (\varphi \wedge \bigoplus[F]\varphi)$

Ind: $[F](\varphi \to \bigoplus\varphi) \to (\varphi \to [F]\varphi)$

$\mathcal{U}_1$: $(\varphi\mathcal{U}\psi) \to \langle F\rangle\psi$

$\mathcal{U}_2$: $(\varphi\mathcal{U}\psi) \leftrightarrow \psi \vee (\neg\psi \wedge \varphi \wedge \bigoplus(\varphi\mathcal{U}\psi))$

Rules: (Id), (Ax), MP and (Nec$[F]$) and (Nec$\bigoplus$)

Tableau Calculus for PLTL

Presence of Induction Axiom $\mathtt{Ind}$ means no finitary cut-free sequent calculus (must guess induction hypothesis)

Cannot just "jump" on $\langle F \rangle \varphi$ because of its interaction with $\bigoplus$ which demands "single steps"

Requires a two pass method: build a model-graph, check that it is contains a model.

Tableau Calculus for PLTL: Pass 1

Stage 0: put $w_0 = Y$

Stage 1: repeatedly apply usual $(\wedge)$ and $(\vee)$ rules together with the following to obtain a downward-saturated node w_0^* in which each non-atomic formula is marked as "done" or is of the form $\bigoplus\varphi$:

$\neg\bigoplus\varphi \rightarrow \bigoplus\neg\varphi$ $\qquad$ $[F]\varphi \rightarrow (\varphi \wedge \bigoplus[F]\varphi)$

$\langle F\rangle\varphi \rightarrow (\varphi \vee \bigoplus\langle F\rangle\varphi)$ $\qquad$ $(\varphi\mathcal{U}\psi) \rightarrow \psi \vee (\neg\psi \wedge \varphi \wedge \bigoplus(\varphi\mathcal{U}\psi))$

Stage 2: Current node is now of the form $\bigoplus X; Z$ where Z contains only atoms, negated atoms, and "done" formulae. Create a $\bigoplus$-successor w_1 containing X.

Stage 3: Saturate w_1 via Stage 1 to get w_1^* and add $w_0^* R_{\bigoplus} w_1^*$ if w_1^* is new, else add $w_0^* R_{\bigoplus} v^*$ for the node v^* which already replicates w_1^*.

Stage 4: If w_1^* is new then repeat and so on until no new $*$-nodes turn up giving a possibly cyclic graph.

Tableau Method for PLTL: Pass 2

An eventuality is a formula $\langle F \rangle \varphi$ or $\varphi \mathcal{U} \psi$

A path is a maximal (cyclic) sequence of nodes starting at the root.

"Maximal" means "cannot avoid repetition" (unwind)

A path fulfills $\langle F \rangle \varphi$ if some node on it contains φ

A path fulfills $\varphi \mathcal{U} \psi$ if some node on it contains ψ and between nodes contain φ

Delete all nodes that contain a pair $\{p, \neg p\}$.

Repeatedly delete all nodes who now do not have an $\bigoplus$-successor.

If some single path fulfills all eventualities contained in its nodes then Y is $\mathbf{PLTL}$-satisfiable, otherwise it is not.

Note: all eventualities on that path must be fulfilled on that path!

Lecture 6: Fix-point Logics

PLTL: linear time temporal logic

CTL: computation tree logic

PDL: propositional dynamic logic

LCK: logic of common knowledge

Look at CTL but using only one relation R rather than $R = \sigma^*$

CTL: Computation Tree Logic

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$ (AP)

Formulae:
$$
\begin{aligned}
\varphi ::= p &\mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \\
&\mid EX\varphi \mid AX\varphi \\
&\mid E(\varphi\, U\, \psi) \mid A(\varphi\, U\, \psi) \\
&\mid E(\varphi\, B\, \psi) \mid A(\varphi\, B\, \psi)
\end{aligned} \qquad \text{(Fml)}
$$

Note: Ep is not a formula!

Unary Modal connectives are: $EX\cdot$ and $AX\cdot$

Binary Modal Connectives are: $E(\cdot\, U\, \cdot)$ $A(\cdot\, U\, \cdot)$ $A(\cdot\, B\, \cdot)$ $E(\cdot\, B\, \cdot)$

NNF: we shall later assume that all formulae are in Negation Normal Form

Semantics of CTL

Transition Frame: is a pair (W, R) where W is a non-empty set of worlds and R is a binary relation over W that is total ($\forall w \in W.\ \exists v \in W.\ w\, R\, v$).

Full path: in a transition frame (W, R) is an infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$ of worlds in W such that $\sigma_i\, R\, \sigma_{i+1}$ for all $i \in \mathbb{N}$.

$\mathcal{B}(w)$: for $w \in W$, $\mathcal{B}(w)$ is the set of all fullpaths in (W, R) which begin at w

Model: $M = (W, R, L)$ is a transition frame (W, R) and a labelling function $L : W \to 2^{\mathsf{AP}}$ so that $L(w)$ is the set of atomic formulae true at w

Seriality: $\mathcal{B}(w)$ is non-empty by seriality

Semantics of CTL

Model: $M = (W, R, L)$ is a transition frame (W, R) and a labelling function $L : W \to 2^{\mathsf{AP}}$ so that $L(w)$ is the set of atomic formulae true at w

World forces formula: $M, w \Vdash \varphi$ defined by induction on shape of φ

$$
\begin{array}{lll}
M, w \Vdash p & \text{iff} & p \in L(w), \text{ for } p \in \mathsf{AP} \\
M, w \Vdash \neg\psi & \text{iff} & M, w \nVdash \psi \\
M, w \Vdash \varphi \wedge \psi & \text{iff} & M, w \Vdash \varphi \ \& \ M, w \Vdash \psi \\
M, w \Vdash \varphi \vee \psi & \text{iff} & M, w \Vdash \varphi \text{ or } M, w \Vdash \psi
\end{array}
$$

Intuition: classical connectives behave as usual at a world

Semantics of CTL

Model: $M = (W, R, L)$ is a transition frame (W, R) and a labelling function $L : W \to 2^{\mathsf{AP}}$ so that $L(w)$ is the set of atomic formulae true at w

World forces formula: $M, w \Vdash \varphi$ defined by induction on shape of φ

$$M, w \Vdash EX\varphi \quad \text{iff} \quad \exists v \in W.\, w\, R\, v \;\&\; M, v \Vdash \varphi$$

$$M, w \Vdash AX\varphi \quad \text{iff} \quad \forall v \in W.\, w\, R\, v \;\Rightarrow\; M, v \Vdash \varphi$$

Intuitions: $EX\varphi$ means "some immediate R-successor forces φ"

Intuitions: $AX\varphi$ means "every immediate R-successor forces φ"

X: stands for neXt i.e. immediate

Semantics of CTL

Model: $M = (W, R, L)$ is a transition frame (W, R) and a labelling function $L : W \to 2^{\mathsf{AP}}$ so that $L(w)$ is the set of atomic formulae true at w

World forces formula: $M, w \Vdash \varphi$ defined by induction on shape of φ

$M, w \Vdash E(\varphi\, U\, \psi)$ iff "some full path from w forces φ until ψ"

$M, w \Vdash A(\varphi\, U\, \psi)$ iff "every full path from w forces φ until ψ"

But: we have not defined what it means for a fullpath to force a formula

Must: express it in terms of a world forcing a formula

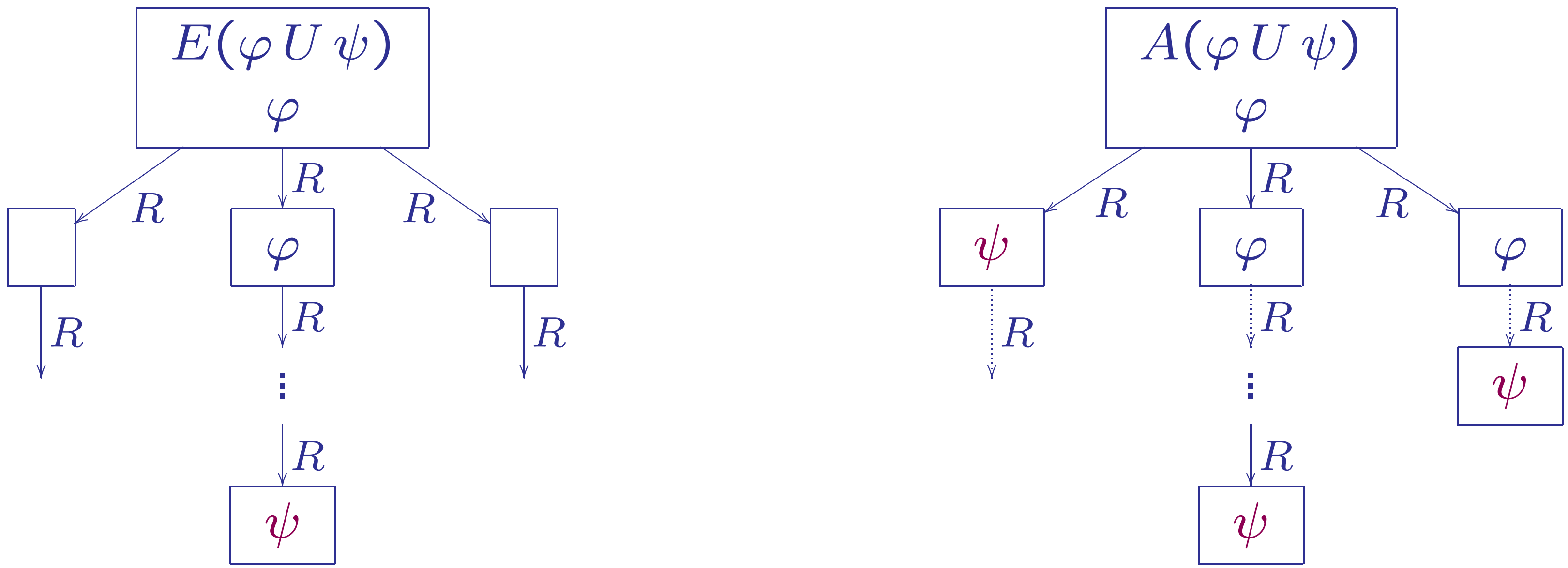
Semantics of CTL

Model: $M = (W, R, L)$ is a transition frame (W, R) and a labelling function $L : W \longrightarrow 2^{\mathsf{AP}}$ so that $L(w)$ is the set of atomic formulae true at w

World forces formula: $M, w \Vdash \varphi$ defined by induction on shape of φ

$$M, w \Vdash E(\varphi\, U\, \psi) \quad \text{iff} \quad \exists \sigma \in \mathcal{B}(w).\, \exists i \in \mathbb{N}.\, [M, \sigma_i \Vdash \psi \,\&\, \forall j < i.\, M, \sigma_j \Vdash \varphi]$$

$$M, w \Vdash A(\varphi\, U\, \psi) \quad \text{iff} \quad \forall \sigma \in \mathcal{B}(w).\, \exists i \in \mathbb{N}.\, [M, \sigma_i \Vdash \psi \,\&\, \forall j < i.\, M, \sigma_j \Vdash \varphi]$$



Semantics of CTL

Model: $M = (W, R, L)$ is a transition frame (W, R) and a labelling function $L : W \to 2^{\mathsf{AP}}$ so that $L(w)$ is the set of atomic formulae true at w

World forces formula: $M, w \Vdash \varphi$ defined by induction on shape of φ

$$M, w \Vdash E(\varphi\, B\, \psi) \quad \text{iff} \quad \exists \sigma \in \mathcal{B}(w).\, \forall i \in \mathbb{N}.\, [M, \sigma_i \Vdash \psi \;\Rightarrow\; \exists j < i.\, M, \sigma_j \Vdash \varphi]$$

"some fullpath from w forces φ before it forces ψ"

$$M, w \Vdash A(\varphi\, B\, \psi) \quad \text{iff} \quad \forall \sigma \in \mathcal{B}(w).\, \forall i \in \mathbb{N}.\, [M, \sigma_i \Vdash \psi \;\Rightarrow\; \exists j < i.\, M, \sigma_j \Vdash \varphi]$$

"every fullpath from w forces φ before it forces ψ"

Note: it is possible that ψ is never forced

Exercises for CTL

Exercise: Show that $M, w \Vdash AX\varphi$ iff $M, w \Vdash \neg EX\neg\varphi$

Exercise: Give semantics for $EF\varphi := E(\top\, U\, \varphi)$ where $\top := p_0 \vee \neg p_0$

Exercise: Give semantics for $AF\varphi := A(\top\, U\, \varphi)$ where $\top := p_0 \vee \neg p_0$

Exercise: Work out the semantics for $AG\varphi := \neg EF\neg\varphi$

Exercise: Work out the semantics for $EG\varphi := \neg AF\neg\varphi$

Exercise: Why can't we define $AG\varphi := A(\varphi\, U\, \bot)$ where $\bot := p_0 \wedge \neg p_0$

Exercise: Why can't we define $EG\varphi := E(\varphi\, U\, \bot)$ where $\bot := p_0 \wedge \neg p_0$

Exercise: Express $AG\varphi$ and $EG\varphi$ in terms of $A(\cdot\, B\, \cdot)$ and $E(\cdot\, B\, \cdot)$ (resp)

Exercises for CTL

Exercise: Show that $\neg E(\varphi\, U\, \psi) \leftrightarrow A((\neg\varphi)\, B\, \psi)$ is CTL-valid

Exercise: Show that $\neg A(\varphi\, U\, \psi) \leftrightarrow E((\neg\varphi)\, B\, \psi)$ is CTL-valid

Exercise: Show that $E(p\, U\, q) \leftrightarrow q \vee (p \wedge EX\, E(p\, U\, q))$ is CTL-valid

Exercise: Show that $A(p\, U\, q) \leftrightarrow q \vee (p \wedge AX\, A(p\, U\, q))$ is CTL-valid

Tableau Rules for CTL using Smullyan's $\alpha-$ and $\beta-$notation

$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}_1$	$\boldsymbol{\alpha}_2$
$\varphi \wedge \psi$	φ	ψ
$E(\varphi\, B\, \psi)$	$\sim\psi$	$\varphi \vee EX\, E(\varphi\, B\, \psi)$
$A(\varphi\, B\, \psi)$	$\sim\psi$	$\varphi \vee AX\, A(\varphi\, B\, \psi)$
$AG\, \varphi$	φ	$AX\, AG\, \varphi$
$EG\, \varphi$	φ	$EX\, EG\, \varphi$

$\boldsymbol{\beta}$	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$
$\varphi \vee \psi$	φ	ψ
$E(\varphi\, U\, \psi)$	ψ	$\varphi \wedge EX\, E(\varphi\, U\, \psi)$
$A(\varphi\, U\, \psi)$	ψ	$\varphi \wedge AX\, A(\varphi\, U\, \psi)$
$EF\, \varphi$	φ	$EX\, EF\, \varphi$
$AF\, \varphi$	φ	$AX\, AF\, \varphi$

Define: $\sim\psi := NNF(\neg\psi)$

Proposition: all instances of $\boldsymbol{\alpha} \leftrightarrow \boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2$ and $\boldsymbol{\beta} \leftrightarrow \boldsymbol{\beta}_1 \vee \boldsymbol{\beta}_2$ are CTL-valid

Note: some of these equivalences require that R is serial/total

Tableau Rules: assuming that all formulae are in Negation Normal Form

$$(\boldsymbol{\alpha})\ \frac{\Gamma; \boldsymbol{\alpha}}{\Gamma; \boldsymbol{\alpha}_1; \boldsymbol{\alpha}_2} \qquad (\boldsymbol{\beta})\ \frac{\Gamma; \boldsymbol{\beta}}{\Gamma; \boldsymbol{\beta}_1 \mid \Gamma; \boldsymbol{\beta}_2} \qquad (EX)\ \frac{\Gamma; EX\varphi; AX\Delta}{\varphi; \Delta}$$

Exercise: if numerator is CTL-satisfiable then so is some denominator

Tableau Calculus for CTL: Phase 1

State Node: a set of formulae of the form $\Lambda, EX\Gamma, AX\Delta$ where Λ contains only atoms and negated atoms

Repeat:

Saturate: repeatedly apply the (α) and (β) rules until none are applicable to give leaves (states) of the form $\Lambda, EX\varphi_1, \cdots, EX\varphi_n, AX\Delta$

Jump: For each state, create n (EX)-children $w_1, \cdots, w_n$ where w_i contains φ_i, Δ

Loop Check: Don't expand a node that duplicates another node

Until no rules are applicable

Tableau Method for CTL: Phase 2 (Almost)

Eventuality: Each formula $E(\varphi\, U\, \psi)$/$A(\varphi\, U\, \psi)$ is an eventuality since it entails that eventually ψ must become true on some/every path

Fulfilled: $E(\varphi\, U\, \psi) \in s$ is fulfilled if there is some path $s_0 = s, s_1, \cdots$ from s such that there exists a k such that $\psi \in s_k$ and $\varphi \in s_j$ for all $j < k$

Fulfilled: $A(\varphi\, U\, \psi) \in s$ is fulfilled if for every path $s_0 = s, s_1, \cdots$ from s there exists a k such that $\psi \in s_k$ and $\varphi \in s_j$ for all $j < k$

Repeat:

- ➤ delete all nodes that contain a pair $\{p, \neg p\}$
- ➤ delete any states with no R-successor (seriality)
- ➤ delete any node that contains an un-fulfilled eventuality

Until: no state is deleted

But this can give the wrong answer as the “unwinding” is more subtle due to branching nature of CTL-models

Tableau Method for CTL: Phase 2

Eventuality: Each formula $E(\varphi\, U\, \psi)$/$A(\varphi\, U\, \psi)$ is an eventuality since it entails that eventually ψ must become true on some/every path

Fulfilled: $E(\varphi\, U\, \psi) \in s$ is fulfilled if there is some path $s_0 = s, s_1, \cdots$ from s such that there exists a k such that $\psi \in s_k$ and $\varphi \in s_j$ for all $j < k$

Fulfilled: $A(\varphi\, U\, \psi) \in s$ is fulfilled if the graph can be unwound in a complicated way (see Emerson)

Repeat:
- ➤ delete all nodes that contain a pair $\{p, \neg p\}$
- ➤ delete any states with no R-successor (seriality)
- ➤ delete any node that contains an un-fulfilled eventuality

Until: no state is deleted

Example: $AGp \to AGp$

$$\neg(AG\,p \to AG\,p)$$

$$AG\,p\ ;\ \neg AG\,p \xrightarrow{nnf} AG\,p\ ;\ EF\,\neg p \xrightarrow{\boldsymbol{\alpha}} p\ ;\ AXAG\,p\ ;\ EF\,\neg p$$

$$p\ ;\ AXAG\,p\ ;\ EF\,\neg p \xrightarrow{\boldsymbol{\beta}_1} p\ ;\ AXAG\,p\ ;\ \neg p$$

$$p\ ;\ AXAG\,p\ ;\ EF\,\neg p \xrightarrow{\boldsymbol{\beta}_2} p\ ;\ AXAG\,p\ ;\ EXEF\,\neg p \xrightarrow{EX} AG\,p\ ;\ EF\,\neg p$$

$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}_1$	$\boldsymbol{\alpha}_2$
$AG\,\varphi$	φ	$AXAG\,\varphi$

$\boldsymbol{\beta}$	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$
$EF\,\varphi$	φ	$EXEF\,\varphi$

$$(EX)\ \frac{\Gamma;\ EX\varphi;\ AX\Delta}{\varphi;\ \Delta}$$

Example: $AGp \rightarrow AGp$ Pruning Phase

$$
\boxed{\neg(AG\,p \rightarrow AG\,p)}
$$

$$
\begin{tikzcd}
\boxed{AG\,p\ ;\ \neg AG\,p} \arrow[d, "nnf"'] & \\
\boxed{AG\,p\ ;\ EF\,\neg p} \arrow[d, "\boldsymbol{\alpha}"'] & \\
\boxed{p\ ;\ AXAG\,p\ ;\ EF\,\neg p} \arrow[dr, "\boldsymbol{\beta}_2"] & \\
& \boxed{p\ ;\ AXAG\,p\ ;\ EXEF\,\neg p} \arrow[d, "EX"'] \\
& \boxed{AG\,p\ ;\ EF\,\neg p}
\end{tikzcd}
$$

Prune the node containing $\{p, \neg p\}$

Example: $AGp \rightarrow AGp$ Pruning Phase

$$\boxed{\neg(AG\,p \rightarrow AG\,p)}$$

$$\begin{tikzcd}
\boxed{AG\,p\,;\ \neg AG\,p} \arrow[d, "nnf"'] & \\
\boxed{AG\,p\,;\ EF\,\neg p} \arrow[d, "\boldsymbol{\alpha}"'] & \\
\boxed{p\,;\ AXAG\,p\,;\ EF\,\neg p} \arrow[dr, "\beta_2"] & \\
& \boxed{p\,;\ AXAG\,p\,;\ EXEF\,\neg p} \arrow[d, "EX"'] \\
& \boxed{AG\,p\,;\ EF\,\neg p}
\end{tikzcd}$$

Prune the root containing $EF\,\neg p$ since no path fulfils $F\,\neg p$

That is, $AG\,p\,;\ EF\,\neg p$ is not CTL-satisfiable.

Hence $AG\,p \rightarrow AG\,p$ is CTL-valid.

Further Reading

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