# Decompositions and EFFICIENT ALGS.

ACT 9.0

## (COURCELLE'S THEOREM)

DEF. $MSO_2$ logic (of graphs) = the language having a variables

- for vertices $(x, y, \ldots)$, edges $(e, f, \ldots)$ and their sets.
- usual constructs $=, x \in X, \wedge, \neg, \ldots$
- ~~the~~ incidence relation of $G$: $inc(v, e)$,
- the quantifiers $\exists x, \exists e, \exists X, \exists F$.

$G \models \varphi$ - graph is model, it is implicit on all formulas

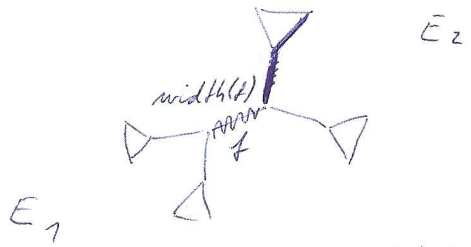
$$HAM \equiv \exists F \Big[ \forall x \exists f, f' \in F \big( f \neq f' \wedge inc(x, f) \wedge inc(x, f') \big)$$

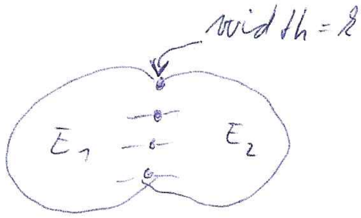
$$\wedge \forall x \forall f, f', f'' \in F \Big( \big( inc(x, f) \wedge inc(x, f') \wedge inc(x, f'') \big) \rightarrow \big( f = f' \vee f = f'' \vee f' = f'' \big) \Big)$$

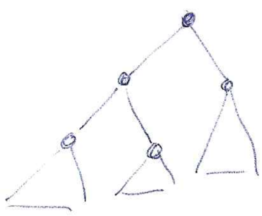
$$\wedge \forall X \Big( \forall x : x \notin X \vee \forall x : x \in X \vee \exists u, v, f \big( inc(u, f) \wedge inc(v, f) \wedge f \in F \wedge u \in X \wedge v \notin X \big) \Big) \Big]$$

**Def:** $k$-bounded graph $\bar{G}$ ($\in \mathcal{U}_k$) is
$\bar{G} = (G, B)$ where $B : \{1, \dots, k\} \to V(G)$
inj (maybe partial)

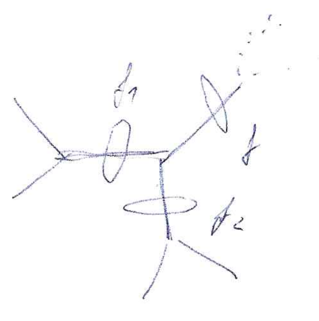
**branch decomp:** cubic tree $T$ plus mapping
$\tau : E(G) \to \text{leaves}(T)$ biject



com. oper. at n is some identif rel on
$\{1, \dots, k\} \times \{f_1, f_2, f_3\}$



# THE CORE OF COURCELLE:

DEF: Let $\varphi$ be a formula ($MSO_2$?).

a $\varphi$-equipped $k$-bd. graph $\overline{G}^\varphi$ is a tuple of

- $k$-bd. gr. $\overline{G}$
- for every vertex(edge) and vertex $x$, an interpretation $q(x) \in V(G)$ $(\in E(G))$,
- and for every set var $X$ an interpretation $q(X) \in 2^{V(G)}$ $\left(\in 2^{E(G)}\right)$.

$\overline{G}^\varphi$ is $\varphi$-partially-equipped if it is allowed $q(x) = \perp$ undefined.

DEF: Having two $\varphi$-part-equipped $k$-bd graphs $\overline{G}_1^\varphi$, $\overline{G}_2^\varphi$, their join $\overline{G}_1 \otimes \overline{G}_2$ has a (full) $\varphi$-equipment if the following is true:

- For every vertex variable $x \in Free(\varphi)$,
  either $\exists i \in \{1, \dots, k\} : q_1(x) = b_1(i) \wedge q_2(x) = b_2(i)$, etc.
  or $q_1(x) = \perp \wedge q_2(x) \in V(G_2) \setminus b_2(\{1, \dots k\})$ or vice versa.
- For every edge var. $f \in Free(\varphi)$, exactly one of $q_1(f), q_2(f)$ is not $\perp$.
- For every vertex-set var $X \in Free(\varphi)$, it holds
  $\forall i = 1, \dots, k : b_1(i) \in q_1(X) \Leftrightarrow b_2(i) \in q_2(X)$.

~~at the~~

The resulting equipment (interpretation) of $Free(\varphi)$ in $\overline{G_1 \otimes G_2}$ is naturally obtained by taking the defined one for elements, and the set union for sets.

NOTATION: $\mathcal{U}_k^\varphi$ is the universe of all $k$-bd. $\varphi$-part.-equipped graphs.

DEF: Cannonical equivalence of $\varphi$ over $\mathcal{U}_k^\varphi$ is $\overline{G}_1^\varphi \simeq_{\varphi,k} \overline{G}_2^\varphi$

if and only if $\forall H^\varphi \in \mathcal{U}_k^\varphi: \overline{G}_1^\varphi \otimes \overline{H}^\varphi \models \varphi \iff \overline{G}_2^\varphi \otimes \overline{H}^\varphi \models \varphi$.

(where $\overline{G}_i^\varphi \otimes \overline{H}^\varphi \models \varphi$ is ~~False~~ UNDEFINED. if the joint $\varphi$-equipmen is undefined.)

(or use both $\varphi$ and $\neg\varphi$)

THM (THE CORE OF COURCELLE)

If $\varphi$ is an $MSO_2$ formula, then for each $k \in \mathbb{N}$, $\simeq_{\varphi,k}$ has finite index (= # of classes).

Proof (by induction on the structure $\varphi$):

BASE:
- $\varphi \equiv (x = y)$: has three classes $k+1$ classes
  $g(x) \neq g(y)$ or $g(x) = g(y) = \bot$
  or $g(x) = g(y) \in V(G)$ makes $k+1$ classes by $\beta^{-1}(g(x))$.
- $\varphi \equiv (f = f')$ has 3 classes
- $\varphi \equiv (x \in X)$ has $\leq$ three times (based on $g(x) = \bot$, or $g(x) \notin g(X)$ def., or $g(x) \in g(X)$)
  $\leq k \cdot 2^k$ classes (based on $\beta^{-1}(g(x))$).
- $\varphi \equiv (f \in F)$ has 3 classes.

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- $\psi \equiv inc(x_1, f)$ has $- g(x) = g(f) = \bot$, or

$g(f) \in E(G)$ and $g(x)$ is not an end of $g(f)$, or

$g(x)$ is an end of $g(f)$ giving let classes, or

$\boxed{g(f) = \bot \text{ and } g(x) \in B^{-1}(\{1, \dots, k\}) \text{ giving } k \text{ classes.}}$

$\leftarrow$ again $k+1$ classes

STEP :

- $\psi \equiv \neg \varphi$ ... troubles :-( ... $\uparrow$

- $\psi \equiv (\psi_1 \wedge \psi_2)$, then $\mathrm{index}(\cong_{\psi,k}) \leq \mathrm{index}(\cong_{\psi_1,k}) \cdot \mathrm{index}(\cong_{\psi_2,k})$.

- (important case)

$\psi \equiv \exists x. \varphi$ ... different equipments.

assume $\bar{G}_1^{\psi} \not\cong_{\psi,k} \bar{G}_2^{\psi}$, meaning that $\exists \bar{H}^{\psi} \in \mathcal{U}_k^{\psi}$ and $\exists u \in V(G_1) \cup \{\bot\}$

s.t. $\bar{G}_1^{\psi}[g(x)=u] \otimes \bar{H}^{\psi} \models \varphi$, but $\forall v \in V(G_2) \cup \{\bot\}$,

it is $\bar{G}_2^{\psi}[g(x)=v] \otimes \bar{H}^{\psi} \not\models \varphi$.

... getting to

$\sim 2^{\mathrm{index}(\cong_{\varphi,k})}$ classes.