Def: The canonical equivalence of ψ over $\mathcal{U}_k^\psi$ is defined as $\bar G_1^\psi \cong_{\psi,k} \bar G_2^\psi$ $(\in \mathcal{U}_k^\psi)$ iff, for all $\bar H^\psi \in \mathcal{W}_k^\psi$, the following holds:

Whenever one of $\bar G_1^\psi \otimes \bar H^\psi$, $\bar G_2^\psi \otimes \bar H^\psi$ has a full ψ-equipment, then both of them do, and, moreover $\bar G_1^\psi \otimes \bar H^\psi \models \psi \Leftrightarrow \bar G_2^\psi \otimes \bar H^\psi \models \psi$.

Thm. For MSO_2 ψ, $\cong_{\psi,k}$ has a finite index

Proof .... base cases OK

Induction on ψ: $\psi = \neg\varphi$ - the same equivalence classes

$\psi = \varphi_1 \wedge \varphi_2$ - $\cong_{\psi,k}$ is a coarsening of $\cong_{\varphi_1,k} \cap \cong_{\varphi_2,k}$

$\psi = \exists x \varphi$: assume $\bar G_1^\psi \not\cong_{\psi,k} \bar G_2^\psi$, then there exists some $\bar H^\psi \in \mathcal{W}_k^\psi$ "distinguishing" them. Firstly if $\bar G_1^\psi \otimes \bar H^\psi$ has a full ψ-equipment while $\bar G_2^\psi \otimes \bar H^\psi$ does not, then also $G_1^\psi[q(x)=\bot] \otimes \bar H^\psi[q(x)=v]$ has a full φ-equipment, while for G_2 it does not.

Hence, $G_1^\psi[q(x)=\bot]$ ~~is not equ~~ $\not\cong_{\varphi,k} G_2^\psi[q(x)=\bot]$ over $\mathcal{U}_k^\varphi$!

Secondly, if $\bar G_1^\psi \otimes \bar H^\psi \models \psi \not\Leftrightarrow \bar G_2^\psi \otimes \bar H^\psi \models \psi$, then (by semantics of $\exists$), up to symmetry, for some $u_1 \in V(\bar G_1 \otimes \bar H)$ ~~such that~~ we have $(\bar G_1^\psi \otimes \bar H^\psi)[q(x)=u_1] \models \varphi$, but for $u_2 \in V(\bar G_2 \otimes \bar H)$ it is $(\bar G_2^\psi \otimes \bar H^\psi)[q(x)=u_2] \not\models \varphi$.

Suppose $u_1 \notin V(\bar G_1)$ ~~Then~~ and denote by $\bar H_1^\psi = \bar H^\psi[q(x)=u_1]$. Then $\bar G_1^\psi[q(x)=\bot] \otimes \bar H_1^\psi \models \varphi$, but $\bar G_2^\psi[q(x)=\bot] \otimes \bar H_1^\psi \not\models \varphi$, and hence $\bar G_1^\psi[q(x)=\bot] \not\cong_{\varphi,k} \bar G_2^\psi[q(x)=\bot]$ over $\mathcal{U}_k^\varphi$!

~~If u_1 is on the boundary of $\bar{G}_1$, then the case is analogical.~~

Now suppose $u_1 \in V(\bar{G}_1)$ but not on the boundary, and denote $\bar{H}_0^{\varphi} = \bar{H}^{\varphi}[q(x) = \bot]$. Then $\bar{G}_1^{\varphi}[q(x)=u_1] \otimes \bar{H}_0^{\varphi} \models \varphi$, but $\bar{G}_2^{\varphi}[q(x)=u_2] \otimes \bar{H}_0^{\varphi} \not\models \varphi$ for every $u_2 \in V(\bar{G}_2)$.

(*) Hence $\{ [\bar{G}_1^{\varphi}[q(x)=u_a]]_{\cong_{\varphi,k}} : u_a \in V(\bar{G}_1) \text{ not boundary} \}$
$\neq \{ [\bar{G}_2^{\varphi}[q(x)=u_a]]_{\cong_{\varphi,k}} : u_a \in V(\bar{G}_2) \text{ not boundary} \}$.

Lastly, suppose u_1 on the boundary of $\bar{G}_1$, and denote $\bar{H}_1^{\varphi} = \bar{H}^{\varphi}[q(x)=u_1']$ where $\beta^{-1}(u_1') = \beta^{-1}(u_1)$. Then $\bar{G}_1^{\varphi}[q(x)=u_1] \otimes \bar{H}_1^{\varphi} \models \varphi$, but $\bar{G}_2^{\varphi}[q(x)=u_2] \otimes \bar{H}_1^{\varphi} \not\models \varphi$ where $\beta^{-1}(u_2) = \beta^{-1}(u_1)$. Hence (*) again. □

The other cases can be proved. $\Downarrow$ $\text{index}(\cong_{\varphi,k}) \leq O(2^{\text{index}(\cong_{\varphi,k})})$

COR1: Every MSO_2-definable decision problem can be, for every $k \in N$, solved by a finite tree automaton running on k-boundaried parse trees.

Proof: A prop. P has closed MSO_2 formulas π, then $\cong_{\pi}$ has finitely many classes, and by the Myhill-Nerode theorem there exists a tree automaton A for accepting $G \models \pi$.

COR2: What can be solved with MSO_2 formulas φ with free variables?:

- ▲ count the ~~sat~~ number of equipments of free ~~var~~ var st. $G \models \varphi(\ldots)$
- ▲ generate all sat. eq. of free variables with polynomial delay
- ▲ solve optim. problems of the form $\max\{\varepsilon(\ldots) : G \models \varphi(\ldots)\}$ where ε is linear evaluational function (such as set cardinality)