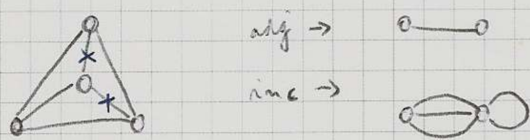
[illegible]

III CONNECTIVITY, STRUCTURE AND MINORS

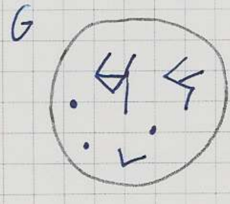
MINOR def. Graph H is a minor of G if H results from a subgraph of G by contraction of edges.

two defs. in one! (2 models)

$\boxed{H \leq G}$



prop. Let H result from G by a sequence of vertex/edge deletions and edge contractions. Then H is a minor of G



idea of proof: We trace back all the contracted edges and get some components of G. We select the spanning tree of each component, other ~~the~~ edges are deleted. Finally we contract all components to vertices. => We can do all the deletions first before all the contractions.

LEMMA: H is a minor of G if and only if there exist a collection of pairwise disjoint connected subgraphs $F_v \subseteq G$, where $v \in V(H)$, such that $uv \in E(H)$ implies some edge in G between $V(F_u)$ and $V(F_v)$.

only for adj. model

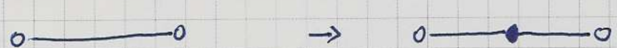
proof: $H \leq G$ means a subgraph $G_1 \subseteq G$ and set $E_1 \subseteq E(G_1)$ to be contracted (to get H). Then every connected component induced by E_1 in G_1 corresponds to one vertex w of H, and we denote it by $F_w \subseteq G_1 \subseteq G$

conv., assume those F_v, $v \in V(H)$. WLOG let each F_v be a tree. Let $G_2 \subseteq G$ be a subgraph on the vertices $\bigcup_{v \in V(H)} V(F_v)$, and assume G_2 is edge-minimal satisfying the assumptions of the lemma. Then H results from G_2 by contractions of all $\bigcup_{v \in V(H)} E(F_v)$. □

LEMMA *: H is a minor of G if and only if there exists $F_v \subseteq G$ for $v \in V(H)$ disjoint connected such that
- if there are $k > 0$ parall. edges between u, v in H, then G has $\geq k$ edges between $V(F_u)$ and $V(F_v)$.
- if there are $l > 0$ loops over u in H, then G has $\geq l$ edges (even loops) between vertices of F_u but not in $E(F_u)$.

proper for incidency model

SUBDIVISION OF AN EDGE



the opposite of subdividing is SUPPRESSING

TOPOLOGICAL MINOR

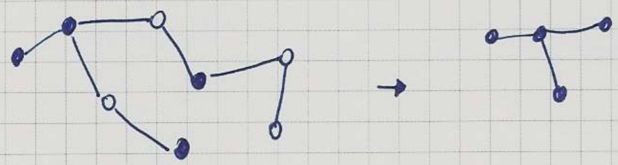
def. A graph H is a subdivision of G if H results from G by replacing some edges of G by new paths (of length >1) that are internally disjoint from G and from each other.

~~We also say that H is a topological minor of G if H is a subdivision of a subgraph~~

H is a topological minor of G if some

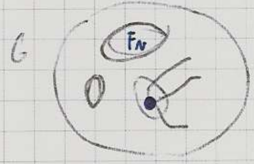
subdivision of H is a subgraph of G.
(The vertices in this subgraph correspond to $V(H)$ are called BRANCHING vertices of this top. minor)

obviously top. minor IS a minor, but not vice versa



PROP. If $\Delta(H) \leq 3$ and H is a minor of G, then H is a topological minor of G.

$\Delta(H)$ in inc. model = number of outgoing lines



BRANCHING VERTEX

LEMMA 2: If H is a minor of G, then there exists H' obtained by a series of splittings in H such that H' is a topol. minor of G.

def. Let $F_v \subseteq G$, $v \in V(H)$ be as in LEMMA 1. We choose $T_v \subseteq F_v$ as the min. interconnecting tree for all the terminals of edges of H ending in v. Then $\Delta(T_v) \geq 2$ and the deg.-2 vertices can be ignored. So, let T'_v result from T_v by supressing the deg.-2 vertices, hence $\cancel{\Delta(T'_v) \geq 3}$. the vertices of T'_v have degree ≥ 3 in G. Then a subgraph isom. to graph T'_v can be obtained in H via splittings of v. We repeat for other vert. in H.

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TREE-STRUCTURED DECOMPOSITION

def. A tree decomposition of a graph G is a pair (T, β) where T is a tree and $\beta : V(T) \to 2^{V(G)}$, such that

- $\bigcup_{x \in V(T)} \beta(x) = V(G)$
- $\forall e \in E(G)\ \cancel{\exists x \in (T)}\ \exists x \in V(T)$ s.t. $e \in \beta(x)$
- $\forall x, y, z \in V(T)$ s.t. y lies on the T path between x and z it holds $\beta(y) \supseteq \beta(x) \cap \beta(z)$ — interpolation property

The sets $\beta(x)$ are called bags of (T, β)

The width of (T, β) is the $\max_{x \in V(T)} |\beta(x)| - 1$.

DEF. The tree-width of G is the minimum width over all tree-decomp. of G.

PROP. G has tree-width 1 if and only if ~~G is a forest~~. a simplification of G is a forest, i.e. iff G has no K_3-minor.

simplification - loops and parallell edges

PROP. The tree-width of G is the same as that of a simplification of G.

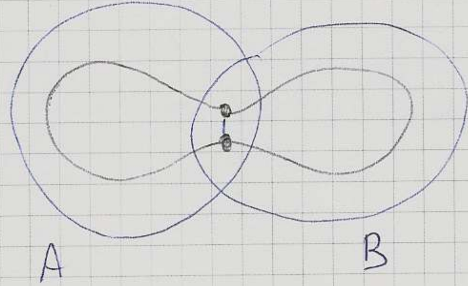
LEMMA3: G has tree-width ≤ 2 if and only if G has no K_4 minor.

LEMMA4: If H is a minor of G, then $tw(H) \leq tw(G)$.

- quite trivial (contractions don't increase tw)

proof of LEMMA 3: $tw(K_4)=3$. and hence $\Rightarrow$ holds.

$\Leftarrow$ 3-connected graphs always have a K_4 minor and so our G has a 2-cut



Make G_A and G_B (both with a new edge on the cut), and recurs. construct tree-decomp. of G_A, G_B of width ≤ 2, then paste together

Rem. SERIES-PARALELL GRAPHS