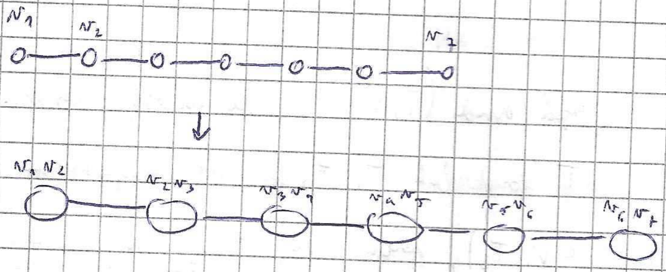
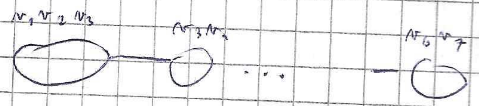
TREE-STRUCTURED DECOMPOSITION

(T, W) T-tree $W = \bigcup_{t \in V(T)} W_t$ $W_t \subseteq V(G)$

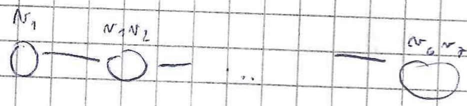
PATHS:



width 1



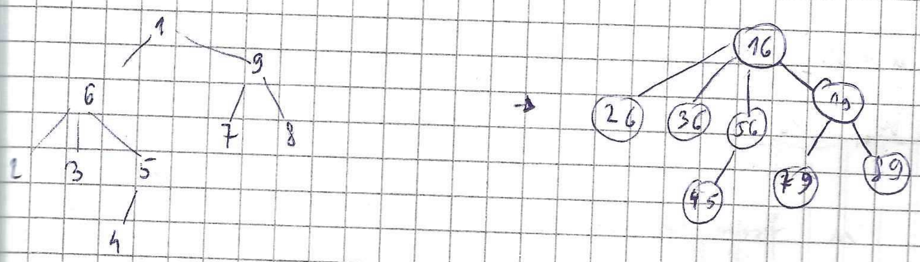
width 2



width 1

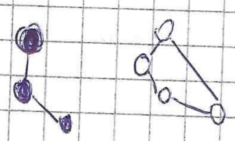
$\Rightarrow$ as many decomposition as we like

TREES:



FOREST

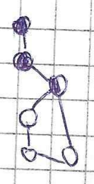
G_1, G_2, G_3 → new root



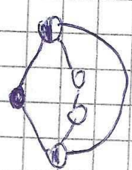
$TW \leq 1$: forests

$TW(G) \leq 2 \Leftrightarrow G$ is series-parallel - fig 1

a)



b)



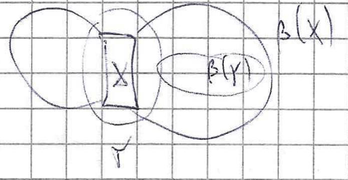
LEMMA 5: $k+1$ cops can always capture the robber on graph G with tree-width$(G)=k$

Proof: by separation lemma

cops are the vertices of some bag

Def. (HAVEN): A haven β of order k assigns to every $X \in [V(G)]^{<k}$ the vertex set of a component of $G \setminus X$ s.t. $X \subseteq Y \in [V(G)]^{<k}$ $\Rightarrow \beta(Y) \subseteq \beta(X)$

havens: strategy for a robber



Def. (bramble)

Two subsets of $V(G)$ touch if they have a vertex in common or G contains edge between them. A set of mutually touching vertex sets (connected) is bramble. $U \subseteq V(G)$ covers bramble $\mathcal{B}$ if it meets every element of $\mathcal{B}$. The least number of vertices to cover $\mathcal{B}$ is the order of $\mathcal{B}$.

Bramble of order n in a grid is the set of crosses $C_{ij} = \{(i,l) \mid l = 1, \dots, n\}$ $\cup \{(l,j) \mid l = 1 \dots n\}$

$\rightarrow n^2$ crosses, order n.

cops •
robber •

Theorem [Seymour, Thomas, 93]

Let G be a graph and $k \geq 1$ an integer. Then the following are equivalent:

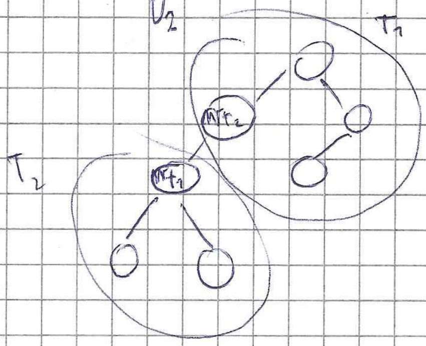
(i) G has a haven of ~~order~~ $\geq k$
(ii) G has a bramble of order $\geq k$
(iii) $<k$ cops cannot catch a robber in G
(iv) $<k$ cops cannot monotonely* capture a robber in G
(v) G has tree-width $\geq k-1$

$tw(G) \le 3 \iff$ G has no minor isomorphic to K_5, 5-prism, octahedron, V8

$tw(K_n) = n-1$

<u>Lemma 1</u> (separation): Let G be a graph and (T,W) its tree decomposition. Let $t_1 t_2$ be an edge of T and let T_1, T_2 be the components of $T - t_1t_2$, $t_1 \in V(T_1)$, $t_2 \in V(T_2)$, then $W_{t_1} \cap W_{t_2}$ separates $\bigcup_{t\in V(T_1)} W_t$ and $\bigcup_{t \in V(T_2)} W_t$ in G.

$= U_1$ $\quad$ $= U_2$



PROOF:

$uv \in E(G)$, $u \in U_1 \setminus U_2$, $v \in U_2 \setminus U_1$

$U_1 \cap U_2 \subseteq W_{t_1} \cap W_{t_2}$

...

<u>Lemma 2</u>: If $H \preceq G$, then $tw(H) \le tw(G)$

Proof: • subgraph $\quad$ dec. of G $(T, \bigcup_{t\in V(T)} W_t) \xrightarrow{\text{then}} (T, \bigcup_{t \in V(T)} (W_t \cap V(H)))$

• edge contraction

H is constructed from G by contracting uv into w

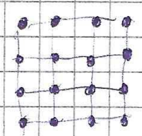
→ replace each u or v by w.

LEMMA 3 : If (T, W) is a tree decomposition of G and K is a complete subgraph of G, then ~~vertices~~ $V(K) \subseteq W_t$ for some $t \in V(T)$

Proof (sketch): Show, given a $W \subseteq V(G)$ there is either a bag W_t, $t \in V(T)$, that contains W, or there are $t_1, t_2 \in V(T)$ such that $W_{t_1} \cap W_{t_2}$ separates some vertices in W

GRIDS ($n \times n$ grid = Cartesian product of two paths of n vertices)

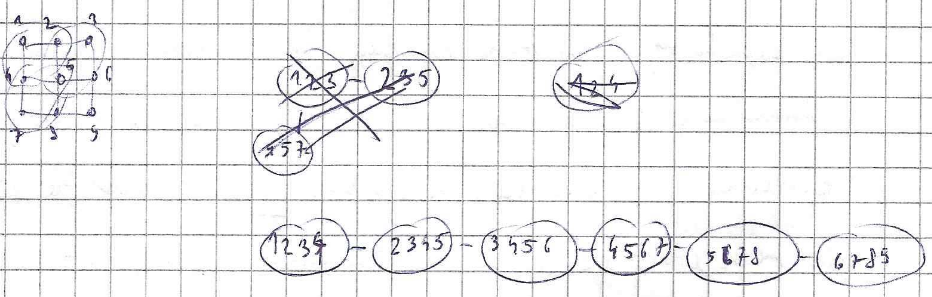
4x4



2x2

tw 2

3x3



tw 3

LEMMA 4 : Tree width of $n \times n$ grid is at most n.

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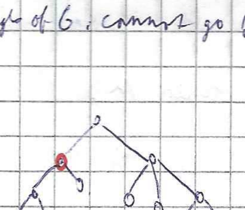
k cops, helicopters

1 robber, move along edge of G, cannot go through a vertex occupied by a cop, moves at infinite speed

• cops
• robber



2 cops on tree
3 cops on a cycle
n cops on a K_n