WELL-QUASI-ORDERING (WQO)

DEF: WQO is binary relation transitive, reflexive and for every infinite seq $x_1, x_2, \ldots$ there exist $i < j$ st. $x_i \leq x_j$. Then (x_i, x_j) is a good pair.

PROP: $\leq$ is WQO iff there is no inf. antichain, nor inf. strictly decreasing chain.

Proof: Let $x_0, x_1, x_2, \ldots$ be an infinite sequence having no good pair.

Colour the pair (i,j) if $i<j$ & $x_i > x_j$ RED otherwise BLUE.

By Ramsei Theorem there exists an infinite subsequence $i_0, i_1, i_2, \ldots$ with all pair of the same colour.

Then $x_{i_0}, x_{i_1}, \ldots$ is inf. antichain or strictly decreasing.

Def: For WQO $\leq$, a set X is downward-closed (lower ideal) if $\forall x \in X$ and $\forall y \leq x : y \in X$.

Prop: If $\leq$ is WQO, then for every down-closed X there is a finite forbidden set Z, such that

$\forall y : y \notin X \iff \exists z \in Z : y \geq z$

Proof (sketch) obstacles infinite many changings
- updating $\Rightarrow$ infinite dec. seq (add & make obsolete)
- enlarging $\Rightarrow$ inf. antichain

DEF: Ordering for sets: $A \leq B$ if there is inj $f: A \to B$
st. $\forall a \in A$: $f(a) \geq a$.

PROP: If $\leq$ is WQO (on elements), then also all finite subsets are WQO by $\leq$.

Proof: Select "minimal" bad sequence $A_0, A_1, A_2, \dots$
(ie. not having good pair) such that $\forall i \geq 0$,
$|A_i|$ is minimal among all bad seq. starting with
$A_0, A_1, \dots A_{i-1}$. Now choose arb. $a_n \in A_n$, $n = 0, 1, \dots$
By WQO there is a inf increasing chain
$a_{n_0} \leq a_{n_2} \leq \dots$, and set $B_i = A_{n_i} \setminus \{a_{n_i}\}$.
The seq $A_0, A_1, \dots, A_{n_0 - 1}, B_0, B_1, \dots$
has $|B_0| < |A_{n_0}|$ and hence it has a good pair
which acc. is $B_i \leq B_j$. But $a_{n_i} \leq a_{n_j}$,
and hence $A_{n_i} \leq A_{n_j}$, contr. $\square$

THM: ~~[illegible]~~
Trees are WQO by topological minor.
Proof: similar to previous one.
We make each tree rooted (anyhow), and define $T_x \leq T_x'$
if a is subdivision of T_x has isomorphic subgr. in T_x' preserving the direction "anyway from root".
Again, we choose a "min" bad sequence $T_0, T_1, T_2, \dots$ s.t. $\forall i \geq 0$ height (T_i)
is minimal possible among all bad seq. starting $T_0, T_1, T_2, \dots T_{i-1}$.
Let D be the set of all r. trees obtained from T_i by deleting the root.
Then D is WQO: Let $W_0, W_1, \dots$ be seq in D, and W_0 be branch of T_n.
The seq $T_0, \dots, T_{n-1}, W_0, W_1, \dots$ is not bad, since height $(W_0) \leq h(T_n)$

So there is a good pair, $W_i \le W_j$.

But then also all finite subsets of D are WQO and the trees T_i are actually finite sets of their branches from D, and so we get $T_i \le T_j$

... a contradiction ↯ $\square$

Look to web / syllabus → choose assignment.