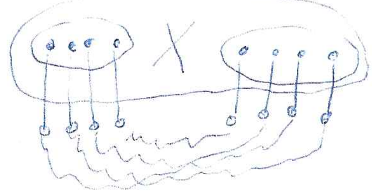
TOWARD GRAPH MINORS THEOREM

DEF.: A set $X \subseteq V(G)$ is externally k-connected if $|X| \geq k$ and for all $Y, Z \subseteq X$ with $|Y| = |Z| \leq k$, there exist $|Y|$ of pairwise-disjoint Y-Z paths in G, which are all internally disjoint from X.



THM: A graph G has tree-width $> t$ if and only if it contains a externally k-connected set for $k = f(t)$

Proof: A partial tree-decomp of width t is such that all bags except possibly some leaf bags have size $\leq t+1$, and no two begs $> t+1$ are neighbours.
We choose a part. tree-decomposition of width t maximizing the number of nodes under the assumption that no two nodes are in an inclusion.
If there is no bag $> t+1$, then $tw(G) \leq t$. OK
So take a leaf bag B of size $t+1$, and let $X \subseteq B$ be the cut separating $G[B]$ from the rest of G.
Then X belongs to the neib. bag of B, and so $|X| \leq k$.
If X is ext. k-connected in G on B, then OK
Otherwise by the def. there are $Y, Z \subseteq X$, $|Y| = |Z| \leq k$, and by Menger, a set $W \subseteq B \setminus X$ which is a cut between Y, Z in $G[B]$; $|W| < |Y|$.

Then we make a new decomposition by adding two new bags $(X \cup w) \setminus z$, $(X \cup w) \setminus y$ and "split" B into two neighbour of the new bags. A CONTRADICTION to maximality ↯ $\square$

THM. (Real): If G has $tw \geq r^{4m^2(r+2)}$ then G has K_m or $r \times r$-grid minor.

THM. ("Kuratowsky for arbitrary surface"): For every surface S, there is a finite number of excluded minors for embeddability into S.

Proof: We show, by contr. that a graph G contains a huge $r \times r$ grid as a minor, cannot be an extended minor for emb. in S.

The rest follows from WQO of bounded tw. graphs.

First, we embed G into some larger surface S' and find a very large subgrid D in G which is flat (the union of its faces is disk) – this D is considered with all "interior" attachements of G.

Let v be a "middle" vertex of D, then v is encircled in the embedded D by very many concentric disjoint cycles $\mathcal{C}$.

By minimality, we embed $G-v$ in S, and find a cycle in $\mathcal{C}$ which is flat. Than we may simply replace the flat side by the corresp. subembeddings of D, which results in a embedding of G in S, a contr.