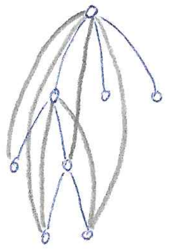
DEF. The <u>closure</u> $cl(T)$ of a rooted tree T is obtained from T by adding all edges $f = \{u, v\}$ where v lies on the path from u to the root.



DEF. The <u>tree-depth</u> of a graph G is the least h such that $G \subseteq cl(F)$ where F is a rooted forest of height h.

THM 1 : The graphs of bounded tree-depth are WQO under induced subgraph.

proof. HOMEWORK

TOWARD GRAPH MINORS THEOREM

KRUSKAL THM (HIGMAN'S LEMMA)

WQO LEMMA ON TREES

WQO MINORS OF TW BOUNDED

IRELEVANT VERTEX TRICK

+ EXCLUDED GRID THEOREM

KURATOWSKI THM. ON ANY SURFICE

DECOMPOSITION OF GRAPHS WITH NO K_m-minor

THE GRAPH MINORS THM.

ALGORITHM FOR LINKAGE

ALGORITMIC GM : TESTING FOR A FIXED VERTEX

NOTATION: Let $(T, (B_t)_{t \in V(T)})$ be a tree-dec. of a graph G.

The *torsos* of G in this decomp. are graphs H_t, $t \in V(T)$ such that H_t is obtained from $G[B_t]$ by adding all the edges $\{x, y\}$ where $x, y \in B_t \cap B_{t'}$ where $\{t, t'\} \in E(T)$.

* SURFACE WITH CUFFS: A cuff = a disk hole in a surface, formally

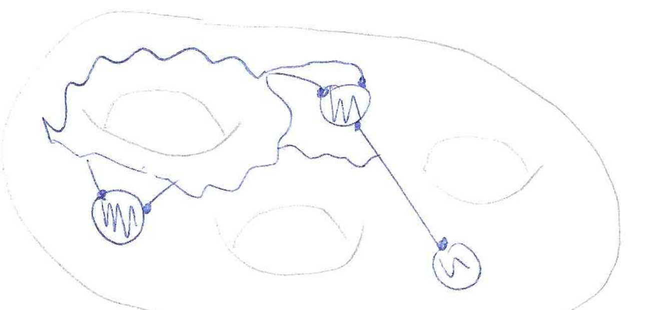
C_i is the image of cont. $f_i : [0,1] \to \Sigma$, $f_i(0) = f_i(1)$: ...

Write $\Sigma - k$ for the surface with k cuffs.

DEF.: A graph H k-nearly embeds into Σ if there exists $X \subseteq V(H)$, $|X| \leq k$, and subgraphs $H_0, H_1, \dots, H_k \subseteq H$ such that $H - X = H_0 \cup H_1 \cup \dots \cup H_k$,

and ① H_0 embeds in $\Sigma - k$ with only vertices mapped into C_i's, form. $\sigma : H_0 \hookrightarrow \Sigma - k$

② $H_1, \dots, H_k$ are pairwise disj., possibly empty, and $H_0 \cap H_i = \sigma^{-1}(C_i)$, $i = 1..k$



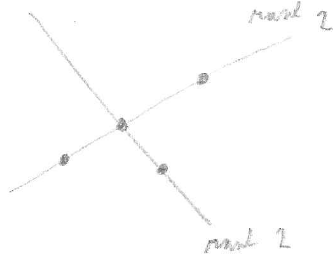
③ each H_i, $i = 1..k$, has a path decomp. $(B_z^i)_{z \in \sigma^{-1}(C_i)}$ st. $z \in B_z^i$

THM 2: For every n there exists a k such that every graph with no K_n-minor has a tree-decomp. whose torsos k-nearly embed on some surfaces to which K_n does not embed.

BRANCH-DECOMPOSITIONS

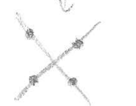
NOTATION: a set function $f: 2^E \to \mathbb{R}$

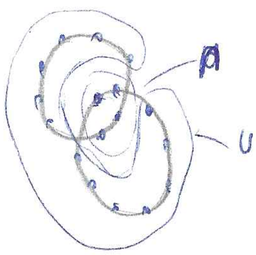
- symmetric $f(X) = f(E \setminus X)$
- submodular $f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$



U rank 3

∩ 0-1





} ~~mod~~ - modular in this case

- integer-valued $\to \mathbb{N}_0$

DEF: Let f be a symmetric set function on E, usually also submodular and integer-valued.

A branch decomposition of E with respect to f is a pair (T, τ), where T is a subcubic tree (no degrees > 3), and τ: ~~map~~ $E \to$ leaves(T) is a bijection.

The width of an edge $e \in E(T)$ is $f(\tau^{-1}(L_e)) = f(\tau^{-1}(\overline{L}_e))$

The width of (T, τ) is the maximum of the widths of its edges, and the branch-width of E wrt f is the min. possible width of a branch-decomp.

EXAMPLE: the branch-width of a graph is defined on ~~the~~ $E(G)$ with $f(X)$, $X \subseteq E(G)$, is the number of vertices shared between X and $E(G) \setminus X$

RANK-WIDTH: Now $E = V(G)$, and $f(X) = \text{rank}\left(\begin{array}{cc} & V \setminus X \\ X & \boxed{\begin{matrix} 0 & 0 & 1 \\ 1 & & \end{matrix}} \end{array} \right)$ over $GF(2)$.