Parciální zlomky

$R(x) = \frac{P(X)}{Q(X)}$ se nazývá racionální lomená funkce, (RLF)

jestliže $P(x)$, $Q(x)$ jsou polynomy.

je-li $\underline{\mathrm{st}\, P(x) < \mathrm{st}\, Q(x)}$ je $\underline{R(x)}$ ryze racionální lom. fce. (RRLF)

Věta: nechť $R(x) = \frac{P(x)}{Q(x)}$ a nechť $P(x), Q(x)$ nemají společné kořeny a $Q(x) = a_n \cdot (x-\alpha_1)^{r_1} \cdots (x-\alpha_n)^{r_n} \cdot ((x-a_1)^2+b_1^2)^{s_1} \cdot \ldots \cdot ((x-a_m)^2+b_m^2)^{s_m}$ je rozklad v $\mathbb{R}$.

Pak existují A_i, B_i, M_i, ... tak, že

$$R(x) = \left(\frac{A_{r_1}}{(x-\alpha_1)^{r_1}} + \ldots + \frac{A_1}{x-\alpha_1}\right) + \ldots + \left(\frac{B_{r_n}}{(x-\alpha_n)^{r_n}} + \ldots + \frac{B_1}{x-\alpha_n}\right) +$$

$$+ \left(\frac{M_{s_1} \cdot x + N_{s_1}}{((x-a_1)^2+b_1)^{s_1}} + \ldots + \frac{M_1 x + N_1}{(x-a_1)^2+b_1}\right) + \ldots +$$

$$+ \left(\frac{U_{s_m} x + V_{s_m}}{((x-a_m)^2+b_m)^{s_m}} + \ldots + \frac{U_1 x + V_1}{(x-a_m)^2+b_m^2}\right)$$

Ⓟř Zapište RLF jako součet polynomu a RRLF, tu pak rozložte na parciální zlomky (koeficienty nepočítejte).

$$R(x) = \frac{x^5 - x^4 + 6x^2 + x - 2}{x^4 - 2x^3}$$

$$\left(x^5 - x^4 + 6x^2 + x - 2\right) : \left(x^4 - 2x^3\right) = x + 1 + \frac{2x^3 + 6x^2 + x - 2}{x^4 - 2x^3}$$

$$R(x) = x + 1 + \frac{2x^3 + 6x^2 + x - 2}{x^3(x-2)} = x + 1 + \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-2}$$

Ⓟř Rozložte na tvar parciálních zlomků:

$$\frac{x+1}{(x^2+1)(x^3+x)} = \frac{x+1}{(x^2+1)^2 \cdot x} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1} + \frac{E}{x}$$

Ⓟⓡ Rozložte na parciální zlomky.

$$\frac{x-1}{x^4+3x^2+2} = \frac{x-1}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} = \frac{x-1}{x^2+1} + \frac{-x+1}{x^2+2}$$

$$\frac{x-1}{(x^2+1)(x^2+2)} = \frac{(Ax+B)(x^2+2)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+2)} = \frac{x^3(A+C)+x^2(B+D)+x(2A+C)+(2B+D)}{(x^2+1)(x^2+2)}$$

porovnáním koeficientů polynomů v čitateli

$$\left.\begin{aligned} x^3&: \quad 0 = A+C \\ x^2&: \quad 0 = B+D \\ x^1&: \quad 1 = 2A+C \\ x^0&: \quad -1 = 2B+D \end{aligned}\right\} \quad \begin{aligned} &A=1,\ C=-1 \\ &B=-1, D=1 \end{aligned}$$

Postup: 1) udělat RRLF z RLF dělením polynomů

2) rozložit polynom jmenovatele v RRLF na kořenové činitele v $\mathbb{R}$ nebo na polynomy druhého stupně, které nemají kořen v $\mathbb{R}$ (mají kořeny v $\mathbb{C}$)

3) odpovídající reálným kořenům

$$\frac{P(x)}{(x-\alpha)^n} = \frac{A}{(x-\alpha)^n} + \frac{B}{(x-\alpha)^{n-1}} + \dots + \frac{C}{(x-\alpha)}$$

odpovídající nereálným kořenům

$$\frac{P(x)}{(x^2+2)^n} = \frac{Ax+B}{(x^2+2)^n} + \frac{Cx+D}{(x^2+2)^{n-1}} + \dots + \frac{Ex+F}{x^2+2}$$

} zjednodušeně řečeno, jinak napsáno ve větě.

Integrace racionální lomené funkce

Ⓘ $\int \frac{A}{x-x_0}\,dx = A\int\frac{dx}{x-x_0} = A\cdot\ln|x-x_0| + C$

Pr $\int \frac{x}{(x-1)(x^2-4)}\,dx = \int\frac{x}{(x-1)(x-2)(x+2)}\,dx = \int\left(\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+2}\right)dx =$

$= A\ln|x-1| + B\ln|x-2| + C\ln|x+2| + D$

Ⓘⅰ $\int \frac{A}{(x-x_0)^n}\,dx = A\int\frac{dx}{(x-x_0)^n} = \left|\begin{matrix} S: \\ x-x_0=t \\ dx=dt\end{matrix}\right| = A\int\frac{1}{t^n}dt = A\cdot\frac{t^{-n+1}}{-n+1} + C = \cancel{\frac{A}{1-n}\cdot(x-x_0)^{1-n}}$

$= \frac{A}{1-n}\cdot\frac{1}{(x-x_0)^{n-1}} + C$

Pr $\int\frac{1}{x(x+1)^3}\,dx = \int\left(\frac{A}{x} + \frac{B}{(x+1)^3} + \frac{C}{(x+1)^2} + \frac{D}{x+1}\right)dx = A\cdot\ln|x| + D\cdot\ln|x+1| - \frac{C}{x+1} - \frac{B}{2(x+1)^2} + E$

Ⓘⅱ $\int\frac{Bx+C}{(x-x_0)^2+a^2}\,dx = \int\frac{\frac{B}{2}(2x-2x_0) + Bx_0+C}{(x-x_0)^2+a^2}\,dx = \frac{B}{2}\int\frac{2x-2x_0}{(x-x_0)^2+a^2}\,dx + (Bx_0+C)\cdot\int\frac{dx}{(x-x_0)^2+a^2}\,dx =$

$= \frac{B}{2}\ln\left((x-x_0)^2+a^2\right) + \frac{Bx_0+C}{a}\operatorname{arctg}\frac{x-x_0}{a} + D$

$\int\frac{dx}{(x-x_0)^2+a^2} = \frac{1}{a^2}\int\frac{dx}{\left(\frac{x-x_0}{a}\right)^2+1} = \left|\begin{matrix}\frac{x-x_0}{a}=t \\ \frac{1}{a}dx=dt\end{matrix}\right| = \frac{1}{a}\int\frac{dt}{t^2+1} = \frac{1}{a}\operatorname{arctg} t + D =$

$= \frac{1}{a}\operatorname{arctg}\frac{x-x_0}{a} + D$

Ⓑ

Př

$$\int \frac{x+1}{(x-1)(x^2-2x+5)}\,dx = \int\left(\frac{A}{x-1} + \frac{Bx+C}{x^2-2x+5}\right)dx =$$

$$= A\cdot \ln|x-1| + \frac{B}{2}\ln\left(x^2-2x+5\right) + \frac{B+C}{2}\,\mathrm{arctg}\,\frac{x-1}{2} + D.$$

$$\int \frac{Bx+C}{x^2-2x+5}\,dx = \int \frac{\frac{B}{2}(2x-2) + B + C}{x^2-2x+5}\,dx = \frac{B}{2}\ln\left(x^2-2x+5\right) + (B+C)\cdot\int\frac{dx}{(x-1)^2+4} =$$

$$= \frac{B}{2}\ln\left(x^2-2x+5\right) + \frac{B+C}{2}\,\mathrm{arc\,tg}\,\frac{x-1}{2} + D$$

IV

$$\int \frac{Bx+C}{\left((x-x_0)^2+a^2\right)^n}\,dx = \int \frac{\frac{B}{2}(2x-2x_0) + Bx_0 + C}{\left((x-x_0)^2+a^2\right)^n}\,dx = \left|\begin{array}{l}(x-x_0)^2+a^2 = t\\ (2x-2x_0)dx = dt\end{array}\right| =$$

$$= \frac{B}{2}\int\frac{dt}{t^n} + (Bx_0+C)\int\frac{dx}{\left((x-x_0)^2+a^2\right)^n} =$$

$$= \frac{B}{2(1-n)}\,\frac{1}{t^{n-1}} + (Bx_0+C)\int\frac{dx}{\left((x-x_0)^2+a^2\right)^n} =$$

$$= \frac{B}{2(1-n)}\,\frac{1}{\left((x-x_0)^2+a^2\right)^{n-1}} +$$

$$\int \frac{dx}{\left((x-x_0)^2+a^2\right)^m} =: K_m$$

$$K_{m+1} = \frac{1}{a^2}\left(\frac{2m-1}{2m} K_m + \frac{1}{2m} \frac{x-x_0}{\left((x-x_0)^2+a^2\right)^m}\right)$$

$$K_1 = \int \frac{dx}{(x-x_0)^2+a^2} = \frac{1}{a} \operatorname{arctg} \frac{x-x_0}{a}$$

Př

$$\int \frac{dx}{(x+1)(x^2+x+1)^2} = \int \frac{A}{x+1}dx + \int \frac{Bx+C}{(x^2+x+1)^2}dx + \int \frac{Dx+E}{x^2+x+1}dx =$$

$$= \boxed{A \cdot \ln|x+1|} + \int \frac{\frac{D}{2}(2x+1) - \frac{D}{2} + E}{x^2+x+1} dx + \int \frac{\frac{B}{2}(2x+1) - \frac{B}{2} + C}{(x^2+x+1)^2} dx$$

$$\int \frac{\frac{D}{2}(2x+1) - \frac{D}{2} + E}{x^2+x+1} dx = \frac{D}{2} \ln(x^2+x+1) - \left(\frac{D}{2} - E\right)\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} =$$

$$\boxed{= \frac{D}{2} \ln(x^2+x+1) - \left(\frac{D}{2} - E\right) \cdot \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{x-\frac{1}{2}}{\frac{2}{\sqrt{3}}}}$$

$$\int \frac{\frac{B}{2}(2x+1) - \frac{B}{2} + C}{(x^2+x+1)^2} dx = \frac{B}{2}\int \frac{dt}{t^2} - \left(\frac{B}{2} - C\right)\int \frac{dx}{\left((x+\frac{1}{2})^2+\frac{3}{4}\right)^2} =$$

$$\boxed{= \frac{B}{2} \cdot \frac{-1}{x^2+x+1} - \left(\frac{B}{2} - C\right) \cdot K_2}$$

$$\boxed{K_2 = \frac{1}{\frac{3}{4}}\left(\frac{2\cdot 1-1}{2\cdot 1} \cdot K_1 + \frac{1}{2\cdot 1} \cdot \frac{x+\frac{1}{2}}{\left((x+\frac{1}{2})^2+\frac{3}{4}\right)^1}\right) = \frac{4}{3}\left(\frac{1}{2} \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + \frac{1}{2} \frac{x+\frac{1}{2}}{x^2+x+1}\right)}$$

Ⓓ

Typické substituce

$R(a^x)$ S: $a^x = t$

Ⓟř $\int_0^\infty \frac{dx}{e^x+e^{-x}} = \left| \begin{matrix} e^x = t \\ e^x dx = dt \end{matrix} \right| = \int \frac{dt}{t(t+t^{-1})} = \int_1^\infty \frac{dt}{t^2+1} = \operatorname{arctg} t + C = \operatorname{arctg} e^x + C$

$\Downarrow$

$dx = \frac{dt}{t}$

$\int_0^\infty = \operatorname{arctg} \infty - \operatorname{arctg} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \underline{\frac{\pi}{4}}$

$R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)$ S: $\frac{ax+b}{cx+d} = t^n$

Ⓟř $\int \sqrt[3]{\frac{x-1}{x+1}} \cdot \frac{1}{x-1}\, dx = \left| \begin{matrix} \frac{x-1}{x+1} = t^3 \Rightarrow & x-1 = t^3x+t^3 \\ & x = \frac{t^3+1}{1-t^3} \Rightarrow x-1 = \frac{2t^3}{1+t^3} \\ & dx = \frac{6t^2}{(1-t^3)^2}\, dt \end{matrix} \right| =$

$= \int t \cdot \cancel{2t^3} \frac{1+t^3}{2t^3} \cdot \frac{6t^2}{(1-t^3)^2}\, dt = 3 \int \frac{dt}{1-t^3} = 3 \int \frac{dt}{(1-t)(1+t+t^2)} = \ldots$

RRLF.

$R\left(x, \sqrt{ax^2+bx+c}\right)$

Ⓟř $\int \frac{x\, dx}{(x+1)\sqrt{-x^2+x+2}} = \int \frac{x\, dx}{(x+1)\sqrt{(2-x)(x+1)}} = \left| \begin{matrix} S: \frac{2-x}{x+1} = t^2 \\ x = \frac{2-t^2}{t^2+1} \quad dx = \frac{-6t}{(t^2+1)^2}\, dt \\ x+1 = \frac{3}{t^2+1}, \quad \sqrt{(2-x)(x+1)} = \sqrt{\frac{2-x}{x+1}(x+1)^2} \end{matrix} \right.$

$= \int \frac{\frac{2-t^2}{t^2+1} \cdot \frac{-6t}{(t^2+1)^2}}{\frac{3}{t+1} \cdot t \cdot \frac{3}{t^2+1}}\, dt = \frac{2}{3} \int \frac{t^2-2}{t^2+1}\, dt = RLF \ldots$

E

Př $$\int \frac{dx}{x+\sqrt{x^2+x+1}} = \left| \begin{array}{l} \sqrt{x^2+x+1} = -x+t \ /^2 \\ \cancel{x^2}+x+1 = \cancel{x^2}-2xt+t^2 \\ x = \frac{t^2-1}{1+2t} \\ dx = \frac{2t^2+2t+2}{(1+2t)^2}\,dt \end{array} \right| =$$

$$= \int \frac{2t^2+2t+2}{(1+2t)^2} \cdot \frac{1}{t}\,dt = \text{RRLF} \ldots$$

$$\int \sin^m x \cos^n x \, dx \quad , m,n \in \mathbb{Z}$$

m liché → S: $\cos x = t$

n liché → $\sin x = t$

m, n sudá → $\operatorname{tg} x = t$

Př $$\int \sin^5 x \underset{(\cos^2 x)^2 \cdot \cos x}{\cos^5 x}\, dx = \left| \begin{array}{l} \sin x = t \\ \cos x\, dx = dt \end{array} \right| = \int t^5 \cdot (1-t^2)^2\, dt = \ldots$$

Př $$\int \frac{\sin^2 x}{\cos^5 x}\, dx = \left| \begin{array}{l} \operatorname{tg} x = t \\ \frac{1}{\cos^2 x}\, dx = dt \end{array} \right| = \int t^2 \cdot (t^2+1)\, dt = \ldots$$

$$\frac{1}{\cos^2 x} = ?$$

$$t^2 = \operatorname{tg}^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$$

Ⓔ

Př $\int \frac{dx}{x+\sqrt{x^2+x+1}} = \left| \begin{array}{l} \sqrt{x^2+x+1} = -x + t \;/^2 \\ \cancel{x^2} + x + 1 = \cancel{x^2} - 2xt + t^2 \\ x = \frac{t^2-1}{1+2t} \\ dx = \frac{2t^2+2t+2}{(1+2t)^2}\, dt \end{array} \right| =$

$= \int \frac{2t^2+2t+2}{(1+2t)^2} \cdot \frac{1}{t}\, dt =$ RRLF. . .

$\int \sin^m x \cos^n x\, dx \quad , m, n \in Z$

m liché → S: $\cos x = t$

n liché → $\sin x = t$

m, n sudá → $\mathrm{tg}\, x = t$

Př $\int \sin^5 x \cos^5 x\, dx = \left| \begin{array}{l} \sin x = t \\ \cos x\, dx = dt \end{array} \right| = \int t^5 \cdot (1-t^2)^2\, dt = \ldots$

$(\cos^2 x)^2 \cdot \cos x$

Př $\int \frac{\sin^2 x}{\cos^6 x}\, dx = \left| \begin{array}{l} \mathrm{tg}\, x = t \\ \frac{1}{\cos^2 x}\, dx = dt \end{array} \right| = \int t^2 \cdot (t^2+1)\, dt = \ldots$

$\frac{1}{\cos^2 x} = ?$

$t^2 = \mathrm{tg}^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$

Ⓕ

$R(\sin x, \cos x)$:

$r(\sin x)\cdot\cos x \to \sin x = t$

$r(\cos x)\cdot\sin x \to \cos x = t$

$r(\mathrm{tg}\, x) \longrightarrow \mathrm{tg}\, x = t \quad \Leftrightarrow R(\sin x, \cos x) = R(-\sin x, -\cos x)$

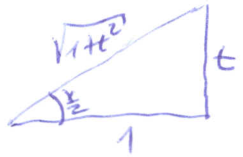
nebo univerzální substituce

$$\boxed{\mathrm{tg}\frac{x}{2} = t}$$

$\frac{x}{2} = \mathrm{arctg}\, t$ $\qquad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$x = 2\,\mathrm{arctg}\, t$ $\qquad \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$dx = 2\cdot\frac{1}{t^2+1}\,dt$

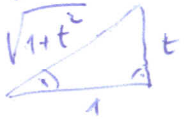


$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$, $\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$

$\Downarrow \qquad \Downarrow$

$\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

Ⓟ$_r$ $\displaystyle\int \frac{(\sin x + 2)\cdot\cos x}{\sin^2 x - 2\sin x + 5}\,dx = \left|\begin{matrix}\sin x = t\\ \cos x\, dx = dt\end{matrix}\right| = \int\frac{t+2}{t^2-2t+5}\,dt = \mathrm{RRLF}\ldots$



Ⓟ$_r$ $\displaystyle\int\frac{\sin x - \cos x}{\sin x + 2\cos x}\,dx = \left|\begin{matrix}\mathrm{tg}\, x = t\\ x = \mathrm{arctg}\, t \to dx = \frac{1}{t^2+1}\,dt\end{matrix}\right.$, $\sin x = \frac{t}{\sqrt{t^2+1}}$, $\cos x = \frac{1}{\sqrt{t^2}}$

$\left(\int\frac{-\sin + \cos x}{-\sin x - 2\cos x} = \frac{\sin x - \cos x}{\sin x + 2\cos x}\right) \to$

$$= \int\frac{\frac{t}{\sqrt{1+t^2}} - \frac{1}{\sqrt{1+t^2}}}{\frac{t}{\sqrt{1+t^2}} + 2\frac{1}{\sqrt{1+t^2}}}\cdot\frac{1}{1+t^2}\,dt = \int\frac{t-1}{(t+2)(t^2+1)}\,dt$$

(Př) $\int \frac{1-\sin x}{1+\cos x}\,dx = \left| \begin{matrix} tg\frac{x}{2} = t \\ + \text{napočítáme} \end{matrix} \right| =$

$$= \int \frac{1-\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot 2 \cdot \frac{1}{1+t^2}\,dt = \int \frac{t^2+1-2t}{1+t^2}\,dt = \left| \text{PLF, nebo} \right| =$$

$$= \int \frac{t^2+1}{1+t^2}\,dt - \int \frac{2t}{1+t^2}\,dt = t - \ln(1+t^2) + C =$$

$$= tg\frac{x}{2} + \ln\left(1+tg^2\frac{x}{2}\right) + C$$

$R(x, \sqrt{a^2-x^2}) \longrightarrow$ S: $x = a\cdot\sin t$ nebo $x = a\cdot\cos t$

$R(x, \sqrt{x^2-a^2}) \longrightarrow$ $x = \frac{a}{\sin t}$ nebo $x = \frac{a}{\cos t}$

$R(x, \sqrt{x^2+a^2}) \longrightarrow$ $x = a\,tg\,t$

(Př) $\int \frac{1}{x^2\sqrt{x^2+a^2}}\,dx = \left| \begin{matrix} x = a\cdot tg\,t \\ dx = \frac{a}{\cos^2 t}\,dt \end{matrix} \right| = \int \frac{1}{a^2\frac{\sin^2 t}{\cos^2 t}} \cdot a\frac{1}{\cos t}\,dt = \int \frac{1\cdot\cos t}{a\cdot\sin^2 t}\,dt$

$$\sqrt{x^2+a^2} = \sqrt{a^2\frac{\sin^2 t}{\cos^2 t}+a^2} =$$

$$= a\sqrt{\frac{\sin^2 t+\cos^2 t}{\cos^2 t}} = a\sqrt{\frac{1}{\cos^2 t}}$$

$= \left| \begin{matrix} \sin t = u \\ \cos t\,dt = du \end{matrix} \right|$ ~~$= \frac{1}{a}\int \frac{du}{u^2} = \frac{1}{a}\cdot\frac{-1}{u} + C = \frac{-1}{a\sin t} + C =$~~

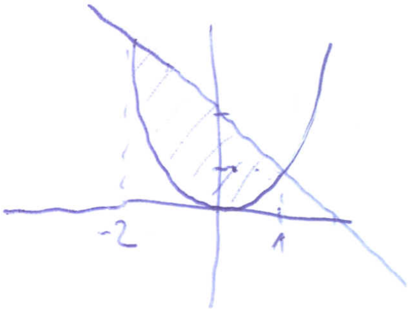
~~$= \frac{-1}{a\sin\operatorname{arctg}\frac{x}{a}}$~~ ~~$a\sin t - \cos t$~~

H

Př) Určete plochu ohraničenou křivkami

$y = x^2$

$x + y = 2 \longrightarrow y = 2 - x$



~~$S = \int_{2}^{1} (x^2 - (2 - x))\,dx =$~~

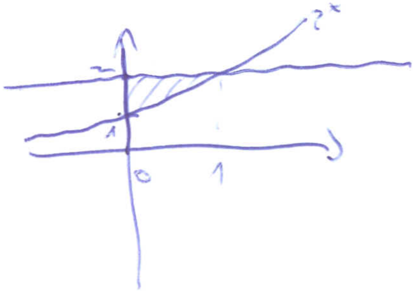
$$S = \int_{-2}^{1} (2 - x - x^2)\,dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} =$$

$$= 2 - \frac{1}{2} - \frac{1}{3} - \left(-4 - 2 + \frac{8}{3}\right) =$$

$$= \frac{9}{2}$$

Př) Plochu ohraničenou

$y = 2^x, \; y = 2, \; x = 0$



$$S = \int_{0}^{1} (2 - 2^x)\,dx = \left[2x - \frac{2^x}{\ln 2}\right]_{0}^{1} =$$

$$= 2 - \frac{1}{\ln 2}$$

délka křivky

$$L = \int_a^b \sqrt{1+f'^2(x)}\,dx$$

P5 $f(x) = x^{\frac{3}{2}}$, $0 \leq x \leq 4$

$f'(x) = \frac{3}{2} x^{\frac{1}{2}}$ $f'^2(x) = \frac{9}{4} x$

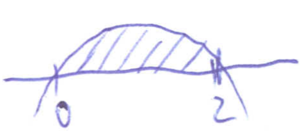
$$L = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx = \left| \begin{array}{c} 1+\frac{9}{4}x = t \\ \frac{9}{4}dx = dt \\ \Downarrow \\ dx = \frac{4}{9}dt \end{array} \right| = \int_1^{10} \sqrt{t}\cdot\frac{4}{9}\,dt =$$

$$= \frac{4}{9}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^{10} = \frac{8}{27}\left(10\cdot\sqrt{10} - 1\right)$$

objem rotačního tělesa (okolo osy x)

$$V = \pi \int_a^b f^2(x)\,dx$$

P6 $y = 2x - x^2$; $y = 0$ kolem osy x



$$V = \pi \int_0^2 (2x - x^2)^2\,dx = \pi \int_0^2 (4x^2 - 4x^3 + x^4)\,dx$$

$$= \pi\left[4\frac{x^3}{3} - 4\frac{x^4}{4} + \frac{x^5}{5}\right]_0^2 = \frac{16}{15}\pi$$

Povrch pláště

$$P = 2\pi \int_a^b f(x)\sqrt{1+f'^2(x)}\,dx$$

valec $\underset{\text{výška}}{v} = x$, R poloměr

tj. $f(x) = R$

$f'(x) = 0$

$$P = 2\pi \int_0^v R\sqrt{1+0}\,dx$$

$$= 2\pi R \int_0^v dx = 2\pi R\,[x]_0^v = \underline{\underline{2\pi R v}}$$

DDÚ | neurčitý integrál

⑯ $\int x(1-x)^{20}\,dx$

Ř: S: $1-x=t$ $\left(-\frac{1}{21}(1-x)^{21}+\frac{1}{22}(1-x)^{22}+C\right)$

	Ř:
① $\int \mathrm{cotg}\,x\,dx$	① $(\ln\lvert\sin x\rvert + C)$
② $\int \mathrm{tg}\,x\,dx$	② $(-\ln\lvert\cos x\rvert + C)$
③ $\int \frac{x}{x^2+a^2}\,dx$	③ $\left(\frac{1}{2}\cdot\ln\lvert x^2+a^2\rvert + C\right)$
④ ~~$\int \mathrm{tg}^2 x$~~ $\int \mathrm{tg}^2 x\,dx$	④ $(\mathrm{tg}\,x - x + C)$
⑤ $\int \frac{1-x}{x^2}\,dx$	⑤ $\left(-\frac{1}{x} - \ln\lvert x\rvert + C\right)$
⑥ $\int \frac{x^2}{1+x^2}\,dx$	⑥ $(x - \mathrm{arctg}\,x + C)$
⑦ $\int \frac{2^{x+2} - 5^{x-1}}{10^x}\,dx$	⑦ $\left(2\frac{\left(\frac{1}{5}\right)^x}{\ln\frac{1}{5}} - \frac{1}{5}\frac{\left(\frac{1}{2}\right)^x}{\ln\frac{1}{2}} + C\right)$
⑧ $\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx$	⑧ $\left(\frac{4}{7}x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + C\right)$
⑨ $\int x^2\cdot\sqrt[3]{1+x^3}\,dx$	⑨ S: $1+x^3=t$ $\left(\frac{1}{4}\sqrt[3]{(1+x^3)^4} + C\right)$
⑩ $\int \frac{e^x}{2+e^x}\,dx$	⑩ S: $2+e^x=t$ $(\ln(2+e^x) + C)$
⑪ $\int \frac{\ln x}{x\sqrt{1+\ln x}}\,dx$	⑪ S: $1+\ln x=t$ $\left(\frac{2}{3}(1+\ln x)^{\frac{3}{2}} - 2(1+\ln x)^{\frac{1}{2}} + C\right)$
⑫ $\int \ln x\,dx$	⑫ per-partes=PP $(x\ln x - x + C)$
⑬ $\int \sin^2 x\,dx$	⑬ PP $\left(\frac{x - \sin x\cos x}{2}\right)$
⑭ $\int x^3 e^{x^2}\,dx$	⑭ nejdřív S: $x^2=t$, pak PP $\left(\frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C\right)$