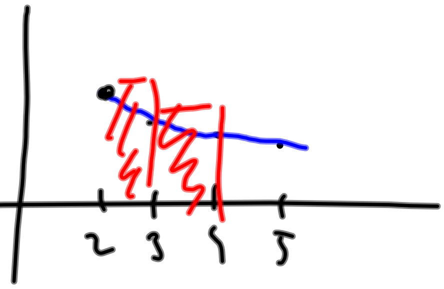
1) Integrální kritérium:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \text{ konverguje} \Leftrightarrow$$

$$\Leftrightarrow \int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx \text{ konverguje}$$

$$f(x) := \frac{1}{x(\ln x)^2}$$



$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \int_{\ln(2)}^{\infty} \frac{1}{t^2}\,dt$$

$$\begin{bmatrix} t = \ln x \\ dt = \frac{1}{x}dx \end{bmatrix} = \lim_{\delta \to \infty} \int_{\ln(2)}^{\delta} \frac{1}{t^2}\,dt =$$

$$= \lim_{\delta \to \infty} \left[-\frac{1}{t}\right]_{\ln(2)}^{\delta} = \ln(2)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^k (\ln(n))^l}$$

pro $k > 1$, $l \geq 0$ konverguje

pro $k \geq 1$, $l > 1$ konverguje

$$\sum_{n=1}^{\infty} \frac{n}{n^2+n+1} \geq \sum_{n=1}^{\infty} \frac{n}{n^2+n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$$

diverguje

$$\int_3^\infty \frac{1}{x^{n+1}}\,dx = \lim_{\delta\to\infty}\left[-\frac{1}{n x^n}\right]_3^\delta =$$

$$= \lim_{\delta\to\infty}\left(\frac{1}{n 3^n} - \frac{1}{n \delta^n}\right) = \frac{1}{n 3^n}$$

$$\sum_{n=1}^\infty \frac{1}{n 3^n} = \sum_{n=1}^\infty \int_3^\infty \frac{1}{x^{n+1}}\,dx \overset{*}{=}$$

$$\overset{*}{=} \int_3^\infty \sum_{n=1}^\infty \frac{1}{x^{n+1}}\,dx$$

* Weierstrasovo kriterium stejnom. konvergence:
najdeme "konvergentní majorantu" řady funkcí $\sum f_n$, $f_n := \frac{1}{x^{n+1}}$ na int. $(3,\infty)$

$$\frac{1}{3^{n+1}} > \frac{1}{x^{n+1}} = f_n(x) \quad \text{na} \quad \langle 3, \infty)$$

$$a \quad \sum_{n=1}^{\infty} \frac{1}{3^{n+1}} \quad \text{konverguje} \Rightarrow$$

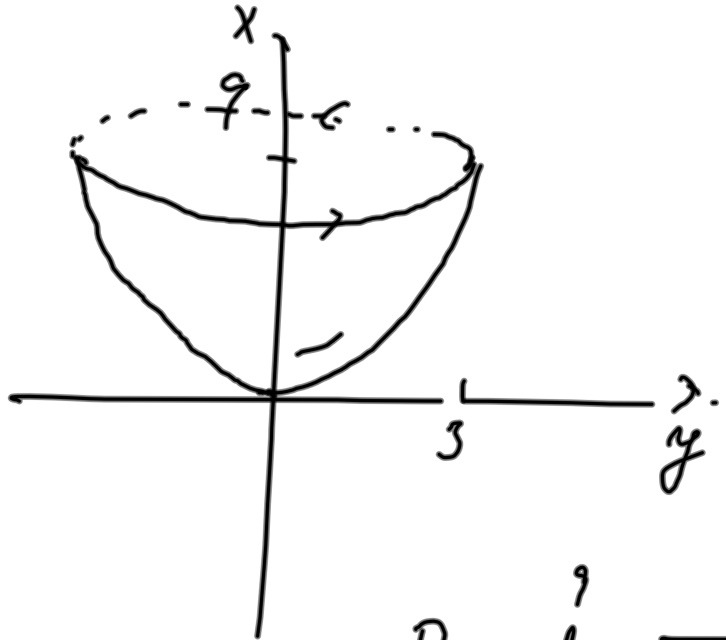
$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{x^{n+1}} \quad \text{konverguje stejnoměrně na } \langle 3, \infty)$$

$$= \int_3^{\infty} \sum_{n=1}^{\infty} \frac{1}{x^{n+1}} \, dx = \int_3^{\infty} \frac{1}{x^2} \left(\frac{1}{1 - \frac{1}{x}} \right) dx =$$

$$= \int_3^{\infty} \frac{1}{x^2 - x} \, dx = \int_3^{\infty} \frac{1}{x(x-1)} \, dx = \int_3^{\infty} \left(\frac{1}{x-1} - \frac{1}{x} \right) dx$$

$$= \lim_{\delta \to \infty} \left[\ln(x-1) - \ln(x) \right]_3^{\delta} = -\ln\left(\frac{2}{3}\right) + \lim_{\delta \to \infty} \ln\left(\frac{\delta - 1}{\delta}\right)$$

$$= \ln\left(\frac{3}{2}\right)$$



$$V = \pi \int_a^b f(x)^2 \, dx$$

$$= \pi \int_0^9 x \, dx =$$

$$y = x^2 \Rightarrow x = \sqrt{y}$$

$$= \pi \left[\frac{x^2}{2} \right]_0^9 = \frac{81\pi}{2}$$

$$P = 2\pi \int_0^9 \sqrt{x} \sqrt{1 + \frac{1}{4x}} \, dx \qquad f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2\sqrt{x}}$$

$$= 2\pi \int_0^9 \sqrt{x + \frac{1}{4}} \, dx =$$

$$= \frac{4\pi}{3} \left[\left(x + \frac{1}{4}\right)^{\frac{3}{2}} \right]_0^9 = \dots$$

$f(x) = \ln(x)$

$$f(x) = f(1) + \sum_{i=1}^{\infty} \frac{f^{(i)}(1)}{i!}(x-1)^i = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \dots$$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f^{(3)}(x) = \frac{2}{x^3}$

$f^{(4)}(x) = -\frac{3!}{x^4}$

$\vdots$

$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}$

$f'(1) = 1$

$f''(1) = -1$

$f^{(3)}(1) = 2$

$\vdots$

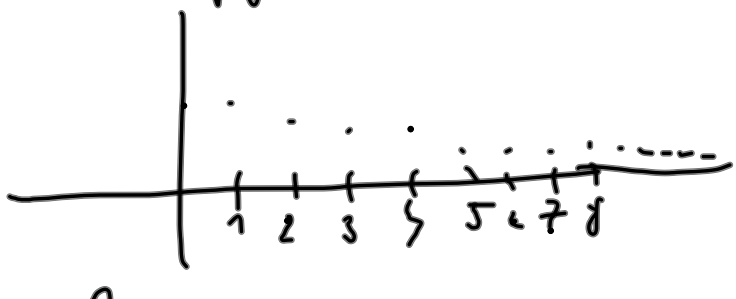
$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

Podílové kritérium

$$\lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+2}}{n+1} (x-1)^{n+1}}{\frac{(-1)^{n+1}}{n} (x-1)^{n}} \right| = \lim_{n \to \infty} \left| \frac{n}{n+1} (x-1) \right| =$$

$$= |x-1| \overset{?}{<} 1 \Rightarrow x \in (0,2)$$

$x = 2$: Konverguje podle Leibnizova kritéria:



$x = 0$:

$$1 = \sqrt{1} = \sqrt{(-1)\cdot(-1)} \overset{*}{=} \sqrt{-1}\cdot\sqrt{-1} = i \cdot i = -1$$

i)

$$f(x) = \frac{1}{1-x} \qquad f(x) = 1 + x + x^2 + x^3 + \dots$$

$$f'(x) = \frac{1}{(1-x)^2} \qquad 1 : (1-x) = 1 + x + x^2 + \dots$$

$$(1-x)$$

$$f''(x) = \frac{2}{(1-x)^3} \qquad -(x - x^2)$$

$$x^2$$

$$f^{(3)}(x) = \frac{2\cdot 3}{(1-x)^4}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

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ii) $f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$f'(x) = -\frac{2x}{(1+x^2)^2}$

$$\begin{aligned}
&1 : (1+x^2) = 1 - x^2 + x^4 - x^6 \\
&-(1+x^2) \\
&\quad -x^2 \\
&\quad -(-x^2 - x^4) \\
&\qquad x^4
\end{aligned}$$

iii) $f(x) = \operatorname{arctg}(x)$ rozvijme v bodě 0.

$f(0) = 0$

$f'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$ $\quad f'(0) = 1$

$f''(x) = -2x + 4x^3 - 6x^5 + 8x^7$ $\quad f''(0) = 0$

$f^{(3)}(x) = -2 + 12x^2 - 6 \cdot 5 x^4 + 8 \cdot 7 x^6 - \dots$

$$f^{(2n)}(0)=0$$

$$f^{(2n+1)}(0)=(2n!)\cdot(-1)^n$$

$$\operatorname{arctg}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}$$

$$f(x)=\int_0^x f'(x)\,dx$$

$$\operatorname{arctg}(x)=\int_0^x\frac{1}{1+x^2}\,dx=\int_0^x\sum_{n=0}^{\infty}(-1)^n x^{2n}=$$

$$=\sum_{n=0}^{\infty}\int_0^x(-1)^n x^{2n}=\sum_{n=0}^{\infty}\int_0^x 1-x^2+x^4-x^6\ldots$$

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$$= \left[\cancel{x} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\right]_0^x =$$

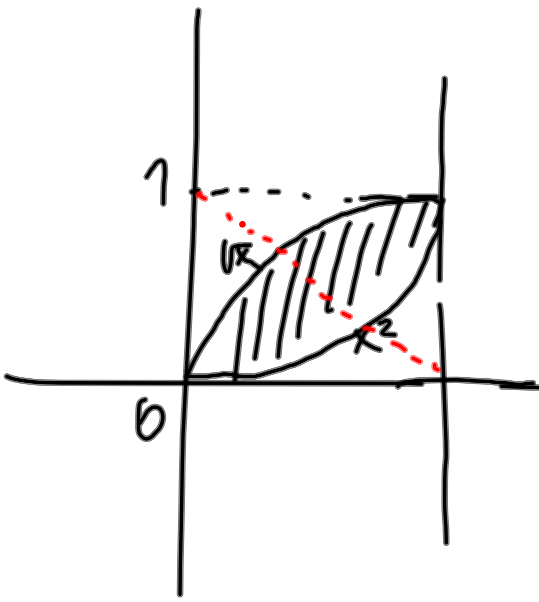
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\int_0^1 \frac{\sin x}{x}\, dx = \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \frac{x^9}{9\cdot 9!} - \cdots\right]_0^1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$= 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} + \frac{1}{9\cdot 9!} -$$



$$\int_0^1 \left(\sqrt{x} - x^2\right) dx =$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 =$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

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