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a) $$a_n = \frac{1}{n + \underbrace{\sqrt{n}+1}_{n^2}} \overset{n>3}{>} \frac{1}{n+n} = \frac{1}{2n}$$

$$n > \sqrt{n}+1$$

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}+1} \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

b) $$a_n = \sqrt{\frac{n^3+5n+1}{n^5-5n^2-1}} > \sqrt{\frac{n^3}{n^5}} = \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \sqrt{\frac{n^3+5n+1}{n^5-5n^2-1}} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

: $\lim\limits_{n\to\infty} \underbrace{\sqrt[n]{a_n}}_{r_n}$

1) $r_n = 2011 \quad \lim\limits_{n\to\infty} r_n = 2011 \Rightarrow r = \frac{1}{2011}$

2) $r_n = \sqrt[n]{2011}, \quad \lim\limits_{n\to\infty} r_n = 1 \Rightarrow r = 1$

3) $r_n = \frac{\sqrt[n]{n-1}}{n^3}, \quad \lim\limits_{n\to\infty} r_n \frac{1}{\infty} = 0$

$$\left(\sum_{n=1}^{\infty} \frac{n-1}{n^{3n}} x^n\right) : \mathbb{R} \to \mathbb{R}$$

4) $r_n = \sqrt[n]{n!}, \quad \lim\limits_{n\to\infty} r_n = \infty$

$$e^{i} = \dots$$

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

$w \in \mathbb{C}$

$$z^{w} := e^{\overset{?}{\overbrace{\ln(z)}}\cdot w}$$

$e^{\ln z} = z$ ... definice

$$e^{z} = e^{a+bi} = e^{a}e^{bi} = \underbrace{e^{a}}_{\substack{\text{abs.}\\ \text{hodnota}}}\underbrace{(\cos b + i\sin b)}_{\substack{\text{komplexní}\\ \text{jednotka}}}$$

$$|z|(\cos\varphi + i\sin\varphi) \overset{?}{=} e^{z}$$

$$\Rightarrow$$

$$\ln z = \ln|z| + i\varphi$$

$$\Rightarrow \quad |z| = e^{a} \Rightarrow a = \ln|z|$$

$$\cos\varphi + i\sin\varphi = \cos b + i\sin b \Rightarrow \varphi = b + 2k\pi$$

$$i^{i} = e^{(\ln i)\cdot i} = e^{\left(i\frac{\pi}{2}\right)\cdot i} = e^{-\frac{\pi}{2}}$$

$$\lim_{x\to 0} \frac{x\cdot e^x}{\sin x} = \lim_{x\to 0} \frac{e^x + xe^x}{\cos x} = 1$$

$$= \lim_{x\to 0} \frac{x}{\sin x}\cdot e^x = \underbrace{\lim_{x\to 0} \frac{x}{\sin x}}_{1} \cdot \underbrace{\lim_{x\to 0} e^x}_{1} = 1$$

$$\lim_{x\to 0} \frac{x^2\cdot e^x}{\sin^2 x} \overset{L.P.}{=} \lim_{x\to 0} \frac{2xe^x + x^2e^x}{\underbrace{2\sin x\cos x}_{\sin 2x}} \overset{L.P.}{=} \lim_{x\to 0} \frac{2e^x + 2xe^x + (2x+x^2)e^x}{2\cos 2x} =$$

$$= \frac{2}{2} = 1$$

$$= \lim_{x\to 0} \frac{x\cdot e^x}{\sin x}\cdot\frac{x}{\sin x} = \underbrace{\lim_{x\to 0} \frac{x\cdot e^x}{\sin x}}_{1} \cdot \underbrace{\lim_{x\to 0} \frac{x}{\sin x}}_{1}$$

$$= 1$$

$$\lim_{x\to 0} \frac{x^3\cdot e^x}{2\sin x - \sin^2 x} \overset{L.P.}{=} \lim_{x\to 0} \frac{3x^2 e^x + x^3 e^x}{2\cos x - \underbrace{2\sin x\cos x}_{\sin 2x}} =$$

$$= 0$$

$$\lim_{x\to 0} \frac{x^3\cdot e^x}{2\sin x - \sin 2x} \overset{L.P.}{=} \lim_{x\to 0} \frac{3x^2 e^x + x^3 e^x}{2\cos x - 2\cos 2x} =$$

$$\overset{L.P.}{=} \lim_{x\to 0} \frac{6xe^x + 3x^2e^x + 3x^2e^x + x^3e^x}{-2\sin x + 4\sin 2x} =$$

$$\overset{L.P.}{=} \lim_{x\to 0} \frac{6e^x + x(\dots)}{-2\cos x + 8\cos 2x} = 1$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln x\cdot\frac{1}{x}} = e^{\lim_{x\to\infty} \ln x\cdot\frac{1}{x}} = e^0 = 1$$

$$\left[\ \lim_{x\to\infty} \frac{\ln x}{x} \overset{L.P.}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0\right.$$

$$\lim_{x\to\infty} |x - \ln x|$$

$$\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty$$

$$\left.\begin{array}{l} \lim_{x\to\infty} f(x) = \infty \\ \lim_{x\to\infty} g(x) = \infty \\ \lim_{x\to\infty} \frac{f(x)}{g(x)} = q > 1 \end{array}\right\} \Rightarrow \lim_{x\to\infty} f(x) - g(x) = \infty$$

$$q = 1 + r \Rightarrow \left(\exists x_0\right) (\forall x > x_0) : \frac{f(x)}{g(x)} > 1 + \frac{r}{2}$$

$$\Leftrightarrow f(x) > \left(1 + \frac{r}{2}\right) g(x)$$

$$\Rightarrow \forall x > x_0 : \ f(x) - g(x) > \left(1 + \frac{r}{2}\right) g(x) - g(x) =$$

$$= \frac{r}{2} g(x)$$

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$$\lim_{x\to\infty}\left(\sqrt{x}-\ln^2 x\right)$$

$$\lim_{x\to\infty}\frac{\sqrt{x}}{\ln^2 x}\overset{L.P.}{=}\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{\frac{2\ln x}{x}}=\lim_{x\to\infty}\frac{\sqrt{x}}{4\ln x}\overset{L.P.}{=}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}}=\lim_{x\to\infty}\frac{1}{8}\sqrt{x}=\infty$$

$$\Rightarrow \lim_{x\to\infty}\left(\sqrt{x}-\ln^2 x\right)=\infty$$

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots \Rightarrow f(0)=a_0$$

$$f'(x)=a_1+2a_2x+3a_3x^2+\cdots \Rightarrow f'(0)\Rightarrow a_1$$

$$f''(x)=2a_2+6a_3x+12a_4x^2+\cdots \Rightarrow f''(0)=2a_2$$

$$\vdots$$

$$f^{(n)}(0)=n!\,a_n$$

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$f(x) = \sin(2x)$ $\qquad$ $f(0) = 0$

$f'(x) = 2\cos(2x)$ $\qquad$ $f'(0) = 2$

$f''(x) = -4\sin(2x)$ $\qquad$ $f''(0) = 0$

$f^{(3)}(x) = -8\cos(2x)$ $\qquad$ $f^{(3)}(0) = -8$

$f^{(4)}(x) = 16\sin(2x)$ $\qquad$ $f^{(4)}(0) = 0$

$f^{(5)}(x) = 32\cos(2x) \Rightarrow |f^{(5)}(x)| \leq 32$

$$T_0^4(\sin 2x) = 0 + 2x + \frac{0}{2!}\cdot x^2 + \frac{-8}{3!}x^3 + \frac{0}{4!}x^4 =$$

$$= 2x - \frac{4}{3}x^3$$

Odhadnutie chyby v bode $\frac{\pi}{4}$:

chyba $\leq \frac{1}{5!} f^{(5)}(c) \left(\frac{\pi}{4}\right)^5 \leq$, $c \in \left(0, \frac{\pi}{4}\right)$

$\leq \frac{32}{5!} \cdot \left(\frac{\pi}{4}\right)^5 \leq 0{,}08$

$$f(x) = \ln(\sin(x))$$

$$f'(x) = \frac{\cos x}{\sin x} = \operatorname{cotg} x$$

$$f''(x) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

$$f^{(3)}(x) = -(\sin x)^{-2} = 2 \cdot \frac{\cos x}{\sin^3 x}$$

$$= 2 \cdot \operatorname{cotg} x \cdot \frac{1}{\sin^2 x}$$

$$f\left(\frac{\pi}{4}\right) = \ln\left(\frac{\sqrt{2}}{2}\right)$$

$$= \ln(\sqrt{2}) - \ln(2) =$$

$$= \frac{1}{2}\ln 2 - \ln(2) =$$

$$= -\frac{1}{2}\ln(2)$$

$$f'\left(\frac{\pi}{4}\right) = 1$$

$$f''\left(\frac{\pi}{4}\right) = -\frac{1}{\frac{1}{2}} = -2$$

$$f^{(3)}\left(\frac{\pi}{4}\right) = 2 \cdot \frac{1}{\frac{1}{2}} = 4$$

$$T^3_{\frac{\pi}{4}}\left(\ln(\sin(x))\right) = -\frac{1}{2}\ln 2 + \left(x - \frac{\pi}{4}\right) -$$

$$- \left(x - \frac{\pi}{4}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{4}\right)^3$$

$$=$$