IA165

Combinatory Logic for Computational Semantics

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Lecture 3

- Continue about the theory of combinators

- Interesting book

"To mock a Mockingbird and Other LOgic Puzzles: including an Amazing Adventure in Combinatory Logic" by Raymond Smullyan (1985)

→ combinator Birds

B from Bluebird, C from Cardinal, I from Identity Bird, K from Kestrel, S from Starling, and so on.

Theory of combinators

- Combinatory base: **S** and **K**

  All combinators can be defined from the two combinators S and K.

- Combinators, called elemantary : **I**, **K**, **B**, **W**, **C**, **S**, **Φ**, **Ψ**
- All combinators work with the elimination (e.) and the introduction (i.) rules.
- All combiantors are defined by β–reduction: $X fgx...$ $\rightarrow_\beta$ $fgx....$

Introduction and elimination rules (in Gentzen style)

- Rules analogues to the rules of the natural deduction of Gentzen
- Rules introduce or eliminate the logical constants

: => of implication, & of conjonction, v of disjonction

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Rule of elimination                          Rule of introduction

p           p=>q                                  q

---------   [e.=>]                           --------------[i.=>]

q                                                       p=>q

p&q                        p&q                     p, q

----[e-&]                  -----[e-&]              ----[i-&]

p                              q                   p&q
```

Comment:

a. With each of the simpler combinators we shall associate a rule of reduction, designated by the symbol for the combinator between parentheses.

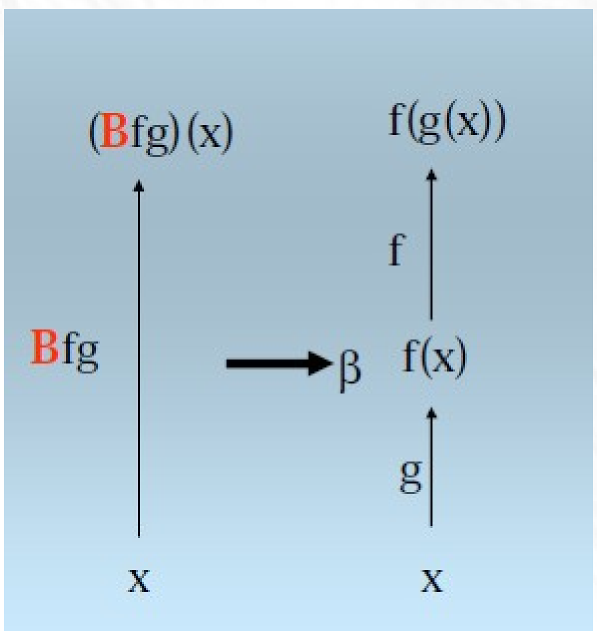
b. This rule states that when the combinator in question is applied sucessively to a (finite) series of variables, the resulting combination reduces to a certain combination of those variables.

c. This reduction will follow from the definition of the combinator by (β−) rule.

d. And we shall use lower case italic letters for unspecified variables. The letters *x*, *y*, *z* will often be used, when in the usual application, the variable is thought of as a function or an argument.

- The combinator B takes two functiors f and g and composes the function g with the argument x. → composition

$$\mathbf{B}fgx \longrightarrow_\beta f(gx)$$

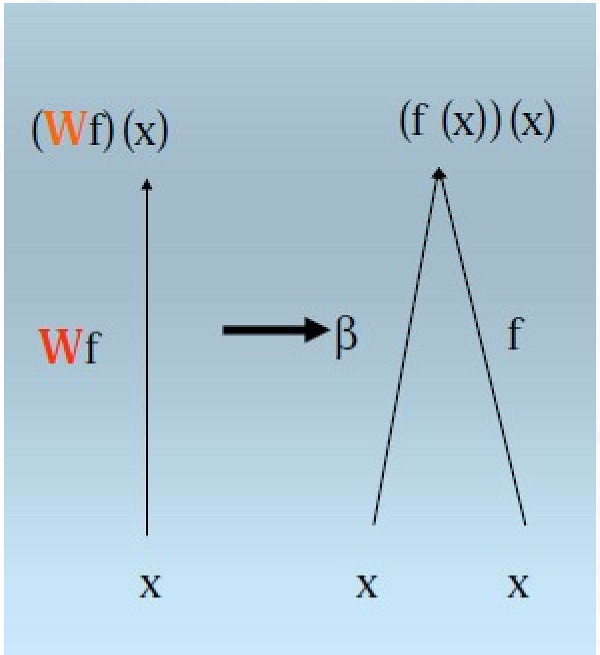


The introduction and elimination rule of the combinator B

$$\frac{(\mathbf{B}fg)(x)}{f(g(x))} \qquad \frac{f(g(x))}{(\mathbf{B}fg)(x)}$$

- The combinator W takes one functor f and applies the functor f to the argument x by duplicating the argument x. → duplication

$$Wfx \longrightarrow_{\beta} fxx$$

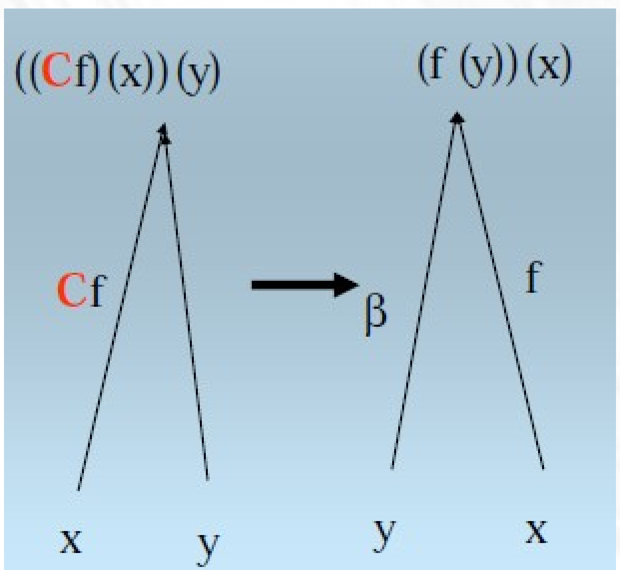


The introduction and elimination rule of the combiantor W

(Wf) (x)	(f(x)) (x)
----------	----------
(f(x)) (x)	(Wf) (x)

- The combinator C takes one functor f and two arguments x and y. The elimination of the combinator C by β-reduction allows to converse the position of the arguement x with y. → conversion

$$\mathbf{C}fxy \longrightarrow_{\beta} fyx$$

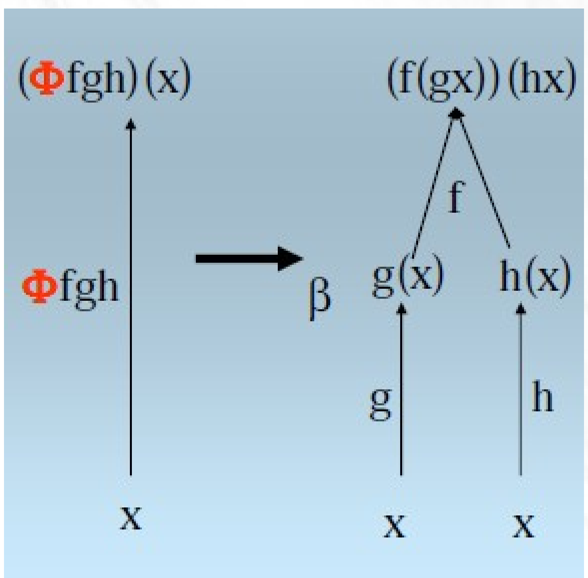


The introduction and elimination rule of the combinator C

$$\frac{((\mathbf{C}f)(x))(y)}{(f(y))(x)} \qquad \frac{(f(y))(x)}{(\mathbf{C}f)(x))(y)}$$

- The combinator **Φ** takes three functors f, g, and h, and the functor g and h become intricate with the argument x. → intrication

$$\Phi fghx \xrightarrow{\beta} f(gx)(hx)$$

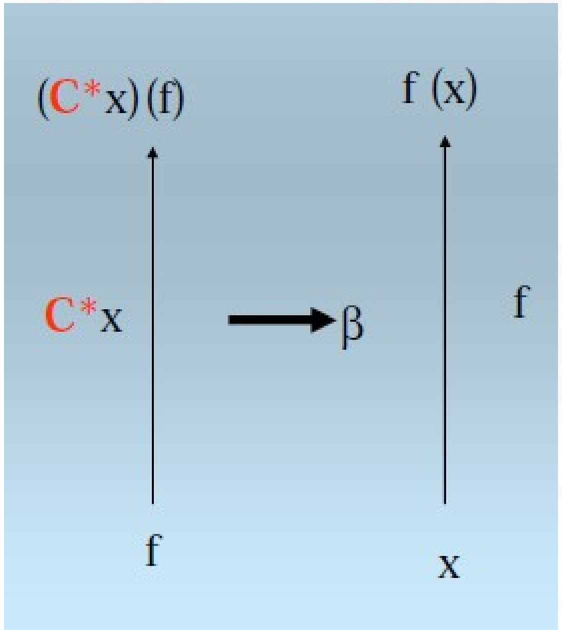


The introduction and elimination rule of the combinator **Φ**

$$\frac{(\Phi fgh)(x)}{f(gx)(hx)} \qquad \frac{f(gx)(hx)}{(\Phi fgh)(x)}$$

- The combinator C* takes an argument x (operand) and transforms it in functor(operator). → transformation of operand to operator

$$(C^{*}x)f \longrightarrow_{\beta} fx$$

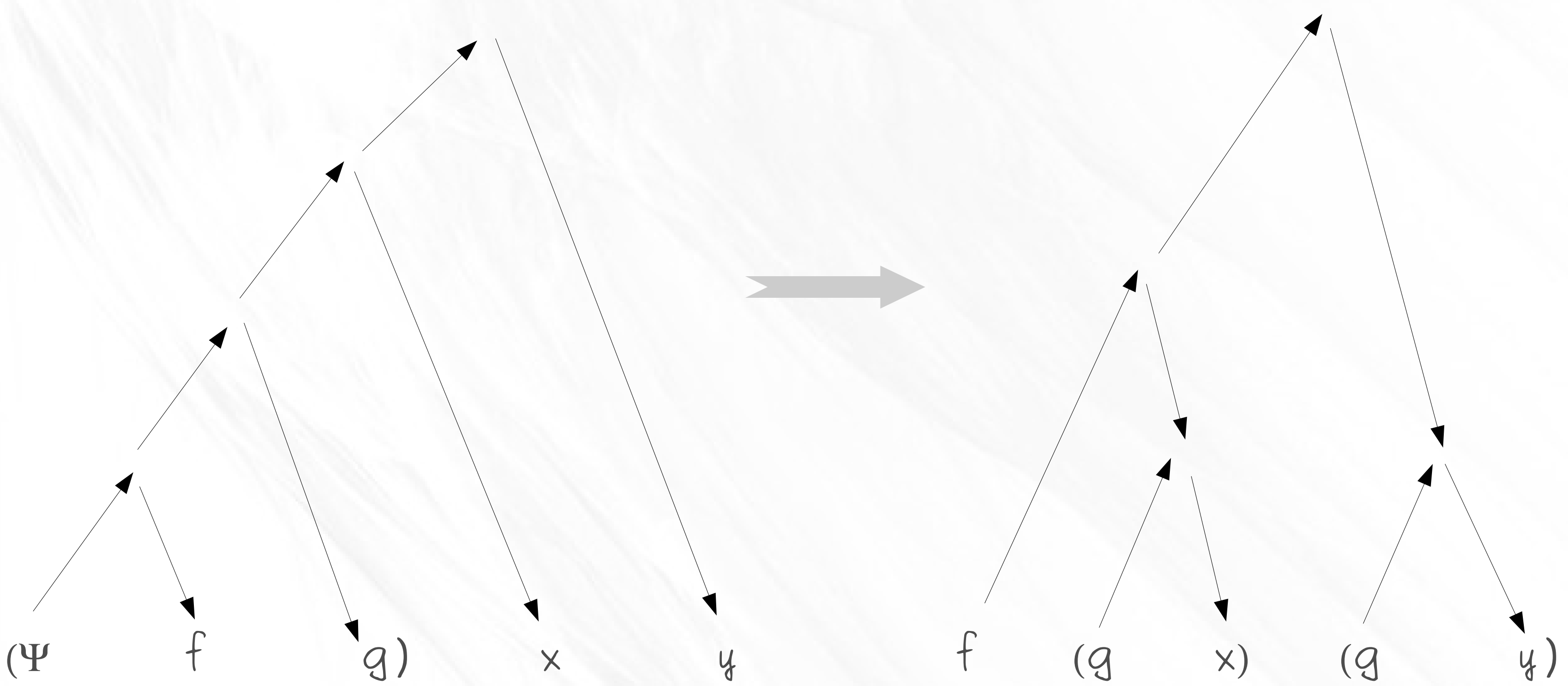


The introduction and elimination rule of the combinator C*

$$\frac{(C^{*}x)(f)}{f(x)} \qquad\qquad \frac{f(x)}{(C^{*}x)(f)}$$

- $\Psi fgxy \twoheadrightarrow_\beta f(gx)(gy)$

the combinator **Ψ** takes two functions f and g and composes with the arguments x and y by distributing the second function g to each of them. → distribution



β-reduction

I(x)	-> β	x	
(Wf)(x)	-> β	(f(x))(x)	= fxx
(Cf)(y)(x)	-> β	(f(x))(y)	= fxy
(Bfg)(x)	-> β	f(g(x))	= f(gx)
(Φfgh)(x)	-> β	f(g(x))(h(x))	= f(gx)(hx)
(Ψfg)(x))(y)	-> β	(f(g(x)))(g(y)	= f(gx)(gy)
(Sfg)(x)	-> β	(f(x))(g(x))	= fx(gx)
(Kf)(x)	-> β	(f(y))(x)	= fyx
(C*x)(f)	-> β	f(x)	= fx

- Examples of β-reduction : $X \geq_\beta Y$

By the scheme of β-reduction [**B**]

$$B(fx)gx \geq_\beta fx(gx)$$

$$X = xy(Bxyz)UT \geq_\beta xy(x(yz))UT = Y$$

By the scheme of β-reduction [**W**]

$$X = B(Wxy)U \geq_\beta B(xyy)U = Y$$

We use here the infix $\leq$ to denote the relation converse to $\geq$.

Interdefinability of simple combinators

- Certain of the combinators can be defined by the others.

  => by the way that the reduction rule for the derived combinator will follow from its definition and the reduction rule for the basic combinators.

Important point

We shall show that (1) **W**, **S**, **Φ** and **ψ** can be so defined in terms of **B**, **C** and either **W** or **S** (see the classwork n°3); (2) that **I** can be defined in terms of **W** and **K**; and (3) that all the combinators on the list can be expressed in terms of **S** and **K** (refer to the classwork n°2: completeness of the S-K base).

Properties of B-Theorems

__Theorem 1.__ *If x is a regular combinaotr and Y a combinator; then* X•Y *applied to a sequence of arguments performs upon them first the transoformation* X*, then the transformation* Y*.*

Example:

C performs a permutation, **W** performs a duplication, **C•W** performs a permutation, then a duplication on the result:

$$(C \bullet W)fxy \geq C(Wf)xy \geq Wfyx \geq fyyx$$

Theorem 2. *The product is associative, i.e.,*

$$X \circ (Y \circ Z) = (X \circ Y) \circ Z$$

Proof

a. $((X \circ Y) \circ Z)f \geq B(BXY)Zf$

$\geq BXY(Zf)$

$\geq X(Y(Zf))$

b. $(X\ (Y\ Z))f \geq BX(BYZ)f$

$\geq X(BYZf)$

$\geq X(Y(Zf))$

Theorem 3. The product is distributive with respect to pre-application of B, i.e.,

$$B(X \circ Y) = BX \circ BY$$

Proof

$$B(X \circ Y)fx \geq (X \circ Y)(fx)$$

$$\geq X(Y(fx)$$

$$\leq X(BYfx)$$

$$\leq BX(BYf)x$$

$$\leq (BX \circ BY)fx$$

Power of combinators

- **Definition**: We define the powers of a combinatory composite by natural induction thus:

  $X^0 =_{def} I$

  $X^1 =_{def} X$

  $X^{n+1} =_{def} X \circ X^n$

  given the combinatorial expression X:

- From these definitions, it is possible to deduce the following X, X^2, X^3, that is, $X, X \circ X, X \circ X \circ X, \ldots$

- Application to the combinators

  $C^2 fxy \geq C(Cf)xy \geq Cfyx \geq fxy$

  $W^2 fx \geq W(Wfx) \geq Wfxx \geq fxxx$

  $K^2 fxy \geq K(Kf)xy \geq Kfy \geq f$

- $B^2 fxyz \equiv B(Bf)xyz \geq Bf(xy)z \geq f((xy)z) \equiv f(xyz)$

<u>Theorem</u>: For any expressions U, X, Y, $Z_1, \ldots, Z_n$,

$$B^n U X Z_1 \ldots Z_n \geq U(X Z_1 \ldots Z_n),$$

$$\Phi^n U X Y Z_1 \ldots Z_n \geq U(X Z_1 \ldots Z_n)(Y Z_1 \ldots Z_n)$$

Summing up-combinators

combinators	β-reduction
B for functional composition	$\mathbf{B}xyz \geq_{\beta} x(yz)$
I for identity	$\mathbf{I}x \geq_{\beta} x$
C for conversion	$\mathbf{C}xyz \geq_{\beta} x(zy)$
W for duplication	$\mathbf{W}xy \geq_{\beta} xyy$
S for distribution	$\mathbf{S}xyz \geq_{\beta} xz(yz)$
K for effacement	$\mathbf{K}xy \geq_{\beta} x$
C* for transposition	$\mathbf{C^*}xy \geq_{\beta} yx$
Φ for intrication	$\mathbf{\Phi}xyzu \geq_{\beta} x(yu)(zu)$
Ψ for distribution	$\mathbf{\Psi}\,xyzu \geq_{\beta} x(yz)(yu)$

Summing up-theorems

S K K x → x

S (K S) K x y z → x(yz)

S (K (S I)) K x y → (y x)

S (K (S I)) (S (K K) I) x y → (y x)

**Completeness of the S-K base

[S K K =I]

[S (K S) K=B]

[S (K (S I)) K=C]

[S (K (S I)) (S (K K) I)=C]

Next week...

- About the Combinators vs λ-conversion and the application of the combinators to natural language analysis