Theorems on finite sets

[IMHO You should read this proof in the book.]

Suppose we are given the set $N = \{1, 2, \dots n\}$. Call a family F of subsets of N an <u>antichain</u> if no set of F contains another set of the family F. Which is the largest antichain?

$|F_k| = \binom{n}{k}$

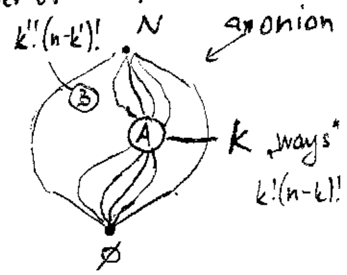
<u>Theorem 1</u>: The size of the largest antichain of N set is $\binom{n}{\lfloor n/2 \rfloor}$ or $\binom{n}{\lceil n/2 \rceil}$ (does not matter).

<u>Proof 1</u>: $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$. We will try to prove that all the subsets have the same size.

Consider $\emptyset = C_0 \subset C_1 \subset \dots \subset N$ $\quad |C_i| = i$ for $i = 0, \dots, n$

There are $n!$ permutations of this chain (total number of chains)

Let $A \in F$ $\quad |A| = k$ $\quad k! \cdot (n-k)!$ of sth.



$$|I| = \sum_{k=0}^{n} m_k \quad \dots \quad \sum_{k=0}^{n} m_k \cdot k! \cdot (n-k)! \leq n!$$

$$\Downarrow$$

$$\sum_{k=0}^{n} \frac{m_k \cdot k! \cdot (n-k)!}{n!} \leq 1$$

$$\frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \sum_{k=0}^{n} m_k \leq 1 \qquad |F| = \sum_{k=0}^{n} m_k \leq \binom{n}{\lfloor n/2 \rfloor}$$

□QED (considering some arcane stuff...)

$$\sum_{A \in F} |A|!(n-|A|)! \leq n! \implies |F| \leq \frac{n!}{\lfloor \frac{n}{2} \rfloor! \cdot \lceil \frac{n}{2} \rceil!}$$

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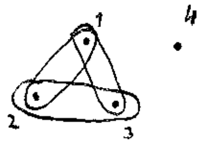


Intersecting family?

$A \in F$

$A_i = N \setminus A \notin F$

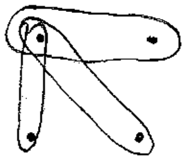
$|F| \leq 2n-1$

$|F_1| = 2^{n-1}$

$n \geq 2k$

1 all $(k-1)$ subsets of $\{2 \dots n\}$

$\boxed{|F_1| = \binom{n-1}{k-1}}$



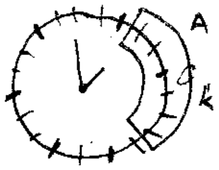
every subset shares one element

hence the formula ↗

Theorem 2: The largest size of an intersecting k-family in an n-set when $n \geq 2k$ is $\binom{n-1}{k-1}$

Proof 2: A circle and points and edges ...

|Erdös|



A_j $(i \geq 2)$

A_i contains $k-1$ inner points

So, cyclic permutations. Here they come.

$(n-1)!$ – number of cyclic perm.] $\pi = a_1 \dots a_n$

$k \cdot (n-1)!$ upper bound

$k!(n-1)!$ [redacted] – number of consecutively appearances (presentations)

$K!$ possibilities of [redacted] (censored).

Then a miracle happens:

$|F_1| \leq \dfrac{k \cdot (n-1)!}{k!(n-k)!} \longrightarrow k(n-1)!(n-1-(k-1))!$ $\nearrow \binom{n-1}{k-1}$ $\square$QED

$x \cdot k!(n-k)! \leq k(n-1)!$

$A_1 \dots A_n \subset X$

$x_1 \dots x_n$

$x_i \in A_i$ for all i

Boys, [redacted] girls and marriage ?! 0_0

For a moment there I thought I was in the wrong class.

So, a "girl" has a set of boys she likes. Is there a way for all the girls to marry a boy they like?

Damn, It ~~too~~ starts to look like a soap opera.

D... did I just read?

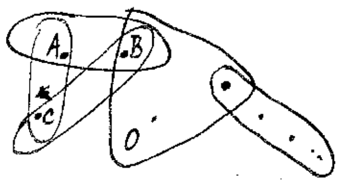
Theorem 3: Let $A_1 \dots A_n$ be a collection of subsets of a finite set X. Then there exists a system of distinct representatives(?) (SDR) iff the union of any m-sets A_i contains at least m elements for $1 \leq m \leq n$

Proof 3 „⇒" By Induction (that's a method, not an author) my work is done here.

Base: for $n = 1$ it **OBVIOUSLY** holds.

1. Step: $n > 1$ $\{A_1 \dots A_n\}$

$1 \le l < n$



Critical family – Size of union equals number of sets.

Case 1: there is no critical family strictly smaller than n. $l < n$

$x \in A_n$ — m sets A_i'

$A_1 \dots A_{n-1}$ — $x_1 \dots x_{n-1}$ of $\{A_1' \dots A_{n-1}'\}$

$A_i' = A_i \setminus \{x\}$ — $x_n = x \Rightarrow$ SDR for original collection

Case 2: there exist a critical family

$A_1 \dots A_l$ is a critical family

$\bigcup_{i=1}^{l} A_i = \tilde{X}$ $\quad |\tilde{X}| = l < n$

$A_1 \dots A_l$ $\quad x_1 \dots x_n \in \tilde{X}$

$x_i \in A$ for $i \le l$

Remaining collection

$A_{l+1} \dots A_n$

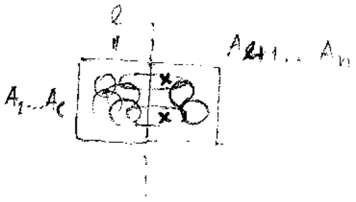
$A_{l+1} \dots A_n$ ⇒ PUFF! ⇒ $\boxed{\square \text{QED}}$

Abra Cadabra!

$\|$

$\boxed{A_{l+1} \dots A_n - \tilde{X}}$ satisfies the collection

If you add any set to critical family, you violate the rule.



„⇐" trivial