CAYLEY'S FORMULA

T: There are n^{n-2} different labeled trees on n vertices.

T_n number of labeled trees

① BIJECTION by Jogol

trees on $N = \{1,\dots,n\}$ ~~left~~ left ○ right □

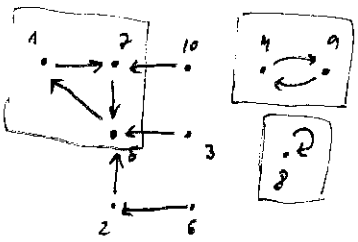
$\mathcal{T}_n = \{(1,\circ,\square)\}$; $|\mathcal{T}_n| = n^2 \cdot T_n$

$|\mathcal{T}_n| = n^n$ bijection : the set N^N of all mappings $N \to N$

$f: N \to N$ $\quad \vec{G_f}$ (?)

! $\underline{\mathcal{T}_n}$ is bijection, not f!

Ex.: $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}$



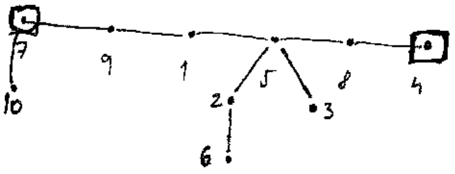
M union of the vertexes with „cycles"

$M \subseteq N$ $\quad M = \{1,4,5,7,8,9\}$

$f/_M = \begin{pmatrix} a, b, \dots & r \\ \underline{f(a), f(b) \dots} & \underline{f(r)} \end{pmatrix}$ restriction of mapping f

let $a < b < \dots < r$ (WLOG) I guess $\quad f/_M = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 7 & 9 & 1 & 5 & 8 & 4 \end{pmatrix}$

We put these „cycle" points on a line and finish trees according to orientation of edges.



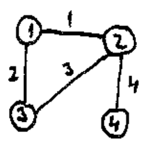
□ QED

② LINEAR ALGEBRA

T_n ... the number of spanning trees in K_n

G ... arbitrary connected simple graph on $V = \{1 \dots n\}$

Incidence matrix



$$\begin{array}{c} \text{edges} \\ B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \text{vertices}$$

→ ~~let C = (-1)·~~ B

$$M = B \cdot B^T = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$ → degrees of vertices

~~1~~ $\tau(G)$ no. of spanning trees $\tau(K) = T_n$

Kirchoff's Theorem: $A(G) = \det M_{ii}$

To prove this, we'll use another theorem:

Binet – Cauchy theorem : $\det(AM) = \dots$ $r\times s$ $s\times r$ $r\le s$???

$$\left|\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{pmatrix}\cdot\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32}\end{pmatrix}\right| = \left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}\cdot\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{vmatrix}\right) + \left(|\dots|\cdot|\dots|\right) + \left(|\dots|\cdot|\dots|\right)$$

$$\det M_{ii} = \cancel{\det} \sum_N (\det N)^2 \qquad N = \text{all submatrices } (n-1)\times(n-1) \text{ of all } C\setminus\{\text{row } i\}$$

$$\Downarrow \quad \underbrace{|\dots|^2 + |\dots|^2 + |\dots|^2}_{N} \qquad \text{NICE!}$$

$$\det N = \begin{cases} \pm 1 & \text{if edges span a tree} \\ 0 & \text{otherwise}\end{cases}$$

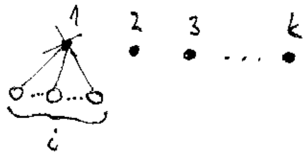
$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = \dots \quad \text{some matrix stuff} \quad + \text{theorem} : \square \text{QED}$$

$$\det M_{ii} = \underbrace{\begin{vmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & -1 & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{vmatrix}}_{n-1} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ -1 & n-1 & \dots & -1 \\ -1 & \dots & \ddots & \vdots \\ -1 & \dots & \dots & n-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & n & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \underbrace{0 \ \dots}_{n-2} & & n \end{vmatrix} = \underline{n^{n-2}} \qquad \square \text{QED}$$

(3) RECURSION (see Recursion) by Riordan & Renyi (?)

A – arbitrary k-set of vertices

$T_{n,k}$ – no. of labeled forests $\{1 \dots n\}$ cons. of k trees where the vertices ~~of~~ of A appears in diff. trees.



Say we remove the least (or the first) label. Then

$$T_{n,k} = \underbrace{\sum_{i=0}^{n-k} \binom{n-k}{i} T_{n-1,k-1+i}}_{} \longrightarrow \text{Counting „fixed" neighbours or sth like that}$$

$$T_{n,k} = k \cdot n^{n-k-1} \quad \text{preposition}$$

$T_{0,0} = 1$, $T_{n,0} = 0$, $T_{n,n} = 1$ (Base case)

$$T_{n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} (k-1+i)(n-1)^{n-1-k-i} \overset{i \to n-k-i}{=} \sum_{i=0}^{n-k} \binom{n-k}{i} (n-1-i)(n-1)^{i-1}$$

$$= \sum_{i=0}^{n-k} \binom{n-k}{i} (n-1)^i - \sum_{i=1}^{n-k} \binom{n-k}{i} \underset{\text{0 does not matter}}{i} (n-1)^i = n^{n-k} - (n-k) \sum_{i=1}^{n-k} \binom{n-1-k}{i-1} (n-1)^{i-1} =$$

$$n^{n-k} - (n-k)\sum_{i=0}^{n-1-k}\binom{n-1-k}{i}(n-1)^{i} = n^{n-k} - (n-k)\,n^{n-k-1} = \underline{k\,n^{n-k-1}}$$

□ QED for $k=1$: $\boxed{n^{n-2}}$

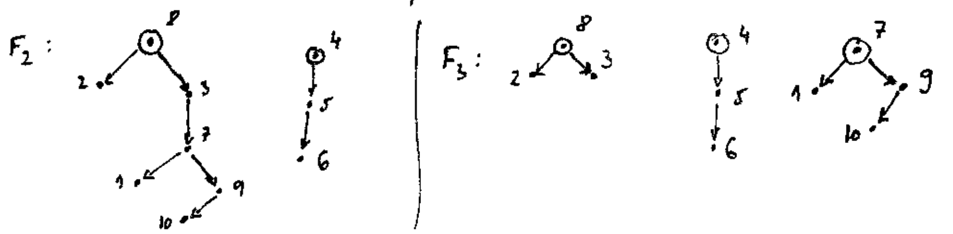
(4) DOUBLE COUNTING by Jim Pitman

rooted forest on $\{1,\dots,n\}$

$\mathcal{F}_{n,k}$ we denote set of all rooted forests that consist of k rooted trees.

$|\mathcal{F}_{n,1}| = n\cdot T_n$ (clear...)

$F_1 \dots F_n$ is „refining" sequence: $F_i \in \mathcal{F}_{n,i}$ and F_i contains F_{i+1} for $\forall i$



Denote:

$N(F_k)$ .. the no. of rooted trees containing F_k

$N^*(F_k)$... no. of refining seq. ending in F_k

Starting at a tree: $N^*(F_k) = N(F_k)\cdot(k-1)!$

Starting at F_k:

$F_k \to F_{k-1}$ add edge $n(k-1)$

$F_{k-1} \to F_{k-2}$ — $n(k-2)$

$\vdots$

$N^*(F_k) = n^{k-1}(k-1)!$

$N(F_k) = n^{k-1}$

$|\mathcal{F}_{n,1}| = n^{n-1}$ □ QED ...