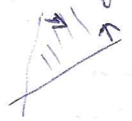
(1) Convex polytopes touching each other

PAIRWISE TOUCHING POLYTOPES

- "to touch" ≡ to intersect, but have disjoint interiors
- polytope × polyhedron

DEF: A polytope (in $\mathbb{R}^d$) is the convex hull of a finite set of points

DEF: A polyhedron (in $\mathbb{R}^d$) is the intersection of finitely many halfspaces



$\rightsquigarrow$ can be unbounded

THEOREM: For any d, a <u>bounded</u> polyhedron is a polytope and vice versa.

↳ difficult, take optimization course!

- THEOREM: For any k, there exist k polyhedra in $\mathbb{R}^3$ s.t. each two of them touch "face-to-face"

DEF: <u>cyclic</u> polytope (in $\mathbb{R}^d$) = the convex hull of $\vec{c(1)}, \vec{c(2)} \dots \vec{c(k)}$ where $\vec{c(t)} = (t, t^2, t^3, \dots t^d)$

LMA: <u>Every two vertices</u> of a cyclic polytope are joined by <u>an edge</u>

- $(t-i)^2(t-j)^2 \geq 0$

$t^4 - (2i+2j)t^3 + (..)t^2 - (\dots)t \; i^2 j^2$

$\vec{k} \cdot (t^4, t^3, t^2, t) \leq 1$

$\rightsquigarrow$ hyperplane $\vec{k} \cdot \vec{c}(t) = 1$

$\vec{k}\vec{c(i)} = 1 = \vec{k}\vec{c(j)}$, but

$k\vec{c(\ell)} < 1$ for all $p \neq i,j$

↳ ij is an edge

<u>proof</u>: Let $\vec{c}(t) = (t, t^2, t^3, t^4)$, and consider the points $\vec{c(1)}, \dots, \vec{c(k)}$. For each $i \neq j \in \{1, .. k\}$ there is $\vec{k}(i,j)$ such that $\vec{k(i,j)} \cdot \vec{c}(i) = 1 = \vec{k(i,j)} \cdot \vec{c}(j)$

while $\forall \ell \in \{1..k\} - \{i,j\}$ it is $\vec{k(i,j)} \cdot \vec{c}(\ell) < 1$. This $\vec{k}$ comes from $(t-i)^2(t-j)^2 \underset{\geq 0}{\neq 0}$

Consequently, there exists a 2-dim disk D_ε (diam ε) in $\mathbb{R}^4$ such that

$\forall \vec{v} \in D_\varepsilon$: $(\vec{k}(i,j) + \vec{v}) \cdot \vec{c}(i) = 1 = (\vec{k(i,j)} + \vec{v}) \cdot \vec{c(j)}$ and

$(\vec{k}(i,j) + \vec{v}) \cdot \vec{c}(\ell) < 1$

POLAR CONSTRUCTION: polytope $\mathrm{hull}(\vec{c(1)}, \dots, \vec{c(k)}) \rightsquigarrow$ polyhedron $\bigcap_i H_i^-$

where H_i^- is def. by $\vec{x} \cdot \vec{c(i)} \leq 1$

Seeing this "dually", the whole set $\vec{k(i,j)} + D_\varepsilon$ belongs to the hyperplanes H_i, H_j, and is strictly inside the halfspaces H_ℓ^- for $\ell \in \{1, \dots k\} - \{i,j\}$, where $H_m^- = \{\vec{x} \in \mathbb{R}^4 : \vec{c(m)} \cdot \vec{x} \leq 1\}$.

It remains to set $B_i = H_i \cap \bigcap \{H_\ell^- : \ell \in \{1, \dots k\} \setminus \{i\}\}$ (a 3-D body)

and project $B_1 \dots B_k$ into $\mathbb{R}^3$.

PROOFS OF THE INFINITY OF PRIME NUMBERS

DEF: A prime number $p \in \mathbb{N}$ such that p is divisible by 1 and p only

PROOF 1: Suppose $\mathbb{P} = \{p_1 \dots p_n\}$ is finite set of all prime numbers.

Take $m = \left(\prod_{i=1}^{n} p_i\right) + 1$ $\rightsquigarrow$ m is not divisible by any prime $\rightsquigarrow$ CONTRADICTION

PROOF 2:

LMA: $\forall n \in \mathbb{N}$ $n, n+1$ are coprime

Take $n > 1$. $N_1 := n$

$N_k := N_{k-1} \cdot (N_{k-1} + 1)$

- N_k has at least k prime factors (for any k)

PROOF 3:

$F_n = 2^{2^n} + 1$

LMA: $F_n = \prod_{k=0}^{n-1} F_k + 2$ (By induction: $F_1 = F_0 + 2$ ✓; $\prod_{k=0}^{n} F_k + 2 = (F_{n}-2) F_n + 2 = (2^{2^n} - 1) \cdot (2^{2^n} + 1) = 2^{2^{n+1}} + 1 = F_{n+1}$)

$\hookrightarrow$ $\forall i, j$ F_i and F_j are coprime

PROOF 4: $\pi(x) :=$ num. of primes $\leq x$

$H_n = \frac{1}{1} + \frac{1}{2} + \dots \frac{1}{n}$ $\quad \ln n+1 < H_n \leq \frac{1}{m} = \prod_{\substack{p \in P \\ p \leq x}} \left(\sum_{k=0}^{\infty} \frac{1}{p^k} \right)$