IV119

20.03. NECESSARY AXIOMS TO KNOW

FOR TWO DISTINCT POINTS THERE EXISTS EXACTLY ONE LINE PASSING THROUGH THEM.

METRIC AXIOMS: $d(X,Y) \geq 0$

$d(X,Y) = d(Y,X)$

$d(X,Z) \leq d(X,Y) + d(Y,Z)$

"BETWEENESS" $B\ x\ y\ z$

- FROM AXIOMS WE HAVE

$\forall X,Y \in L : \exists Z \in L : d(X,Y) = d(X,Z) + d(Z,Y)$

THEOREM 1: IN ANY CONFIGURATION OF n POINTS IN THE PLANE, NOT ALL ON A LINE, THERE IS A LINE WHICH CONTAINS EXACTLY TWO OF THE POINTS.

(J. J. SYLVESTER)

EXAMPLE FOR $n=4$

PROOF:

$\mathcal{P}$ - SET OF ALL POINTS $|\mathcal{P}| = n$

$\mathcal{L}$ - SET OF ALL LINES $\exists X,Y \in \mathcal{P} : L \in \mathcal{L} : X,Y \in L$

$$\mathbb{A} = \{(P,L) \in \mathcal{P} \times \mathcal{L} \mid P \notin L\}$$

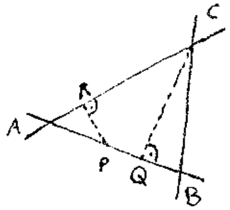
WE DEFINE ORDERING ON $\mathbb{A}$:

$(P,L) \leq (Q,k) \Leftrightarrow d(P,L) \leq d(Q,k)$

NOTE THAT $\forall (P,L) \in \mathbb{A} : d(P,L) > 0$

LET THIS BE THE PAIR WITH THE MINIMUM DISTANCE (P_0, L_0)

CLAIM: L_0 IS THE LINE WE ARE LOOKING FOR



- LEMMA: L_0 CANNOT CONTAIN A THIRD POINT
- CONSTRUCT $Q \notin \mathcal{P}$ $\overleftrightarrow{AB} \equiv L_0$ $C \equiv P_0$

$R \notin \mathcal{P}$
- WE WANT TO PROVE $d(P,R) \leq d(Q,C)$

$\left.\begin{array}{l} \triangle AQC \sim \triangle ARP \\ d(A,P) \leq d(A,Q) \end{array}\right\} \Rightarrow d(P,R) \leq d(Q,C)$ CONTRADICTION $\square$

(PROOF BY COXETER)

„A WICKED PICTURE"

- HOWEVER THE CIRCULAR LINE IS NOT ALLOWED BY THE AXIOMS

THEOREM 2: LET P BE A SET OF $n \geq 3$ POINTS IN THE PLANE, NOT ALL ON A LINE. THEN THE SET $\mathcal{L}$ OF LINES PASSING THROUGH AT LEAST TWO POINTS CONTAINS AT LEAST n LINES

(ERDŐS, DE BRUIJN)

PROOF: BY INDUCTION.

BASE $n=3$

INDUCTION

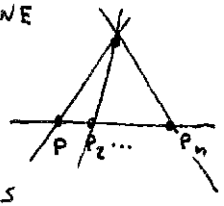
I. HYPOTHESIS: $|\mathcal{P}| = n \Rightarrow |\mathcal{L}| \geq n$

I. STEP: $|\mathcal{P}| = n+1$, $\mathcal{P} = \{P, P_2, \dots, P_n, Q\}$

L_0 SUCH THAT $P, Q \in L_0$

- REMOVE Q: $\mathcal{P}' = \mathcal{P} - \{Q\}$ THEN $|\mathcal{P}'| = n$

TWO CASES: AFTER REMOVING Q ALL POINTS MAY OR MAY NOT BE ON ONE LINE



$\square$

- NOW WE START TO THINK IN TERMS OF SETS

THEOREM 3: LET $\mathbb{X}$ BE A SET OF $n \geq 3$ ELEMENTS AND LET $A_1, \dots, A_m$ BE PROPER SUBSETS OF $\mathbb{X}$ SUCH THAT EVERY PAIR OF ELEMENTS OF $\mathbb{X}$ IS CONTAINED IN PRECISELY ONE SET A_i $(1 \leq i \leq m)$. THEN $m \geq n$ HOLDS.

(MOTZKIN, CONWAY)

EXAMPLE: $\mathbb{X} = \{1,2,3\}$

$A_1 = \{1,2\}$, $A_2 = \{1,3\}$, $A_3 = \{2,3\}$

PROOF 1: - LET r_x DENOTE NUMBER OF SETS CONTAINING x

- OBSERVATION $2 \leq r_x < m$

$\underset{n \geq 3}{\Uparrow}$ $\underset{\text{DIRICHLET'S PRINCIPLE}}{\Uparrow}$

EXAMPLE: $\mathbb{X} = \{1,2,3,4,5\}$

$x = 3$, $A_i = \{1,5\} \Rightarrow$ FOR SOME DISTINCT j,k $\{1,3\} \subseteq A_j$, $\{5,3\} \subseteq A_k$

$x \notin A_i \Rightarrow r_x \geq |A_i|$

BY CONTRADICTION

$m < n$

$m|A_i| < r_x n$

$m(|A_i| - n) < n(r_x - m)$

$m(n - |A_i|) > n(m - r_x)$ $(*)$

$$M = 1 = \sum_{x \in \mathbb{X}} \frac{1}{n} = \sum_{x \in \mathbb{X}} \sum_{A_i : x \notin A_i} \frac{1}{n(m - r_x)}$$

$$N = 1 = \sum_{A_i} \frac{1}{m} = \sum_{A_i} \sum_{x : x \notin A_i} \frac{1}{m(n - |A_i|)}$$

Q: IS THAT EVEN LEGAL?
A: NO, I'LL BE PUT IN JAIL.

FROM $(*)$ WE WOULD HAVE $N < M$.
CONTRADICTION $\square$

IT IS MAGIC, BUT IT HOLDS. TWO GUYS CLAIM IT!

PROOF 2: $B = (x_i; A_1, \dots, A_m)$ B IS A MATRIX

(LINEAR ALGEBRA)

$$B_{x_i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

EXAMPLE $X = \{1,2,3\}$

$A_1 = \{1,2\}$ $A_2 = \{2,3\}$ $A_3 = \{1,3\}$

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$B \cdot B^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

EIGENVALUES (OF A SQUARE MATRIX)

λ SUCH THAT $Ax = \lambda x$ FOR SOME VECTOR x (EIGENVECTOR)

- WE CAN FIND THEM USING

$$\det(A - I \cdot \lambda) = 0$$

$$B \cdot B^T = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{M_1} + \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{M_2}$$

$$\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^3 \Rightarrow \lambda = 1$$

$$\det(B \cdot B^T - I \cdot \lambda) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = 0 \qquad \lambda^3 - 3\lambda^2 = 0$$

$$\lambda^2(\lambda - 3) = 0$$

M_1 - POSITIVE-DEFINITE MATRIX

M_2 - SEMI-DEFINITE MATRIX

$\to M_1 + M_2$ - POSITIVE-DEFINITE

$\text{rank}(A)$ - NUMBER OF LINEARLY INDEPENDENT COLUMNS & ROWS OF A / MINIMUM OF THE

$$\text{rank}(B \cdot B^T) = n \Rightarrow \text{rank}(B) = n \Rightarrow n \le m$$

WE CANNOT INCREASE THE NUMBER OF ROWS

THIS CONCLUDES THE PROOF. □

THEOREM ~~PROOF~~ 3A: IF WE DECOMPOSE A COMPLETE GRAPH K_n INTO m CLIQUES (GRAPH THEORY) DIFFERENT FROM K_n SUCH THAT EVERY EDGE IS IN A UNIQUE CLIQUE, THEN $m \ge n$. (NOTE THAT IT IS REFORMULATED THEOREM 3)

(CLIQUE IS A COMPLETE SUBGRAPH OF A GRAPH)

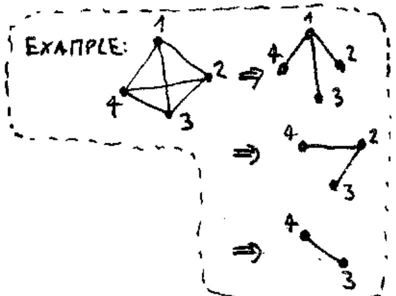
PROOF:

$K_n = (V, E)$ $V = \{1, \dots, n\}$

THIS PROOF IS CONSTRUCTIVE AND LEADS TO A DECOMPOSITION ALGORITHM.

- TAKE THE COMPLETE BIPARTITE GRAPH JOINING 1 TO ALL OTHER VERTICES

EXAMPLE:



- REPEAT JOINING FOR $j \in \{2, \dots, n\}$ WITH VERTICES $\{j+1, \dots, n\} \Rightarrow K_{1,n-2} \dots K_{1,1}$

$$m = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} \ge n$$

SUM OF EDGES PER CLIQUE

ALTHOUGH WE ONLY NEED IT FOR $n \ge 3$

□

CAN WE DO ANY BETTER THAN THE RESULT OF THEOREM 3A? NO.

THEOREM 5: IF K_n IS DECOMPOSED INTO COMPLETE BIPARTITE SUBGRAPHS $H_1, \dots, H_n$ THEN $m \ge n-1$.

PROOF BY CONTRADICTION

H_i CAN BE SPLIT INTO SETS OF VERTICES L_i, R_i

$i \to x_i$

WE ASSIGN A VARIABLE TO EACH i

$$\sum_{i<j\le m} x_i x_j = \sum_{k=1}^{m} \left(\sum_{a \in L_k} x_a \cdot \sum_{b \in R_k} x_b \right)$$

SUPPOSE THAT $m < n-1$, WE'LL CONSTRUCT A SYSTEM OF LINEAR EQs

$$x_1 + \dots + x_n = 0$$

$$\sum_{a \in L_k} x_a = 0 \qquad k = 1 \dots m$$

THE DIMENSION OF THE SPACE OF SOLUTIONS IS NON-ZERO

ASSUME c IS A NON-TRIVIAL SOLUTION

$c = (c_1, \dots, c_n)$

$$0 = \left(\sum_{i=1}^{n} c_i\right)^2 = \sum_{i=1}^{n} c_i^2 + 2 \overbrace{\sum_{i<j\le m} c_i c_j}^{=0} = \sum_{i=1}^{n} c_i^2 > 0$$

AT LEAST ONE OF c_i IS NON-ZERO

$m \ge n-1$ FOLLOWS. □