LINES IN THE PLANE AND DECOMPOSITIONS OF GRAPHS

Incidence axioms (actually just one of them)

- Let A, B be distinct points, then there is exactly one line containing both of them.

Metric axioms

(Metric) notation of distance $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$

- $d(x,y) \geq 0$ $\quad$ ($= 0$ iff $x = y$)
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$ $\quad$ – triangle inequality

Order axioms (Tarski's axioms)

Notation: „betweeness" $\quad Bxyz \equiv$ y is "between" x and y, y is a point on the line segment xz

„congruence" $\quad wx \equiv yz$ means $d(w,x) = d(y,z)$

Theorem 1: In any configuration of n points in the plane, not all on a line, there is a line which contains exactly two of the points.

// Introduction (some cool pics and so on...) | $\mathbb{P}$ - set of points, $\mathbb{L}$ - set of lines

Let $\mathbb{A} = \{(P,\ell) \in \mathbb{P} \times \mathbb{L} \mid P \notin \ell\}$

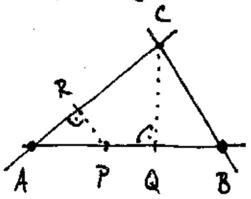
Choose (P_0, l_0) such that $\forall P \in \mathbb{P}, \ell \in \mathbb{L}$ $(P \notin \ell)$: $\underline{d(P_0, l_0) \leq d(P, \ell)}$

$\hookrightarrow d(P,\ell) \overset{def}{=} d(P, L_0) : L_0 \in \ell \wedge \forall L \in \ell : d(P, L_0) \leq d(P, L)$

Claim: l_0 is the line we are looking for.

// Show the proof

// When proving that „created" distance is less or equal, use similarity of triangles



$Q, R \notin \mathbb{P}$: $\quad P \in AQ \wedge R \in AC \Rightarrow |\sphericalangle CAQ| = |\sphericalangle RAP|$

from def of $d(P,\ell)$: $|\sphericalangle AQC| = |\sphericalangle ARP| = 90°$

Hence ~~$\triangle ACQ \cong$~~ $\underline{\triangle AQC \sim \triangle ARP}$

Since $AR \leq AC \wedge AP \leq AQ \Rightarrow \underline{\underline{PR \leq CQ}}$ $\quad \square$ QED

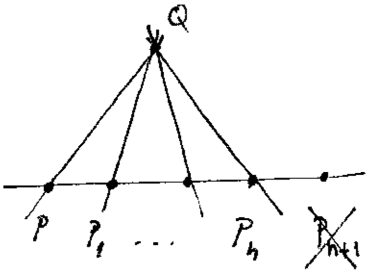
⋮

$\square$ QED

THEOREM 2: Let $\mathbb{P}$ be a set of $n \geq 3$ points in the plane, not all on a line. Then the set $\mathbb{L}$ of lines passing through at least two points contains at least n points.

// Very easy proof by induction

// There is possibly a mistake in the book



on

we have n P's + 1 Q

=> $n+1$ points

⋮

□ QED

THEOREM 3: Let $\mathbf{X}$ be a set of $n \geq 3$ elements, and let $A_1 \dots A_m$ be proper subsets of $\mathbf{X}$, such that every pair of elements of $\mathbf{X}$ is contained in precisely one set A_i. Then $m \geq n$ holds.

NOTATION: for $x \in \mathbb{X}$: r_x be the number of sets A_i containing x.

PROOF 3A: Assumption: $\underline{2 \leq r_x < m}$

If $x \notin A_i \Rightarrow r_x \geq |A_i|$

Let $m < n \rightarrow m|A_i| < r_x n \quad / - mn$

$$m\left(|A_i| - n\right) < n\left(r_x - m\right) \quad / \cdot (-1)$$

$$m\left(n - |A_i|\right) > n\left(m - r_x\right)$$

- $$1 = \sum_{x \in X} \frac{1}{n} = \sum_{x \in X} \sum_{A_i : x \notin A_i} \frac{1}{n\left(m - r_x\right)} = A$$

- $$1 = \sum_{A_i} \frac{1}{m} = \sum_{A_i} \sum_{x : x \notin A_i} \frac{1}{m\left(n - |A_i|\right)} = B$$

$m\left(n - |A_i|\right) > n\left(m - r_x\right) \Rightarrow A > B \Rightarrow \underline{1 > 1}$ contradiction

□ QED

PROOF 3.2 B = incidence matrix $(X;\ A_1 \dots A_m)$

where $B_{xi} = \begin{cases} 1 \Leftarrow x \in A_i \\ 0 \Leftarrow x \notin A_i \end{cases}$

// Don't forget to show examples!

$$BB^T = \begin{pmatrix} r_{x_1} & 1 & \cdots & 1 \\ 1 & r_{x_2} & \cdots & 1 \\ 1 & 1 & \ddots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & r_{x_n} \end{pmatrix} = \underbrace{\begin{pmatrix} r_{x_1}-1 & 0 & \cdots & 0 \\ 0 & r_{x_2}-1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & r_{x_n}-1 \end{pmatrix}}_{\text{positive definite : } \lambda \geq 0} + \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}_{\text{semi-definite } \lambda \in \{n, 0\}}$$

BB^T is positive definite $\Rightarrow$ rank$(BB^T) = n$

Hence rank(B) is at least n $\Rightarrow$ $n \leq m$ □ QED

NOTES : Eigenvectors and eigenvalues for A (square matrix)

$A \cdot \mathbf{v} = \lambda \mathbf{v}$ → "characteristic value"

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = 0$$

$\det\left(A_{n \times n} - \mathbb{I}_n \lambda\right) = 0$

~~only for square~~

$$(1-\lambda)^3 + 2 - 3(1-\lambda) = 0$$
$$1 - 3\lambda + 3\lambda^2 - \lambda^3 + 2 - 3 + 3\lambda = 0$$
$$\lambda^3 - 3\lambda^2 = 0$$
$$\lambda^2(\lambda - 3) = 0 \quad \boxed{\begin{matrix} \lambda = 0 \\ \lambda = 3 = n \end{matrix}}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = 0$$

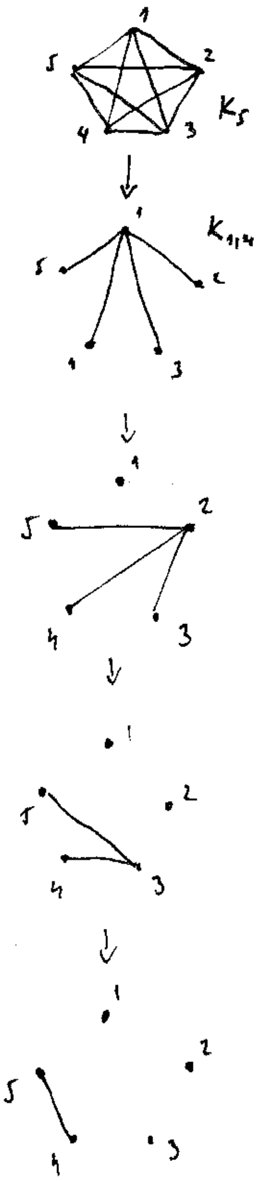
$$(1-\lambda)^3 = 0 \rightarrow \boxed{\lambda = 1 \geq 0} \ : \lambda = n-2$$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \det \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)^3 + 2 - 3(2-\lambda) = 0$$
$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \quad \boxed{\begin{matrix} \lambda_1 = 4 \\ \lambda_{2,3} = 1 \end{matrix} \ \geq 0}$$

THEOREM 3: If we decompose a complete graph K_n into m cliques different
(modified) from K_n, such that every edge is ~~on~~ in a unique clique, then $m \geq n$

PROOF 3.3: $X \sim \cancel{K_n}\ V$ $\quad \left[K_n = (V, E)\right]$
$A_i \sim$ cliques



How to decompose graph:

- number the vertices $\{1 \dots n\}$
- take the complete bipartite graph joining 1 to all other vertices $\{2 \dots n\}$
  - → we obtain graph $K_{1,n-1}$ (star)
- repeat joining for $j \in \{2 \cdots n\}$ with $\{j \dots n\}$
  - → we obtain graphs $K_{1,n-2} \dots K_{1,1}$

This decomposition uses $n-1$ complete bipartite graphs
Hence $m = \sum_{i=1}^{n-1} i = \underline{\frac{1}{2}n(n-1) \geq n}$ $\quad (n \geq 3)$ $\qquad$ □QED

Proof of ~~above~~ the claim above: (induction)

- Base: $n = 3$: $\underline{3 \geq 3}$ holds
- I. Hypothesis: $\frac{1}{2}n(n-1) \geq n$
- I. Step: $m = n+1$: $\frac{1}{2}m(m-1) = \frac{1}{2}n(n+1) = \frac{1}{2}(n^2+n) = \frac{1}{2}(n^2-n+2n)$

$$= \underline{\tfrac{1}{2}(n-1)n + n \geq n+1} = m \quad (n \geq 3) \qquad \square\text{QED}$$

V.

THEOREM 4: If K_n is decomposed into complete biparite ~~graf~~ subgraphs $H_1 \dots H_m$, then $m \geq n+1$

PROOF 4: $K_n = (V, E)$ $V = \{1 \dots n\}$

Let R_j, L_j be the defining vertex sets of complete biparite graph H_j $j \in \{1 \dots m\}$

To every vertex i we assoc. var. x_i.

$H_1 \dots H_m$ decompose K_n:

$$(1) \quad \sum_{i<j\leq m} x_i \cdot x_j = \sum_{k=1}^{m} \left(\sum_{a \in L_k} x_a \cdot \sum_{b \in R_k} x_b \right)$$

Suppose that $m < n-1$

Then
$$x_1 + \dots + x_n = 0$$
$$\sum_{a \in L_k} x_a = 0 \quad (k = 1 \dots m)$$

has fewer equations then variables and hence there exist a non-trivial solution $c = (c_1 \dots c_n)$

$$\sum_{i<j\leq m} c_i c_j = 0 \; *$$

$$0 = (c_1 + \dots + c_n)^2 = \sum_{i=1}^{n} c_i^2 + 2 \sum_{i<j\leq n} c_i c_j = \sum_{i=1}^{n} c_i^2 > 0 \quad \text{(contradiction)}$$

$\square$ QED

$$* \quad \sum_{i<j\leq n} c_i \cdot c_j = \sum_{k=1}^{m} \left(\underbrace{\sum_{a \in L_k} c_a}_{0} \cdot \sum_{b \in R_k} c_b \right) = 0$$