THE PIDGEON HOLE PRINCIPLE

If n objects are placed in r boxes, where $r < n$, then at least one of the boxes contains more than one object.

 $n = 6$, $r = 5$ „ obvious observation"

N, R $\qquad |N| = n > r = |R|$

$f: N \to R \qquad (\exists a \in R)(|f^{-1}(a)| \geq 2)$

stronger: $|f^{-1}(a)| \geq \left\lceil \frac{n}{r} \right\rceil$ $\qquad$ Contr. $\forall a \in R : |f^{-1}(a)| < \frac{n}{r}$

$n < r\frac{n}{r}$

$\underline{n < n}$ contradiction

① Numbers

Claim: Consider numbers $1, 2 \ldots 2n$ and take any $n+1$ of them.
Then there are two among those $n+1$ numbers which are coprime.

Proof: Obviously, there are some of the numbers which are consequental.
(If it wasn't true, we'd need $\geq 2n+2$ numbers) ~~which~~

$(\forall k \in \mathbb{N})(\ GCD(k, k+1) = 1\ )$

Claim transformation: ... such that one divides the other.

$A \subseteq \{1, 2, \ldots, 2n\} \qquad |A| = n+1$

$a \in A \qquad a = 2^k \underbrace{m} \qquad \underline{n}$ different odd parts

$\square$ QED

② Sequences

Claim: In any sequence $a_1, a_2, \ldots, a_{mn+1}$ of $(mn+1)$ distinct $\mathbb{R}$ numbers,
there exists an increasing subsequence $a_{i_1} < a_{i_2} < \ldots < a_{i_{m+1}}$ $(i_1 < i_2 < \ldots < i_{m+1})$
of length $m+1$ or a decreasing subsequence $a_{j_1} > a_{j_2} > \ldots > a_{j_{m+1}}$ $(j_1 < j_2 < \cdots < j_{m+1})$
of length $m+1$ or both.

Proof: $a_i \mapsto t_i$ $\quad t_i$ is the length of the longest inc. subsequence starting at a_i

I. $t_i \geq m+1$

II. $\forall i : t_i \leq m$

$$f : \overbrace{\{a_1, a_2, \ldots, a_{m+1}\}}^{P} \to \overbrace{\{1, 2, \ldots m\}}^{Q}$$

$\exists s \in Q \qquad f(a_i) = s \quad$ for $\frac{nm}{m} + 1 = \underline{n+1}$

$a_{j1}, a_{j2}, \ldots a_{jn+1} \qquad (\cancel{a_j}\ j1 < j2 < \ldots < jn+1)$

Cons. $\quad a_{j}, a_{j+1} \quad : \quad \underline{a_j < a_{j+1}}$

This is a contradiction. Or it (at least) leads to some.

□QED

③ SUMS

Claim: Suppose we are given n integers $a_1 \ldots a_n$, which needn't be distinct. Then there is always a set of consecutive numbers $a_{k+1}, a_{k+2} \ldots a_\ell$ whose sum $\sum_{i=k+1}^{\ell} a_i$ is a multiple of n.

Proof:

$N = \{0, 1, \ldots, n\}$

$R = \{0, 1, 2, \ldots, n-1\}$

$\exists$ $f : N \to R$

$f(m)$ is the remainder of $a_1 + a_2 + \ldots + a_m$ over n.

Since $|N| = n+1 > n = |R|$

$a_1 + \ldots + a_k \;,\; a_1 + \ldots + a_k + \ldots + a_\ell \qquad (k < \ell)$

have the same remainder.

$$\underbrace{\sum_{i=1}^{\ell} a_i - \sum_{i=1}^{k}}_{\text{this is a multiple}} = \boxed{\sum_{i=k+1}^{\ell} a_i} \qquad \text{(we are considering an empty sum } (k=0) \text{ too!)}$$

this is too

□QED

ALGORITHM

Problem: Given n points $P_1, P_2, \dots P_n$ on a line, find a pair of them with the maximal distance such that ~~their~~ there is no other point lying between them. (Input is not in sorted order)

Distances: $\langle \pi_1, \dots, \pi_n \rangle$ Reordering of input seq. in sorted order.

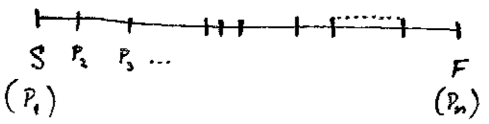
- Sorting can be done in $O(n \log n)$, but this problem can be solved in $O(n)$

Observation: max. dist. should be at least $\frac{\pi_n - \pi_1}{n}$ (obviously)

- can be proved by contradiction

$$\pi_n - \pi_1 = \sum_{i=1}^{n-1} \pi_{i+1} - \pi_1 < \sum_{i=1}^{n-1} \dots \dots \dots \quad \text{Meh, } \square \text{QED}$$

$$\frac{\pi_n - \pi_1}{n-1} \leq \max$$



$\langle a_1, a_2 \rangle, \langle a_2, a_3 \rangle, \dots, \langle a_{n-1}, a_n \rangle$ distances

Our algorithm will remember left-most and right-most points

left[1...n-1]

right[1... n-1]

(too many lines in the picture using 2 bit colors, I've got only 1 bit !!)

right-most point of an interval

↕ length

left-most. p. of the next non-empty interval

can be done $\underline{\underline{O(n)}}$:)