PIGEON-HOLE PRINCIPLE:

IF n OBJECTS ARE PLACED IN r BOXES, WHERE $r < n$, THEN AT LEAST ONE OF THE BOXES CONTAINS MORE THAN ONE OBJECT.

IN LANGUAGE OF MAPPINGS, IT READS:
LET N AND R BE TWO FINITE SETS WITH

$$|N| = n > r = |R|,$$

and let $f: N \to R$ BE A MAPPING. THEN THERE EXISTS SOME $a \in R$ WITH $|f^{-1}(a)| \geq 2$.
STRONGER INEQUALITY:
THERE EXISTS SOME $a \in R$ WITH

$$|f^{-1}(a)| \geq \left\lceil \frac{n}{r} \right\rceil$$

IN FACT, OTHERWISE WE WOULD HAVE $|f^{-1}(a)| < \frac{n}{r}$ FOR ALL a, AND HENCE $n = \sum_{a \in R} |f^{-1}(a)| < r \cdot \frac{n}{r} = n$, WHICH CANNOT BE.

1. NUMBERS

CLAIM 1.1: CONSIDER THE NUMBERS $1, 2, 3, \ldots, 2n$, and TAKE ANY $n+1$ OF THEM. THEN THERE ARE TWO AMONG THESE $n+1$ NUMBERS WHICH ARE RELATIVELY PRIME.

RELATIVELY PRIME = COPRIME = GCD IS 1

OBVIOUS. THERE MUST BE TWO NUMBERS WHICH ARE ONLY 1 APART, AND HENCE RELATIVELY PRIME.

IN 1..2n NUMBERS WE CAN HAVE ONLY n NUMERS, WHERE NONE OF THEM ARE CONSECUTIVE. CONSIDER FOLLOWING:

$\{2, 4, 6, \ldots, 2n\}$.

CLAIM 1.2: SUPPOSE AGAIN $A \subseteq \{1, 2, \ldots, 2n\}$ WITH $|A| = n+1$. THEN THERE ARE ALWAYS TWO NUMBERS IN A SUCH THAT ONE DIVIDES THE OTHER

PROOF: WRITE EVERY NUMBER $a \in A$ IN THE FORM $a = 2^k \cdot m$, WHERE m IS AN ODD NUMBER BETWEEN 1 AND $2n-1$. SINCE THERE ARE $n+1$ NUMBERS IN A, BUT ONLY n DIFFERENT ODD PARTS, THERE MUST BE TWO NUMBERS IN A, WHICH HAVE THE SAME ODD PART. HENCE ONE IS A MULTIPLE OF THE OTHER. $\square$

NOTE. FOR $|A| = n$, IT IS NOT TRUE. CONSIDER:

$$A = \{n+1, n+2, \ldots, 2n\}.$$

2. SEQUENCES

CLAIM 2.1: IN ANY SEQUENCE $a_1, a_2, \ldots, a_{mn+1}$ OF $mn+1$ DISTINCT REAL NUMBERS, THERE EXISTS AN INCREASING SUBSEQUENCE

$$a_{i_1} < a_{i_2} < \cdots < a_{i_{m+1}} \quad (i_1 < i_2 < \ldots < i_{m+1})$$

OF LENGTH $m+1$, OR A DECREASING SUBSEQUENCE

$$a_{j_1} > a_{j_2} > \ldots > a_{j_{n+1}} \quad (j_1 < j_2 < \ldots < j_{n+1})$$

OF LENGTH $n+1$, OR BOTH.

PROOF: ASSOCIATE TO EACH a_i THE NUMBER l_i WHICH IS THE LENGTH OF THE LONGEST INCREASING SUBSEQUENCE STARTING AT a_i.

I. IF $l_i \geq m+1$ THEN WE HAVE AN INCREASING SUBSEQUENCE OF LENGTH $m+1$.

II. SUPPOSE THAT $l_i \leq m$ FOR ALL i. THE FUNCTION ~~MAPPING~~

~~$f: a_i \mapsto l_i$ MAPPING $\{a_1, \dots, a_{mn+1}\}$ TO $\{1, \dots, m\}$~~

$$f : \{a_1, a_2, \dots, a_{mn+1}\} \to \{1, 2, \dots, m\}$$
$$a_i \mapsto l_i$$

TELLS US BY (1) THAT THERE IS SOME $s \in \{1, \dots, m\}$ SUCH THAT $f(a_i) = s$ FOR $\frac{mn}{m} + 1 = n+1$ NUMBERS a_i.

LET $a_{j_1}, a_{j_2}, \dots, a_{j_{n+1}}$, $(j_1 < j_2 < \dots < j_{n+1})$ BE THESE NUMBERS.

NOW LOOK AT TWO CONSECUTIVE NUMBERS $a_{j_i}, a_{j_{i+1}}$. IF $a_{j_i} < a_{j_{i+1}}$, THEN WE WOULD OBTAIN AN INCREASING SUBSEQUENCE OF LENGTH s STARTING AT $a_{j_{i+1}}$ AND CONSEQUENTLY AN INCREASING SUBSEQUENCE OF LENGTH $s+1$ STARTING AT a_{j_i}, WHICH CANNOT BE SINCE $f(a_{j_i}) = s$. WE THUS OBTAIN A DECREASING SUBSEQUENCE $a_{j_1} > a_{j_2} > \dots > a_{j_{n+1}}$ OF LENGTH $n+1$. □

3. SUMS:

CLAIM: SUPPOSE WE ARE GIVEN n INTEGERS $a_1, \dots, a_n$, WHICH NEED NOT BE DISTINCT. THEN, THERE IS ALWAYS A SET OF CONSECUTIVE NUMBERS $a_{k+1}, a_{k+2}, \dots, a_l$ WHOSE SUM $\sum_{i=k+1}^{l} a_i$ IS A MULTIPLE OF n.

<u>PROOF:</u> FOR THE PROOF WE SET $N=\{0,1,\dots,n\}$ AND $R=\{0,1,\dots,n-1\}$. CONSIDER THE MAP $f: N \to R$, WHERE $f(m)$ IS THE REMAINDER OF $a_1+\dots+a_m$ UPON DIVISION BY n. SINCE $|N| = n+1 > n = |R|$, IT FOLLOWS THAT THERE ARE TWO SUMS $a_1+\dots+a_k$, $a_1+\dots+a_l$ $(k<l)$ WITH THE SAME REMMINDER, WHERE THE FIRST SUM MAY BE THE EMPTY SUM DENOTED BY 0. IT FOLLOWS THAT

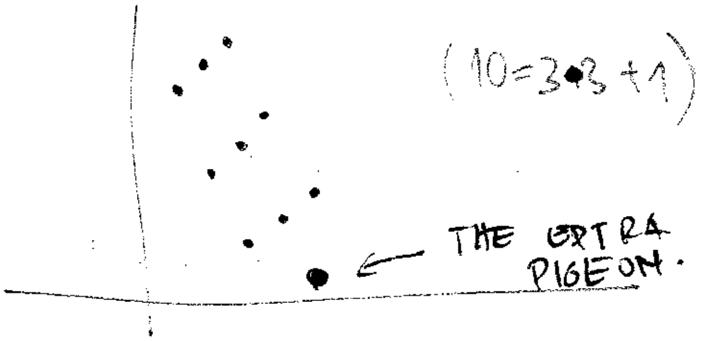
$$\sum_{i=k+1}^{l} a_i = \sum_{i=1}^{l} a_i - \sum_{i=1}^{k} a_i$$

HAS REMAINDER 0 - END OF THE PROOF. $\square$

2. SEQUENCES

FOR $m \cdot n$, NO LONGER TRUE.

CONSIDER WORST CASE



WE GO BY PLACING n NUMB. TUPLES OF ~~[illegible]~~ IN INC. ORDER, WE CAN DO m OF THOSE. WE ARE LEFT WITH THE LAST $m \cdot n+1$ th NUMBER - THE EXTRA PIGEON.