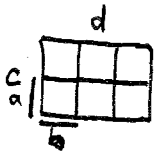
Tiling Rectangles

THEOREM 1: Whenever a rectangle is tiled by rectangles all of which have at least one side of integer length, then the tiled rectangle has at least one side of integer length

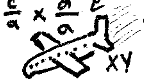
PROOF 1.A

- by NICOLAAS GOVERT DE BRUIJN
- Motivation ~~[illegible]~~ example (fast version ⏩)

$$a,b,c,d \in \mathbb{Z} \Rightarrow \underbrace{\left(a/c \vee a/d\right)} \wedge \left(b/c \vee b/d\right)$$



scaling $\frac{1}{a}$ $\nearrow 1 \times \frac{b}{a}$, $\searrow \frac{c}{a} \times \frac{d}{a}$ — $c \vee d$ is integer



- For any rectangle T in the X-Y plane with sides parallel to the axes

$$\iint_T e^{2\pi i(x+y)}\, dA = 0 \iff \text{at least one side of } T \text{ has integer length}$$

„~~[illegible]~~ fly, you fools" - Gandalf;

$$\int_c^d \int_a^b e^{2\pi i(x+y)}\, dx\, dy = \underbrace{\int_a^b e^{2\pi i x}\, dx}_{0 \iff b-a \in \mathbb{R}} \cdot \int_c^d e^{2\pi i y}\, dy$$

$$\int_a^b e^{2\pi i x}\, dx = \left[\frac{1}{2\pi i} e^{2\pi i x}\right]_a^b = \frac{1}{2\pi i}\left(e^{2\pi i b} - e^{2\pi i a}\right) = \frac{e^{2\pi i a}}{2\pi i a}\left(e^{2\pi i(b-a)} - 1\right)$$

$$e^{2\pi i(b-a)} - 1 = 0 \iff e^{2\pi i(b-a)} = 1$$

#include <euler.h>

$$e^{ix} = \cos x + i \sin x$$

$$\underbrace{\cos 2\pi(b-a)}_{1} + \underbrace{i \sin 2\pi(b-a)}_{0} = 1$$

$$\cos 2\pi(b-a) = 1 \iff 2\pi(b-a) = k2\pi \quad k \in \mathbb{Z}$$

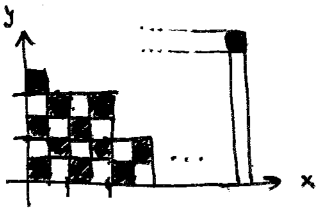
$$k = b-a \Rightarrow b-a \in \mathbb{Z}$$

$$\Rightarrow \sin 2\pi k = 0 \quad (k \in \mathbb{Z})$$

□ QED (kind of)

PROOF 1.B ∘ by RICHARD ROCHBERG & SHERMAN STEIN (Thank you guys)

„Chess board"



Each „segment" (rect.) has equal amount of black & white color.

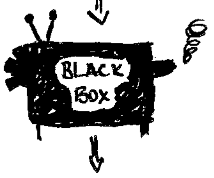
(I didn't hear the whole proof, I was busy colouring rectangles)

<u>Contradiction</u>:

~~min~~

$$\min\left(x, \tfrac{1}{2}\right)\cdot \min\left(y, \tfrac{1}{2}\right) + \max\left(x-\tfrac{1}{2}, 0\right)\cdot \{\max\left(y-\tfrac{1}{2}, 0\right) > \tfrac{1}{2}xy$$

$\Downarrow$



#include <black_box.h>

$\Downarrow$

□QED

PROOF 1.C ∘ by MIKE PATERSON

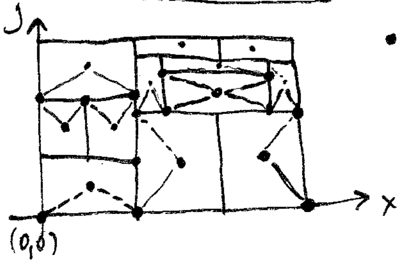
C - set of corners in the tiling for which both coord. are integers

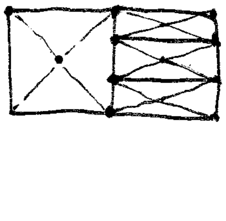
T - set of tiles

*le wild picture with many rectangles

=> <u>BIPARTITE GRAPH</u>

- it has an even number of edges
- •∈C each vertex T is on 2 edges
- when you start (0,0), you may „travel" from this point by the graph to „the other side" (of the graph), which means … □QED ☺ 2.



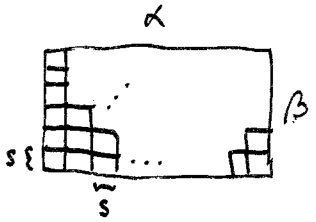


THEOREM 2: A rectangle can be tiled with squares iff the ratio of its side lengths is a rational number.

PROOF 2.A ∘ by MAX DEHN

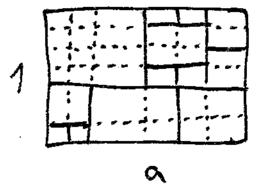
„$\Leftarrow$" $R: \alpha \times \beta$ $\quad \frac{\alpha}{\beta} \in \mathbb{Q}^+ \Rightarrow \frac{\alpha}{\beta} = \frac{p}{q} \quad (p, q \in \mathbb{N} - \{0\})$

$$\frac{\alpha}{p} = \frac{\beta}{q} = s$$



We can tile rect. R with $(p \times q)$ sqares of area $s \times s$.

„$\Rightarrow$"



using scaling

← these are squares... they were meant to be !

Contradiction: let $a \notin \mathbb{Q}$

$$A = \{1, a, a_1, \dots, a_n\} \subseteq \mathbb{R}$$

$V(A)$ (vector space) (rational coefficients)

Base: $B = \{\underline{1, a}, b \dots b_m\}$ (1 isn't enough, since from $1 \cdot q \in \mathbb{Q}$ ($q \in \mathbb{Q}$) we can't get $a \in \mathbb{R}$)

$f: B \to \mathbb{R}$

$f(1) = 1 \qquad f(a) = -1 \qquad f(b_i) = 0 \quad i \geq 3$

Extension:

$f(q_1 b_1 + \dots + q_n b_n) = q_1 f(b_1) + \dots + q_n f(b_n) \qquad q_1 \dots q_n \in \mathbb{Q}$

With the function above we can define area of rectangle

$c, d \in V(A) \qquad \text{area}\left(\square^{d}_{c}\right) = f(c) \cdot f(d)$

[illegible]

1) $\text{area}\left(\square\square^{a}_{c_1 \, c_2}\right) = \text{area}\left(\square^{d}_{c_1}\right) + \text{area}\left(\square^{d}_{c_2}\right)$ (trivial) HOLDS

2) $\text{area}(R) = \sum_{\text{squares}} \text{area}(\square)$

(each square has defined area, since all of them are generated by $V(A)$)

$\text{area}(R) = f(a) \cdot f(1) = -1$

$\text{area}\left(\square_{t}\right) = f^2(t) \geq 0 \Rightarrow \sum_{\text{squares}} \text{area}(\square) \geq 0 \neq 1 = \text{area}(R)$

this is contradiction (obviously)

Hence, $a \in \mathbb{Q}$. ⟶ □QED