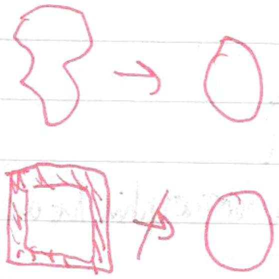
Classification of surfaces

~~Stat~~

Def: Homeomorfism :⇔ "continuous definition"
$R \sim S$



R is homeomorphic to S in a topological space Ω iff $\exists$ continous mapping h: ~~$\Omega \times$~~ $[0,1] \to$ ~~Ω~~ 2^{Ω} such that $h(0) = R$ and $h(1) = S$.

→ THM: Jordan–Schönflies theorem: A homeomorphism of a simple closed curve (loop) onto another can be extended into homeomorphism of a plane.

intuition: Homeomorfism of a curve somehow preserves homeomorphism of faces.

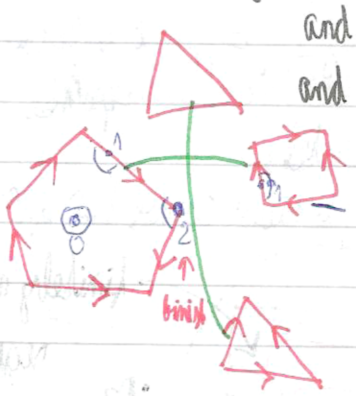
DEF: A surface is a connected compact Hausdorff topological space, locally homeomorphic to a plane (small neighbourhood → open discs).

doesn't have boundary
never ends
no irregular points
it looks nice :)

THM: Every surface can be triangulated (is homeomorfic to triangle = triangle)

← 2-cell because of 2D :)

DEF: 2-cell embedding – choose a finite set of pairwise disjoint polygons in R
and an orientation on their edges (any)
and an perfect matching on the set of edges



– on the union of these polygons, identify the matching pairs of edges in the chosen direction (assume length 1)

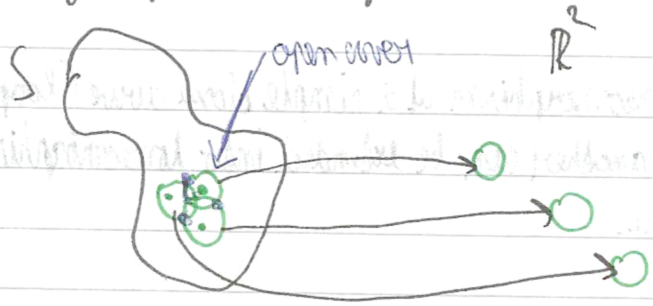
DEF: triangulated surface = created as 2-cell embedding of triangles

LEM: Any connected 2-cell embedding is a surface.

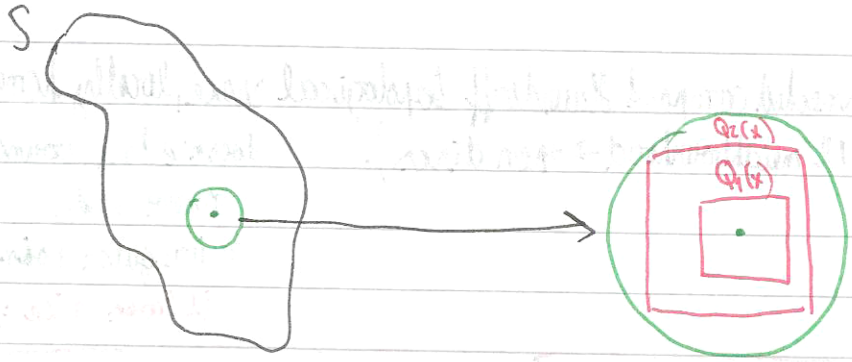
finish!

- connected ✓
- compact (
- locally homeomorphic to a plane (previous image) ✓

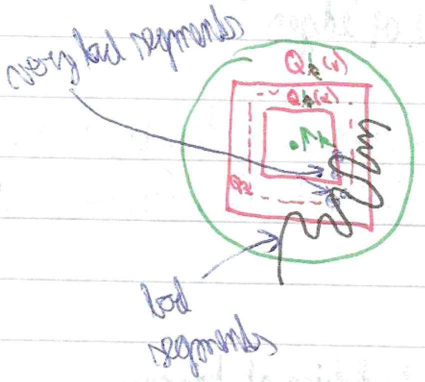
CLAIM: any surface => triangulation



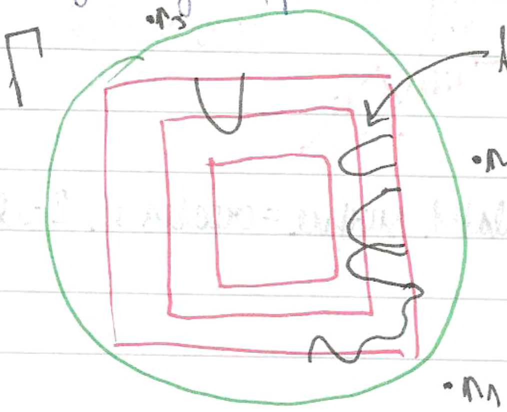
- for every point we have a its neighbourhood <=> open d



- we choose finite subcover from all $x \in S$ by interior of $Q_1[x_i]$, $i = 1, \dots, n$
- by induction on k, boundaries $(Q_1[x_j])$ $j = 1, \dots, k$ intersect in finite points

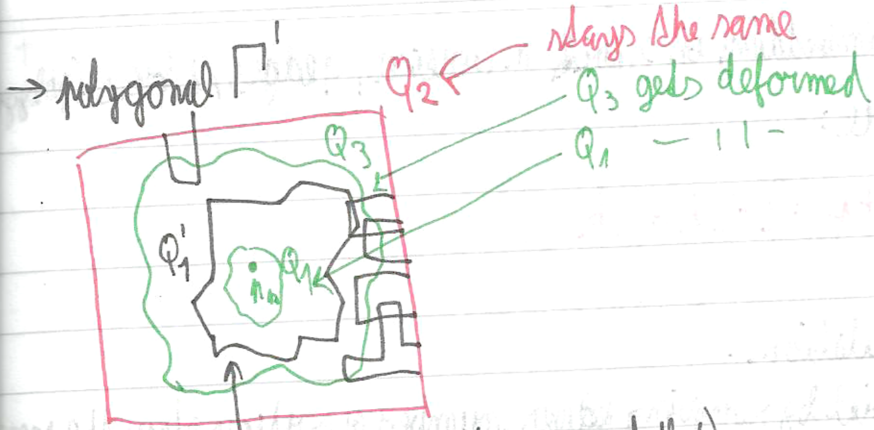


- there are only finitely many very bad, because curve has finite leng



finitely many bad segments

=> finitely many "easy" intersec



(we choose such that)
we draw Q_1' polygonal arc, which intersects only the very bad segments finitely many times

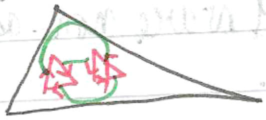
⟶ this become Q_1 (new) and completes the induction

$\Rightarrow \bigcup_{j=1,\dots,m}$ boundaries$(Q_1[r_j])$ is a finite graph on S and we may triangulate it.

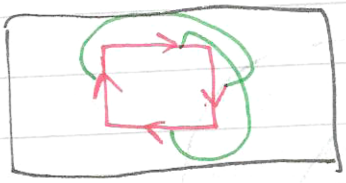
Types of surfaces

DEF: a) orientable S_h; $h \geq 0$ (S_0 is the sphere
b) non-orientable N_k; $k \geq 1$

S_h is obtained from the sphere by adding h handles



N_k is obtained from the sphere by adding k crosscaps



for homework :)
LEM: If S is obtained from a sphere by adding h handles, k crosscaps and ℓ antihandles then: $S \sim S_h$ if $k = \ell = 0$

or $S \sim N_{2h+2\ell+k}$

Th: If a multigraph G is a 2-cell embedding in S with n vertices, q edges, f faces (polygo…
then $S \sim S_h \vee S \sim N_k$ where:

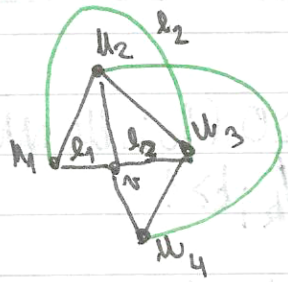
$$n - q + f = 2 - 2h = 2k$$

Proof: Assume G simple triangulation.
Show $n - q + f \leq 2$ (trivial by removing edges, number of vertices stays the same, at most 1 face is removed, it is true in the end: $n-(n-1)+1=$…

Suppose $G \not\sim S_h$, $G \not\sim N_k$ and:

1.) $2 - n + q - f$ as small as possible
2.) (1) applies, now minimal n
3.) (1),(2) applies, $\delta(G)$ minimal

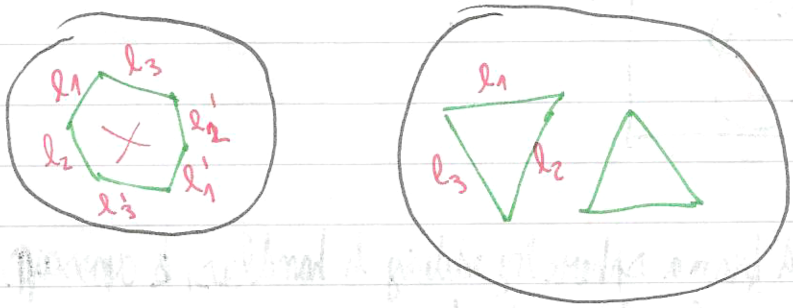
What is the minimal $\delta(G)$? By contradiction $\delta(G) \geq 3$



Then $\delta(G) \geq 3$, $|V(G)| \geq 4$.

Let $\deg(v) = \delta(G)$: If two neighbours u_1, u_3 of v are non-adjacent, then replace $\overline{vu_2}$ by $\overline{u_1u_3}$ getting deg $\delta(G) - 1$.

Otherwise, cut the surface along $\triangle_{v u_1 u_3}$ and see that this is a crosscap, handle or antihandle.



THM: a surface is exactly one of $S_h, h \geq 0$ or $N_k, k>0$.

- we need to show: that $S_i \not\simeq S_j$ for $i \neq j$

$N_i \not\simeq N_j$ for $i \neq j$

$S_i \not\simeq N_j$ for ~~any~~ $i \neq 2j$

LEM: $S_h \not\simeq S_{h'}$ $h' < h$

$N_k \not\simeq N_{k'}$ $k' < k$

$S_h \not\simeq N_k$ for $k \neq 2h$

Proof: Triangulation G, with n, q, f.

$2q = 3f$ (from 2-cell embedding :))

~~$2q = \sum_{i=1}^{n} d_i$ (sum of the degrees)~~

$$n - q + f = 2 - 2h \qquad\qquad n - q + f = 2 - k$$

$$3n - 3q + 3f = 6 - 6h \qquad\qquad 3n - 3q + 3f = 6 - 3k$$

$$3n - 3q + 2q = 6 - 6h \quad \text{or} \quad 3n - 3q + 2q = 6 - 3k$$

$$\underline{q = 3n - 6 + 6h} \qquad\qquad \underline{q = 3n - 6 + 3k}$$

∎

LEM: S_h has no one-sided closed curve.

CLAIM: N_k has a one-sided closed curve.