13.3.2012

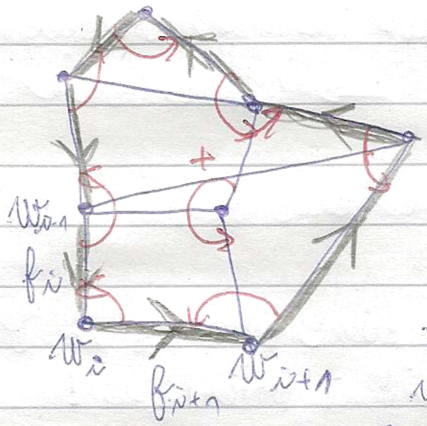
How to embed a graph into a surface?

Def.: An embedding of a graph G into a surface $\mathcal{S}$ is a mapping $V(G) \to$ points of $\mathcal{S}$, $E(G) \to$ arcs in $\mathcal{S}$, such that (seen arcs as "open"): the images of $V(G) \cup E(G)$ are pairwise disjoint.
(loosorded incidence modes of a graph!)
Then G is seen a subset $G \subseteq \mathcal{S}$.

Def.: a face of embedded $G \subseteq \mathcal{S}$ is an arcwise-connected component of $\mathcal{S} \setminus G$.

Edmonds-Ringel rotation principle:

Def.: Rotation system of a graph G is a set of a cyclic permutations $\{\pi_v : v \in V(G)\}$



Claim:
a rotation system determine a 2-cell embedding of G.

~~polygons~~ - define closed facial walks as follows:
pick $f_1 \in E(G)$; $f_1 = w_0 w_1$, then
$f_{i+1} = w_i w_{i+1} \mp \pi(w_i w_{i-1}) = \pi(-f_i)$,
until $f_{k+1} = f_1$ (as directed)
$\Rightarrow$ apply 2-cell-emb. construction on the facial-walk polygons.

Claim: This 2-cell embedding is orientable.
- from picture; same side of edges of polygons.

Def.: An embedding $G \subseteq S$ is cellular if every face is homeom. to the open disc.

THM: Let G be a connected multigraph with ≥ 1 edge cellulary embedded $G \subseteq S$ in orientable surface S.
Then this embedding is homeomorphic to the 2-cell embedding of G given by the rotation system of G "as seen on S".

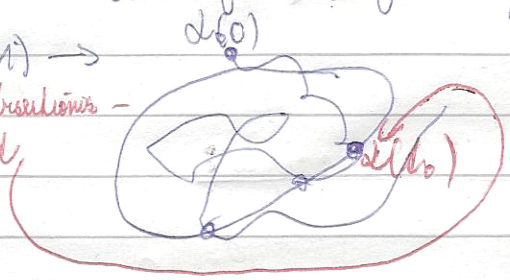
Proof:

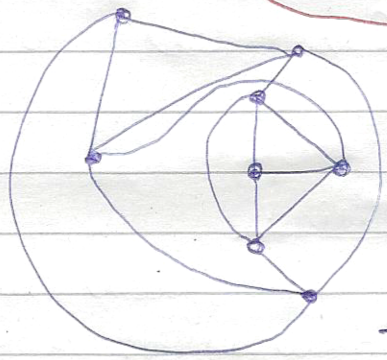
(1) define the cyclic order of arcs of the points

$\alpha: [0,1) \to$

minimal # intersections -

- compact set





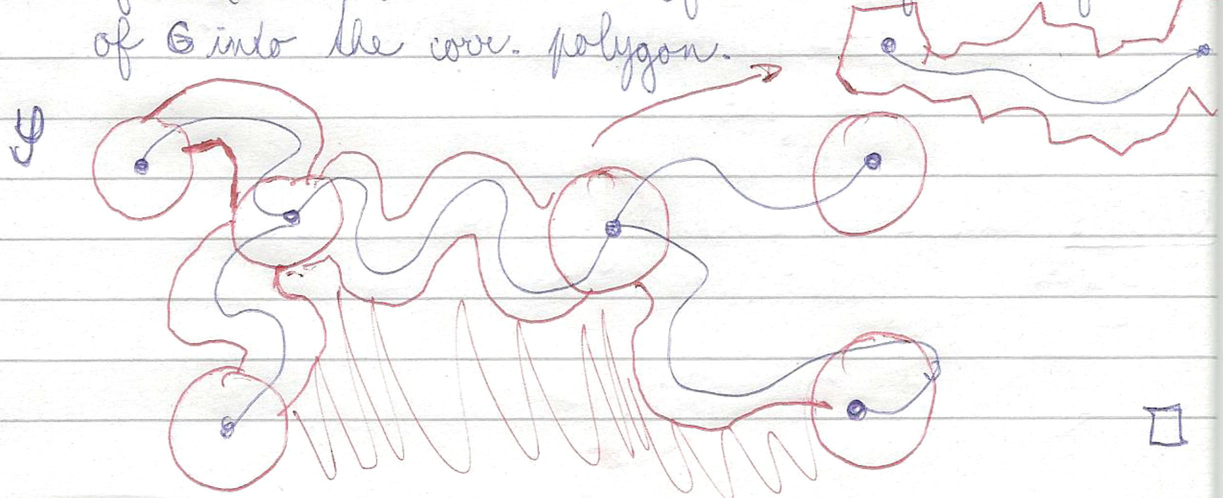
$\longrightarrow$ find nonplanar extension $\to$

$\longrightarrow$ Homework !

- make sure this def. is consistent !

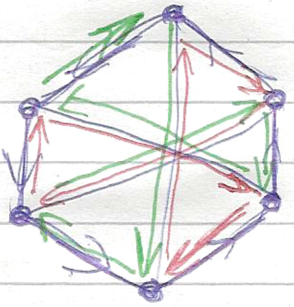
(2.) derive from (1) the rot. system of G as in S
- this is consistent since S is orientable !

(3.) prove that the der. 2-cell-emb. is homeom. to $G \subseteq S$.
- find (and prove) a cont. deformation of each face of G into the corr. polygon.



□

$K_{3,3}$ – 3 faces all 6 edges $\Rightarrow$ genus: $6 - 9 + 3 = 0 = 2 - 2g$



– example K_7 embedding: ....

Nonorientable embeddings of graphs

Def.: An embedding scheme of a (multi) graph G is a rotation system of G plus a signature $\lambda: E(G) \to \{+, -\}$.
$\lambda(e)$ means for $e = uv$, whether the rotations around u and v "agree on clockwise".

Claim: An embedding scheme uniquely defines a 2-cell-embedding via facial walks as follows:

$$f_1 = w_0 w_1 \in E(G),\quad f_{i+1} = \pi^{\lambda(f_i)}(w_i w_{i-1}) = \pi^{\lambda(f_i)}(-f_i).$$

THM: G a conn. multigraph, ≥ 1 edges. A cellular embedding of G in a surface S is uniquely (up to homeomorphi.) determined by the corresp. embedding scheme of G.