Transition Matrices

We want to apply this method of computing $\mathbf{A}^k$ to the analysis of a certain type of physical system that can be described by means of the following kind of mathematical model. Suppose that the sequence

$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \ldots \tag{4}$$

of n-vectors is defined by its **initial vector** $\mathbf{x}_0$ and an $n \times n$ **transition matrix A** in the following manner:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k \qquad \text{for each } k \geq 0. \tag{5}$$

We envision a physical system—such as a population with n specified subpopulations—that evolve through a sequence of successive *states* described by the vectors in (4). Then our goal is to calculate the kth *state vector* $\mathbf{x}_k$. But using (5) repeatedly, we find that

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0, \quad \mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 = \mathbf{A}^2\mathbf{x}_0, \quad \mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 = \mathbf{A}^3\mathbf{x}_0,$$

and in general that

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0. \tag{6}$$

Thus our task amounts to calculating the k power $\mathbf{A}^k$ of the transition matrix $\mathbf{A}$.

LE 2 Consider a metropolitan area with a *constant* total population of 1 million individuals. This area consists of a city and its suburbs, and we want to analyze the changing urban and suburban populations. Let C_k denote the city population and S_k the suburban population after k years. Suppose that each year 15% of the people in the city move to the suburbs, whereas 10% of the people in the suburbs move to the city. Then it follows that

$$\begin{aligned} C_{k+1} &= 0.85C_k + 0.10S_k \\ S_{k+1} &= 0.15C_k + 0.90S_k \end{aligned} \tag{7}$$

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$$\underbrace{\begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \\ \tau_1 & 0 & 0 & 0 & 0 \\ 0 & \tau_2 & 0 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 & 0 \\ 0 & 0 & 0 & \tau_4 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} f_1x_1 + f_2x_2 + \ldots + f_5x_5 \\ \tau_1 x_1 \\ \tau_2 x_2 \\ \tau_3 x_3 \\ \tau_4 x_4 \end{pmatrix}$$

$$\det(A - \lambda E) = \begin{vmatrix} f_1-\lambda & f_2 & f_3 & f_4 & f_5 \\ \tau_1 & -\lambda & 0 & 0 & 0 \\ 0 & \tau_2 & -\lambda & 0 & 0 \\ 0 & 0 & \tau_3 & -\lambda & 0 \\ 0 & 0 & 0 & \tau_4 & -\lambda \end{vmatrix} =$$

$$= +f_5 \cdot \tau_1\tau_2\tau_3\tau_4 - \lambda \cdot \det(A_4 - \lambda E) =$$

$$= f_5 \cdot \tau_1\tau_2\tau_3 \cdot \tau_4 - \lambda\left(-f_4 \cdot \tau_1\tau_2\tau_3 - \lambda(\ldots)\right)$$

$$= f_5 \tau_1\tau_2\tau_3 \tau_4 + \lambda(\quad) + \ldots + \lambda^4 f_1 - \lambda^5$$

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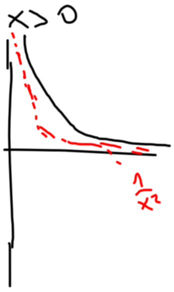
$$\frac{a}{\lambda} + \frac{b}{\lambda^2} + \ldots + \frac{e}{\lambda^5}$$ pro $\lambda > 0$ kladná klesající ↘

$f(x) = \frac{1}{x}$ $x > 0$



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$$x = a_1 x_1 + \ldots + a_m x_m$$

$$A \cdot x = a_1 \cdot \underbrace{A \cdot x_1}_{= \lambda_1 \cdot x_1} + \ldots + a_m \cdot \underbrace{A \cdot x_m}_{= \lambda_m \cdot x_m}$$

$$A^k \cdot x = a_1 \lambda_1^k x_1 + \ldots a_m \lambda_m^k x_m,$$

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$$A = \begin{pmatrix} 0 & 2 & 2 \\ a & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$a \ldots$ úmrtnost v 1. skupině

stagnace $A \cdot x^* = 1 \cdot x^* \Leftrightarrow x^*$ je vl. vektor příslušný vl. číslu 1.

$$\det(A - \lambda E) = \begin{vmatrix} -\lambda & 2 & 2 \\ a & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$\det(A - 1 \cdot E) = \begin{vmatrix} -1 & 2 & 2 \\ a & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} =$$

$$= -1 + 2a + 2a = 0$$

$\underline{a = \frac{1}{4}}$ ⇒ úmrtnost je $\frac{3}{4}$

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$$T \cdot x_0 = x$$

$$\begin{pmatrix} t_{11} & t_{12} & \ldots \\ t_{21} & & \\ \vdots & & \\ t_{n1} & & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} t_{1j} \cdot x_j \\ \sum_{j=1}^{n} t_{2j} \cdot x_j \\ \vdots \end{pmatrix}$$

$\Sigma = 1$ $\Sigma = 1$ ... $\Sigma = 1$ $\Sigma = ?$

$$\det(T - E) = 0 \Leftrightarrow |T - 1 \cdot E| = 0$$

$\Leftrightarrow 1$ je vl. číslo T

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$$f(n+2) = f(n+1) + f(n)$$

$f(0)=1$	$3\cdot f(0)=3$	$g(0)=-1$
$f(1)=1$	3	$g(1)=3$
2	6	$g(2)=2$
3	9	$g(3)=5$
5	15	$g(4)=7$
8	24	
je řeš.	je řeš.	je řeš.

$\Rightarrow f+g$ je řeš.

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$$f(n+2) = f(n+1) + f(n)$$

$$f(n) = \lambda^n \qquad \lambda \neq 0$$

$$\lambda^{n+2} = \lambda^{n+1} + \lambda^n \quad \Big| : \lambda^n$$

$$\lambda^2 = \lambda + 1$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2} = \begin{cases} \frac{1+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \end{cases}$$

$$A \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + B \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$f(0)=1 \iff A \cdot (\quad)^0 + B \cdot (\quad)^0 = 1$$

$$f(1)=1 \iff A \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

$$\left.\begin{aligned} A+B &= 1 \\ (A+B)\tfrac{1}{2} + (A-B)\tfrac{\sqrt{5}}{2} &= 1 \end{aligned}\right\} \quad (A-B)\cdot\frac{\sqrt{5}}{2} = \frac{1}{2}$$

$$A - B = \frac{1}{\sqrt{5}}$$

$$A + B = 1$$

$$2A = \frac{\sqrt{5}+1}{\sqrt{5}}$$

$$A = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$B = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

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