Transition Matrices

We want to apply this method of computing $\mathbf{A}^k$ to the analysis of a certain type of physical system that can be described by means of the following kind of mathematical model. Suppose that the sequence

$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \ldots \tag{4}$$

of n-vectors is defined by its **initial vector** $\mathbf{x}_0$ and an $n \times n$ **transition matrix A** in the following manner:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k \qquad \text{for each } k \geq 0. \tag{5}$$

We envision a physical system—such as a population with n specified subpopulations—that evolve through a sequence of successive *states* described by the vectors in (4). Then our goal is to calculate the kth *state vector* $\mathbf{x}_k$. But using (5) repeatedly, we find that

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0, \quad \mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 = \mathbf{A}^2\mathbf{x}_0, \quad \mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 = \mathbf{A}^3\mathbf{x}_0,$$

and in general that

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0. \tag{6}$$

Thus our task amounts to calculating the k power $\mathbf{A}^k$ of the transition matrix **A**.

EXAMPLE 2 Consider a metropolitan area with a *constant* total population of 1 million individuals. This area consists of a city and its suburbs, and we want to analyze the changing urban and suburban populations. Let C_k denote the city population and S_k the suburban population after k years. Suppose that each year 15% of the people in the city move to the suburbs, whereas 10% of the people in the suburbs move to the city. Then it follows that

$$\begin{aligned} C_{k+1} &= 0.85C_k + 0.10S_k \\ S_{k+1} &= 0.15C_k + 0.90S_k \end{aligned} \tag{7}$$

for each $k \geq 0$. (For instance, next year's city population C_{k+1} will equal 85% of this year's city population C_k plus 10% of this year's suburban population S_k.) Thus the metropolitan area's *population vector* $\mathbf{x}_k = \begin{bmatrix} C_k & S_k \end{bmatrix}^T$ satisfies

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k \quad \text{and hence} \quad \mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 \tag{8}$$

(for each $k \geq 0$) with *transition matrix*

$$\mathbf{A} = \begin{bmatrix} 0.85 & 0.10 \\ 0.15 & 0.90 \end{bmatrix}.$$

The characteristic equation of the transition matrix $\mathbf{A}$ is

$$\begin{aligned}
\left(\frac{17}{20} - \lambda\right)\left(\frac{9}{10} - \lambda\right) - \left(\frac{3}{20}\right)\left(\frac{1}{10}\right) &= 0; \\
(17 - 20\lambda)(9 - 10\lambda) - 3 &= 0; \\
200\lambda^2 - 350\lambda + 150 &= 0; \\
4\lambda^2 - 7\lambda + 3 &= 0; \\
(\lambda - 1)(4\lambda - 3) &= 0.
\end{aligned}$$

Thus the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = 0.75$. For $\lambda_1 = 1$, the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ is

$$\begin{bmatrix} -0.15 & 0.10 \\ 0.15 & -0.10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so an associated eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$. For $\lambda_2 = 0.75$, the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ is

$$\begin{bmatrix} 0.10 & 0.10 \\ 0.15 & 0.15 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so an associated eigenvector is $\mathbf{v}_2 = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$. It follows that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where

$$\mathbf{P} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix},$$

and

$$\mathbf{P}^{-1} = \tfrac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}.$$

Now suppose that our goal is to determine the long-term distribution of population between the city and its suburbs. Note first that $(\frac{3}{4})^k$ is "negligible" when k is sufficiently large; for instance, $(\frac{3}{4})^{40} \approx 0.00001$. It follows that if $k \geq 40$, then the formula $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$ yields

$$\begin{aligned}
\mathbf{A}^k &= \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (\frac{3}{4})^k \end{bmatrix}\left(\tfrac{1}{5}\right)\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \\
&\approx \tfrac{1}{5}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \\
&= \tfrac{1}{5}\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} = \tfrac{1}{5}\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}.
\end{aligned} \tag{9}$$

Hence it follows that, if k is sufficiently large, then

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 \approx \frac{1}{5}\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix}$$
$$= (C_0 + S_0)\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix},$$

because $C_0 + S_0 = 1$ (million), the constant total population of the metropolitan area. Thus, our analysis shows that, irrespective of the initial distribution of population between the city and its suburbs, the long-term distribution consists of 40% in the city and 60% in the suburbs. ■

Remark The result in Example 2—that the long-term situation is independent of the initial situation—is characteristic of a general class of common problems. Note that the transition matrix $\mathbf{A}$ in (8) has the property that *the sum of the elements in each column is* 1. An $n \times n$ matrix with nonnegative entries having this property is called a **stochastic matrix**. It can be proved that, if $\mathbf{A}$ is a stochastic matrix having only positive entries, then $\lambda_1 = 1$ is one eigenvalue of $\mathbf{A}$ and $|\lambda_i| < 1$ for the others. (See Problems 39 and 40.) Moreover, as $k \to \infty$, the matrix $\mathbf{A}^k$ approaches the constant matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_1 & \cdots & \mathbf{v}_1 \end{bmatrix},$$

each of whose identical columns is the eigenvector of $\mathbf{A}$ associated with $\lambda_1 = 1$ that has the sum of its elements equal to 1. The 2×2 stochastic matrix $\mathbf{A}$ of Example 1 illustrates this general result, with $\lambda_1 = 1$, $\lambda_2 = \frac{3}{4}$, and $\mathbf{v}_1 = (\frac{2}{5}, \frac{3}{5})$. ■

Predator-Prey Models

Next, we consider a predator-prey population consisting of the foxes and rabbits living in a certain forest. Initially, there are F_0 foxes and R_0 rabbits; after k months, there are F_k foxes and R_k rabbits. We assume that the transition from each month to the next is described by the equations

$$\begin{aligned} F_{k+1} &= 0.4F_k + 0.3R_k \\ R_{k+1} &= -rF_k + 1.2R_k, \end{aligned} \tag{10}$$

where the constant $r > 0$ is the "capture rate" representing the average number of rabbits consumed monthly by each fox. Thus

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k \quad \text{and hence} \quad \mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0, \tag{11}$$

where

$$\mathbf{x}_k = \begin{bmatrix} F_k \\ R_k \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 0.4 & 0.3 \\ -r & 1.2 \end{bmatrix}. \tag{12}$$

The term $0.4F_k$ in the first equation in (10) indicates that, without rabbits to eat, only 40% of the foxes would survive each month, while the term $0.3R_k$ represents the growth in the fox population due to the available food supply of rabbits. The term $1.2R_k$ in the second equation indicates that, in the absence of any foxes, the rabbit population would increase by 20% each month. We want to investigate the long-term behavior of the fox and rabbit populations for different values of the capture rate r of rabbits by foxes.

The characteristic equation of the transition matrix $\mathbf{A}$ in (12) is

$$\begin{aligned}(0.4-\lambda)(1.2-\lambda)+(0.3)r&=0;\\(4-10\lambda)(12-10\lambda)+30r&=0;\\100\lambda^2-160\lambda+(48+30r)&=0.\end{aligned}$$

The quadratic formula then yields the equation

$$\begin{aligned}\lambda&=\frac{1}{200}\left[160\pm\sqrt{(160)^2-(400)(48+30r)}\right]\\&=\frac{1}{10}\left(8\pm\sqrt{16-30r}\right),\end{aligned}\tag{13}$$

which gives the eigenvalues of $\mathbf{A}$ in terms of the capture rate r. Examples 3, 4, and 5 illustrate three possibilities (for different values of r) for what may happen to the fox and rabbit populations as k increases:

- F_k and R_k may approach constant nonzero values. This is the case of *stable limiting populations* that coexist in equilibrium with one another.
- F_k and R_k may both approach zero. This is the case of *mutual extinction* of the two species.
- F_k and R_k may both increase without bound. This is the case of a *population explosion.*

EXAMPLE 3
Stable Limiting Population

If $r=0.4$, then Equation (13) gives the eigenvalues $\lambda_1=1$ and $\lambda_2=0.6$. For $\lambda_1=1$, the system $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ is

$$\begin{bmatrix}-0.6 & 0.3\\-0.4 & 0.2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},$$

so an associated eigenvector is $\mathbf{v}_1=\begin{bmatrix}1 & 2\end{bmatrix}^T$. For $\lambda_2=0.6$, the system $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ is

$$\begin{bmatrix}-0.2 & 0.3\\-0.4 & 0.6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},$$

so an associated eigenvector is $\mathbf{v}_2=\begin{bmatrix}3 & 2\end{bmatrix}^T$. It follows that $\mathbf{A}=\mathbf{PDP}^{-1}$, where

$$\mathbf{P}=\begin{bmatrix}1 & 3\\2 & 2\end{bmatrix},\quad \mathbf{D}=\begin{bmatrix}1 & 0\\0 & 0.6\end{bmatrix},\quad\text{and}\quad \mathbf{P}^{-1}=-\tfrac{1}{4}\begin{bmatrix}2 & -3\\-2 & 1\end{bmatrix}.$$

We are now ready to calculate $\mathbf{A}^k$. If k is sufficiently large that $(0.6)^k \approx 0$ (for instance, $(0.6)^{25} \approx 0.000003$), then the formula $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$ yields

$$\begin{aligned}\mathbf{A}^k &= \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (0.6)^k \end{bmatrix}(-\tfrac{1}{4})\begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix} \\ &\approx -\tfrac{1}{4}\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix} \\ &= -\tfrac{1}{4}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix} \\ &= \tfrac{1}{4}\begin{bmatrix} -2 & 3 \\ -4 & 6 \end{bmatrix}.\end{aligned}$$

Hence it follows that if k is sufficiently large, then

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 = \tfrac{1}{4}\begin{bmatrix} -2 & 3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 3R_0 - 2F_0 \\ 6R_0 - 4F_0 \end{bmatrix}$$

—that is,

$$\begin{bmatrix} F_k \\ R_k \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{where} \quad \alpha = \tfrac{1}{4}(3R_0 - 2F_0). \tag{14}$$

Assuming that the initial populations are such that $\alpha > 0$ (that is, $3R_0 > 2F_0$), (14) implies that, as k increases, the fox and rabbit populations approach a stable situation in which there are twice as many rabbits as foxes. For instance, if $F_0 = R_0 = 100$, then when k is sufficiently large, there will be 25 foxes and 50 rabbits. ■

EXAMPLE 4
Mutual Extinction

If $r = 0.5$, then Equation (13) gives the eigenvalues $\lambda_1 = 0.9$ and $\lambda_2 = 0.7$. For $\lambda_1 = 0.9$, the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ is

$$\begin{bmatrix} -0.5 & 0.3 \\ -0.5 & 0.3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so an associated eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 3 & 5 \end{bmatrix}^T$. For $\lambda_2 = 0.7$, the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ is

$$\begin{bmatrix} -0.3 & 0.3 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so an associated eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. It follows that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, with

$$\mathbf{P} = \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad \text{and} \quad \mathbf{P}^{-1} = -\tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ -5 & 3 \end{bmatrix}.$$

Now both $(0.9)^k$ and $(0.7)^k$ approach 0 as k increases without bound ($k \to +\infty$). Hence if k is sufficiently large, then the formula $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$ yields

$$\mathbf{A}^k = \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} (0.9)^k & 0 \\ 0 & (0.7)^k \end{bmatrix} (-\tfrac{1}{2}) \begin{bmatrix} 1 & -1 \\ -5 & 3 \end{bmatrix}$$
$$\approx -\tfrac{1}{2} \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so

$$\begin{bmatrix} F_k \\ R_k \end{bmatrix} = \mathbf{A}^k \begin{bmatrix} F_0 \\ R_0 \end{bmatrix} \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{15}$$

Thus F_k and R_k both approach zero as $k \to +\infty$, so both the foxes and the rabbits die out—mutual extinction occurs. ■

EXAMPLE 5
Population Explosion

If $r = 0.325$, then Equation (13) gives the eigenvalues $\lambda_1 = 1.05$ and $\lambda_2 = 0.55$. For $\lambda_1 = 1.05$, the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ is

$$\begin{bmatrix} -0.650 & 0.30 \\ -0.325 & 0.15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Each equation is a multiple of $-13x + 6y = 0$, so an associated eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 6 & 13 \end{bmatrix}^T$. For $\lambda_2 = 0.55$, the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ is

$$\begin{bmatrix} -0.150 & 0.30 \\ -0.325 & 0.65 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so an associated eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$. It follows that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ with

$$\mathbf{P} = \begin{bmatrix} 6 & 2 \\ 13 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1.05 & 0 \\ 0 & 0.55 \end{bmatrix}, \quad \text{and} \quad \mathbf{P}^{-1} = -\tfrac{1}{20} \begin{bmatrix} 1 & -2 \\ -13 & 6 \end{bmatrix}.$$

Note that $(0.55)^k$ approaches zero but that $(1.05)^k$ increases without bound as $k \to +\infty$. It follows that if k is sufficiently large, then the formula $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$ yields

$$\mathbf{A}^k = \begin{bmatrix} 6 & 2 \\ 13 & 1 \end{bmatrix} \begin{bmatrix} (1.05)^k & 0 \\ 0 & (0.55)^k \end{bmatrix} (-\tfrac{1}{20}) \begin{bmatrix} 1 & -2 \\ -13 & 6 \end{bmatrix}$$
$$\approx -\tfrac{1}{20} \begin{bmatrix} 6 & 2 \\ 13 & 1 \end{bmatrix} \begin{bmatrix} (1.05)^k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -13 & 6 \end{bmatrix}$$
$$= -\tfrac{1}{20} \begin{bmatrix} (6)(1.05)^k & 0 \\ (13)(1.05)^k & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -13 & 6 \end{bmatrix},$$

and therefore

$$\mathbf{A}^k \approx -\tfrac{1}{20}(1.05)^k \begin{bmatrix} 6 & -12 \\ 13 & -26 \end{bmatrix}. \tag{16}$$

Hence, if k is sufficiently large, then

$$\begin{aligned} \mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 &\approx -\tfrac{1}{20}(1.05)^k \begin{bmatrix} 6 & -12 \\ 13 & -26 \end{bmatrix} \begin{bmatrix} F_0 \\ R_0 \end{bmatrix} \\ &= -\tfrac{1}{20}(1.05)^k \begin{bmatrix} 6F_0 - 12R_0 \\ 13F_0 - 26R_0 \end{bmatrix}, \end{aligned}$$

and so

$$\begin{bmatrix} F_k \\ R_k \end{bmatrix} \approx (1.05)^k \gamma \begin{bmatrix} 6 \\ 13 \end{bmatrix}, \quad \text{where} \quad \gamma = \tfrac{1}{20}(2R_0 - F_0). \tag{17}$$

If $\gamma > 0$ (that is, if $2R_0 > F_0$), then the factor $(1.05)^k$ in (17) implies that the fox and rabbit populations both increase at a rate of 5% per month, and thus each increases without bound as $k \to +\infty$. Moreover, when k is sufficiently large, the two populations maintain a constant ratio of 6 foxes for every 13 rabbits. It is also of interest to note that the monthly "population multiplier" is the larger eigenvalue $\lambda_1 = 1.05$ and that the limiting *ratio* of populations is determined by the associated eigenvector $\mathbf{v}_1 = \begin{bmatrix} 6 & 13 \end{bmatrix}^T$. ■

In summary, let us compare the results in Examples 3, 4, and 5. The critical capture rate $r = 0.4$ of Example 3 represents a monthly consumption of 0.4 rabbits per fox, resulting in stable limiting populations of both species. But if the foxes are greedier and consume more than 0.4 rabbits per fox monthly, then the result is extinction of both species (as in Example 4). If the rabbits become more skilled at evading foxes, so that less than 0.4 rabbits per fox are consumed each month, then both populations grow without bound, as in Example 5.