4.6 Related Rates

What you'll learn about

- Related Rate Equations
- Solution Strategy
- Simulating Related Motion

. . . and why

Related rate problems are at the heart of Newtonian mechanics; it was essentially to solve such problems that calculus was invented.

Related Rate Equations

Suppose that a particle $P(x, y)$ is moving along a curve C in the plane so that its coordinates x and y are differentiable functions of time t. If D is the distance from the origin to P, then using the Chain Rule we can find an equation that relates dD/dt, dx/dt, and dy/dt.

$$D = \sqrt{x^2 + y^2}$$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

Any equation involving two or more variables that are differentiable functions of time t can be used to find an equation that relates their corresponding rates.

EXAMPLE 1 Finding Related Rate Equations

(a) Assume that the radius r of a sphere is a differentiable function of t and let V be the volume of the sphere. Find an equation that relates dV/dt and dr/dt.

(b) Assume that the radius r and height h of a cone are differentiable functions of t and let V be the volume of the cone. Find an equation that relates dV/dt, dr/dt, and dh/dt.

SOLUTION

(a) $V = \frac{4}{3}\pi r^3$ Volume formula for a sphere

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

(b) $V = \frac{\pi}{3}r^2 h$ Cone volume formula

$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2 \cdot \frac{dh}{dt} + 2r\frac{dr}{dt} \cdot h\right) = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Now try Exercise 3.

Solution Strategy

What has always distinguished calculus from algebra is its ability to deal with variables that change over time. Example 1 illustrates how easy it is to move from a formula relating static variables to a formula that relates their rates of change: simply differentiate the formula implicitly with respect to t. This introduces an important category of problems called *related rate problems* that still constitutes one of the most important applications of calculus.

We introduce a strategy for solving related rate problems, similar to the strategy we introduced for max-min problems earlier in this chapter.

Strategy for Solving Related Rate Problems

1. **Understand the problem.** In particular, identify the variable whose rate of change you *seek* and the variable (or variables) whose rate of change you *know*.
2. **Develop a mathematical model of the problem.** Draw a picture (many of these problems involve geometric figures) and label the parts that are important to the problem. *Be sure to distinguish constant quantities from variables that change over time.* Only constant quantities can be assigned numerical values at the start.

continued

3. **Write an equation relating the variable whose rate of change you seek with the variable(s) whose rate of change you know.** The formula is often geometric, but it could come from a scientific application.
4. **Differentiate both sides of the equation implicitly with respect to time t.** Be sure to follow all the differentiation rules. The Chain Rule will be especially critical, as you will be differentiating with respect to the parameter t.
5. **Substitute values for any quantities that depend on time.** Notice that it is only safe to do this *after* the differentiation step. Substituting too soon "freezes the picture" and makes changeable variables behave like constants, with zero derivatives.
6. **Interpret the solution.** Translate your mathematical result into the problem setting (with appropriate units) and decide whether the result makes sense.

We illustrate the strategy in Example 2.

EXAMPLE 2 A Rising Balloon

A hot-air balloon rising straight up from a level field is tracked by a range finder 500 feet from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 radians per minute. How fast is the balloon rising at that moment?

SOLUTION

We will carefully identify the six steps of the strategy in this first example.

Step 1: Let h be the height of the balloon and let θ be the elevation angle.

We seek: dh/dt

We know: $d\theta/dt = 0.14$ rad/min

Step 2: We draw a picture (Figure 4.55). We label the horizontal distance "500 ft" because it does not change over time. We label the height "h" and the angle of elevation "θ." Notice that we do not label the angle "$\pi/4$," as that would freeze the picture.

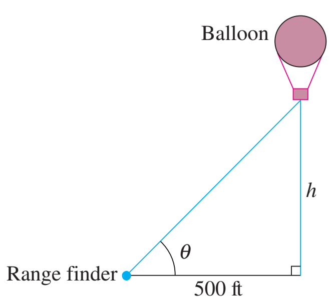


Figure 4.55 The picture shows how h and θ are related geometrically. We seek dh/dt when $\theta = \pi/4$ and $d\theta/dt = 0.14$ rad/min. (Example 2)

Step 3: We need a formula that relates h and θ. Since $\dfrac{h}{500} = \tan\theta$, we get $h = 500\tan\theta$.

Step 4: Differentiate implicitly:

$$\frac{d}{dt}(h) = \frac{d}{dt}(500\tan\theta)$$
$$\frac{dh}{dt} = 500\sec^2\theta\,\frac{d\theta}{dt}$$

Step 5: Let $d\theta/dt = 0.14$ and let $\theta = \pi/4$. (Note that it is now safe to specify our moment in time.)

$$\frac{dh}{dt} = 500\sec^2\left(\frac{\pi}{4}\right)(0.14) = 500(\sqrt{2})^2(0.14) = 140.$$

Step 6: At the moment in question, the balloon is rising at the rate of 140 ft/min.

Now try Exercise 11.

Unit Analysis in Example 2

A careful analysis of the units in Example 2 gives

$$dh/dt = (500\text{ ft})(\sqrt{2})^2(0.14\ \text{rad/min}) = 140\ \text{ft}\cdot\text{rad/min}.$$

Remember that radian measure is actually dimensionless, adaptable to whatever unit is applied to the "unit" circle. The linear units in Example 2 are measured in feet, so "ft · rad" is simply "ft."

EXAMPLE 3 A Highway Chase

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

continued

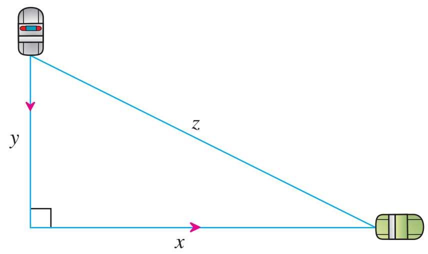


Figure 4.56 A sketch showing the variables in Example 3. We know dy/dt and dz/dt, and we seek dx/dt. The variables x, y, and z are related by the Pythagorean Theorem: $x^2 + y^2 = z^2$.

SOLUTION

We carry out the steps of the strategy.

Let x be the distance of the speeding car from the intersection, let y be the distance of the police cruiser from the intersection, and let z be the distance between the car and the cruiser. Distances x and z are increasing, but distance y is decreasing; so dy/dt is negative.

We seek: dx/dt

We know: $dz/dt = 20$ mph and $dy/dt = -60$ mph

A sketch (Figure 4.56) shows that x, y, and z form three sides of a right triangle. We need to relate those three variables, so we use the Pythagorean Theorem:

$$x^2 + y^2 = z^2$$

Differentiating implicitly with respect to t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}, \text{ which reduces to } x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}.$$

We now substitute the numerical values for x, y, dz/dt, dy/dt, and z (which equals $\sqrt{x^2 + y^2}$):

$$(0.8)\frac{dx}{dt} + (0.6)(-60) = \sqrt{(0.8)^2 + (0.6)^2}(20)$$
$$(0.8)\frac{dx}{dt} - 36 = (1)(20)$$
$$\frac{dx}{dt} = 70$$

At the moment in question, the car's speed is 70 mph. ***Now try Exercise 13.***

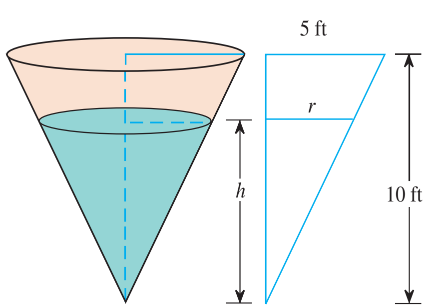


Figure 4.57 In Example 4, the cone of water is increasing in volume inside the reservoir. We know dV/dt and we seek dh/dt. Similar triangles enable us to relate V directly to h.

EXAMPLE 4 Filling a Conical Tank

Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

SOLUTION 1

We carry out the steps of the strategy. Figure 4.57 shows a partially filled conical tank. The tank itself does not change over time; what we are interested in is the changing cone of *water* inside the tank. Let V be the volume, r the radius, and h the height of the cone of water.

We seek: dh/dt

We know: $dV/dt = 9$ ft^3/min

We need to relate V and h. The volume of the cone of water is $V = \frac{1}{3}\pi r^2 h$, but this formula also involves the variable r, whose rate of change is not given. We need to either find dr/dt (see Solution 2) or eliminate r from the equation, which we can do by using the similar triangles in Figure 4.57 to relate r and h:

$$\frac{r}{h} = \frac{5}{10}, \text{ or simply } r = \frac{h}{2}.$$

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

continued

Differentiate with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

Let $h = 6$ and $dV/dt = 9$; then solve for dh/dt:

$$9 = \frac{\pi}{4}(6)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at 0.32 ft/min.

SOLUTION 2

The similar triangle relationship

$$r = \frac{h}{2} \text{ also implies that } \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$

and that $r = 3$ when $h = 6$. So, we could have left all three variables in the formula $V = \frac{1}{3}\pi r^2 h$ and proceeded as follows:

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)$$

$$= \frac{1}{3}\pi\left(2r\left(\frac{1}{2}\frac{dh}{dt}\right)h + r^2\frac{dh}{dt}\right)$$

$$9 = \frac{1}{3}\pi\left(2(3)\left(\frac{1}{2}\frac{dh}{dt}\right)(6) + (3)^2\frac{dh}{dt}\right)$$

$$9 = 9\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi}$$

This is obviously more complicated than the one-variable approach. In general, it is computationally easier to simplify expressions as much as possible *before* you differentiate.

Now try Exercise 17.

Simulating Related Motion

Parametric mode on a grapher can be used to simulate the motion of moving objects when the motion of each can be expressed as a function of time. In a classic related rate problem, the top end of a ladder slides vertically down a wall as the bottom end is pulled horizontally away from the wall at a steady rate. Exploration 1 shows how you can use your grapher to simulate the related movements of the two ends of the ladder.

EXPLORATION 1 **The Sliding Ladder**

A 10-foot ladder leans against a vertical wall. The base of the ladder is pulled away from the wall at a constant rate of 2 ft/sec.

1. Explain why the motion of the two ends of the ladder can be represented by the parametric equations given on the next page.

continued

$$X1T = 2T$$
$$Y1T = 0$$
$$X2T = 0$$
$$Y2T = \sqrt{10^2 - (2T)^2}$$

2. What minimum and maximum values of T make sense in this problem?

3. Put your grapher in parametric and simultaneous modes. Enter the parametric equations and change the graphing style to "0" (the little ball) if your grapher has this feature. Set Tmin=0, Tmax=5, Tstep=5/20, Xmin=−1, Xmax=17, Xscl=0, Ymin=−1, Ymax=11, and Yscl=0. You can speed up the action by making the denominator in the Tstep smaller or slow it down by making it larger.

4. Press GRAPH and watch the two ends of the ladder move as time changes. Do both ends seem to move at a constant rate?

5. To see the simulation again, enter "ClrDraw" from the DRAW menu.

6. If y represents the vertical height of the top of the ladder and x the distance of the bottom from the wall, relate y and x and find dy/dt in terms of x and y. (Remember that $dx/dt = 2$.)

7. Find dy/dt when $t = 3$ and interpret its meaning. Why is it negative?

8. In theory, how fast is the top of the ladder moving as it hits the ground?

Figure 4.58 shows you how to write a calculator program that animates the falling ladder as a line segment.

```
PROGRAM:LADDER
:For (A, 0, 5, .25)
:ClrDraw
:Line(2,2+√ (100–
(2A)²), 2+2A, 2)
:If A=0:Pause
:End
```

```
WINDOW
  Xmin=2
  Xmax=20
  Xscl=0
  Ymin=1
  Ymax=13
  Yscl=0
  Xres=1
```

Figure 4.58 This 5-step program (with the viewing window set as shown) will animate the ladder in Exploration 1. Be sure any functions in the "Y=" register are turned off. Run the program and the ladder appears against the wall; push ENTER to start the bottom moving away from the wall.
For an enhanced picture, you can insert the commands ":Pt-On(2,2+$\sqrt{(100-(2A)^2)}$,2)" and ":Pt-On(2+2A,2,2)" on either side of the middle line of the program.

Quick Review 4.6 *(For help, go to Sections 1.1, 1.4, and 3.7.)*

In Exercises 1 and 2, find the distance between the points A and B.

1. $A(0, 5)$, $B(7, 0)$ $\sqrt{74}$

2. $A(0, a)$, $B(b, 0)$ $\sqrt{a^2 + b^2}$

In Exercises 3–6, find dy/dx.

3. $2xy + y^2 = x + y$ $\dfrac{1 - 2y}{2x + 2y - 1}$

4. $x \sin y = 1 - xy$ $-\dfrac{y + \sin y}{x + x\cos y}$

5. $x^2 = \tan y$ $2x\cos^2 y$

6. $\ln(x + y) = 2x$ $2x + 2y - 1$

In Exercises 7 and 8, find a parametrization for the line segment with endpoints A and B.

7. $A(-2, 1)$, $B(4, -3)$

7. One possible answer: $x = -2 + 6t$, $y = 1 - 4t$, $0 \le t \le 1$.

8. $A(0, -4)$, $B(5, 0)$

8. One possible answer: $x = 5t$, $y = -4 + 4t$, $0 \le t \le 1$.

In Exercises 9 and 10, let $x = 2\cos t$, $y = 2\sin t$. Find a parameter interval that produces the indicated portion of the graph.

9. The portion in the second and third quadrants, including the points on the axes. One possible answer: $\pi/2 \le t \le 3\pi/2$

10. The portion in the fourth quadrant, including the points on the axes. One possible answer: $3\pi/2 \le t \le 2\pi$

4. (a) $\frac{dP}{dt} = 2RI\frac{dI}{dt} + I^2\frac{dR}{dt}$ **(b)** $0 = 2RI\frac{dI}{dt} + I^2\frac{dR}{dt}$, or, $\frac{dR}{dt} = \left(-\frac{2R}{I}\right)\left(\frac{dI}{dt}\right) = \left(-\frac{2P}{I^3}\right)\left(\frac{dI}{dt}\right)$

Section 4.6 Exercises

In Exercises 1–41, assume all variables are differentiable functions of t.

1. *Area* The radius r and area A of a circle are related by the equation $A = \pi r^2$. Write an equation that relates dA/dt to dr/dt. $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

2. *Surface Area* The radius r and surface area S of a sphere are related by the equation $S = 4\pi r^2$. Write an equation that relates dS/dt to dr/dt. $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$

3. *Volume* The radius r, height h, and volume V of a right circular cylinder are related by the equation $V = \pi r^2 h$.

(a) How is dV/dt related to dh/dt if r is constant? $\frac{dV}{dt} = \pi r^2\frac{dh}{dt}$

(b) How is dV/dt related to dr/dt if h is constant? $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt}$

(c) How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant? $\frac{dV}{dt} = \pi r^2\frac{dh}{dt} + 2\pi rh\frac{dr}{dt}$

4. *Electrical Power* The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.

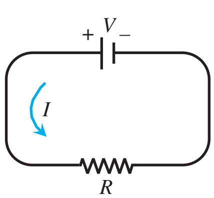
(a) How is dP/dt related to dR/dt and dI/dt?

(b) How is dR/dt related to dI/dt if P is constant?

5. *Diagonals* If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$. How is ds/dt related to dx/dt, dy/dt, and dz/dt? See below.

6. *Area* If a and b are the lengths of two sides of a triangle, and θ the measure of the included angle, the area A of the triangle is $A = (1/2)\,ab\sin\theta$. How is dA/dt related to da/dt, db/dt, and $d\theta/dt$? $\frac{dA}{dt} = \frac{1}{2}\left(b\sin\theta\frac{da}{dt} + a\sin\theta\frac{db}{dt} + ab\cos\theta\frac{d\theta}{dt}\right)$

7. *Changing Voltage* The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote time in sec.



(a) 1 volt/sec

(b) $-\frac{1}{3}$ amp/sec

(c) $\frac{dV}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$

(d) $\frac{dR}{dt} = \frac{3}{2}$ ohms/sec. R is increasing since $\frac{dR}{dt}$ is positive.

(a) What is the value of dV/dt?

(b) What is the value of dI/dt?

(c) Write an equation that relates dR/dt to dV/dt and dI/dt.

(d) Writing to Learn Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing? Explain.

8. *Heating a Plate* When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/sec. At what rate is the plate's area increasing when the radius is 50 cm? π cm^2/sec

5. $\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{x^2 + y^2 + z^2}}$

9. *Changing Dimensions in a Rectangle* The length ℓ of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $\ell = 12$ cm and $w = 5$ cm, find the rates of change of See page 255.

(a) the area, **(b)** the perimeter, and

(c) the length of a diagonal of the rectangle.

(d) Writing to Learn Which of these quantities are decreasing, and which are increasing? Explain.

10. *Changing Dimensions in a Rectangular Box* Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates: **(a)** 2 m^3/sec **(b)** 0 m^2/sec **(c)** 0 m/sec

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

Find the rates at which the box's **(a)** volume, **(b)** surface area, and **(c)** diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

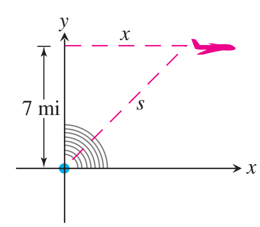
11. *Inflating Balloon* A spherical balloon is inflated with helium at the rate of 100π ft^3/min.

(a) How fast is the balloon's radius increasing at the instant the radius is 5 ft? 1 ft/min

(b) How fast is the surface area increasing at that instant? 40π ft^2/min

12. *Growing Raindrop* Suppose that a droplet of mist is a perfect sphere and that, through condensation, the droplet picks up moisture at a rate proportional to its surface area. Show that under these circumstances the droplet's radius increases at a constant rate. See page 255.

13. *Air Traffic Control* An airplane is flying at an altitude of 7 mi and passes directly over a radar antenna as shown in the figure. When the plane is 10 mi from the antenna ($s = 10$), the radar detects that the distance s is changing at the rate of 300 mph. What is the speed of the airplane at that moment? $\frac{dx}{dt} = \frac{3000}{\sqrt{51}}$ mph ≈ 420.08 mph



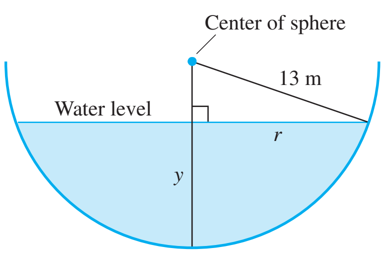
14. *Flying a Kite* Inge flies a kite at a height of 300 ft, the wind carrying the kite horizontally away at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her? 20 ft/sec

15. *Boring a Cylinder* The mechanics at Lincoln Automotive are reboring a 6-in. deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.? $\frac{19\pi}{2500} \approx 0.0239$ in^3/min

16. ***Growing Sand Pile*** Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Give your answer in cm/min.
(a) $\frac{1125}{32\pi} \approx 11.19$ cm/min **(b)** $\frac{375}{8\pi} \approx 14.92$ cm/min

17. ***Draining Conical Reservoir*** Water is flowing at the rate of 50 m^3/min from a concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. **(a)** How fast is the water level falling when the water is 5 m deep? **(b)** How fast is the radius of the water's surface changing at that moment? Give your answer in cm/min.
(a) $\frac{32}{9\pi} \approx 1.13$ cm/min **(b)** $-\frac{80}{3\pi} \approx -8.49$ cm/min

18. ***Draining Hemispherical Reservoir*** Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y units deep.

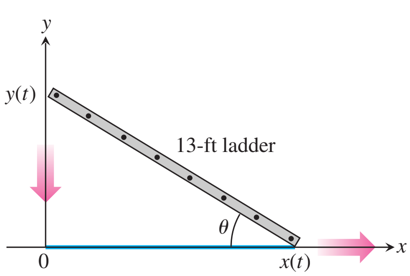


(a) At what rate is the water level changing when the water is 8 m deep? $-\frac{1}{24\pi} \approx -0.01326$ m/min or $-\frac{25}{6\pi} \approx -1.326$ cm/min

(b) What is the radius r of the water's surface when the water is y m deep? $r = \sqrt{169 - (13 - y)^2} = \sqrt{26y - y^2}$

(c) At what rate is the radius r changing when the water is 8 m deep?
$-\frac{5}{288\pi} \approx -0.00553$ m/min or $-\frac{125}{72\pi} \approx -0.553$ cm/min

19. ***Sliding Ladder*** A 13-ft ladder is leaning against a house (see figure) when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

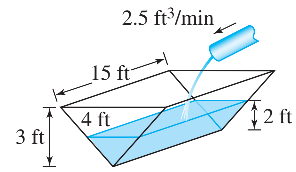


(a) How fast is the top of the ladder sliding down the wall at that moment? 12 ft/sec

(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing at that moment? $-\frac{119}{2}$ ft^2/sec

(c) At what rate is the angle θ between the ladder and the ground changing at that moment? -1 radian/sec

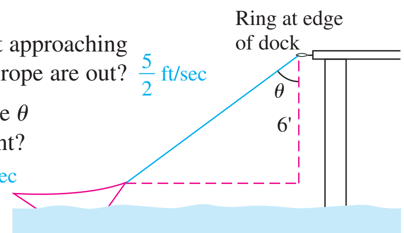
20. ***Filling a Trough*** A trough is 15 ft long and 4 ft across the top as shown in the figure. Its ends are isosceles triangles with height 3 ft. Water runs into the trough at the rate of 2.5 ft^3/min. How fast is the water level rising when it is 2 ft deep? $\frac{1}{16}$ ft/min



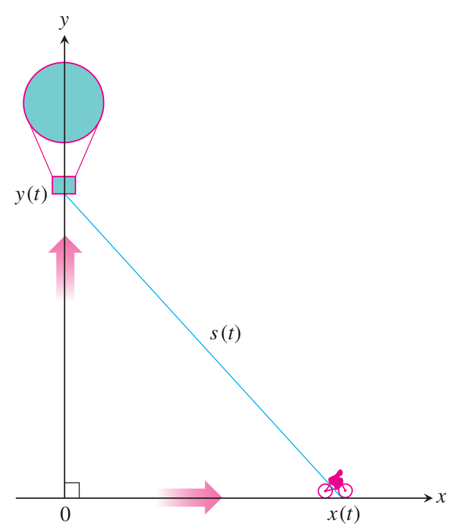
21. ***Hauling in a Dinghy*** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow as shown in the figure. The rope is hauled in at the rate of 2 ft/sec.

(a) How fast is the boat approaching the dock when 10 ft of rope are out? $\frac{5}{2}$ ft/sec

(b) At what rate is angle θ changing at that moment? $-\frac{3}{20}$ radian/sec



22. ***Rising Balloon*** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance between the bicycle and balloon increasing 3 sec later (see figure)? 11 ft/sec



In Exercises 23 and 24, a particle is moving along the curve $y = f(x)$.

23. Let $y = f(x) = \dfrac{10}{1 + x^2}$.
(a) $\frac{24}{5}$ cm/sec **(b)** 0 cm/sec **(c)** $-\frac{1200}{160{,}801} \approx -0.00746$ cm/sec

If $dx/dt = 3$ cm/sec, find dy/dt at the point where

(a) $x = -2$. **(b)** $x = 0$. **(c)** $x = 20$.

24. Let $y = f(x) = x^3 - 4x$.

If $dx/dt = -2$ cm/sec, find dy/dt at the point where

(a) $x = -3$. **(b)** $x = 1$. **(c)** $x = 4$.

(a) -46 cm/sec **(b)** 2 cm/sec **(c)** -88 cm/sec

25. ***Particle Motion*** A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (in meters) increases at a constant rate of 10 m/sec. How fast is the angle of inclination θ of theline joining the particle to the origin changing when $x = 3$? 1 radian/sec

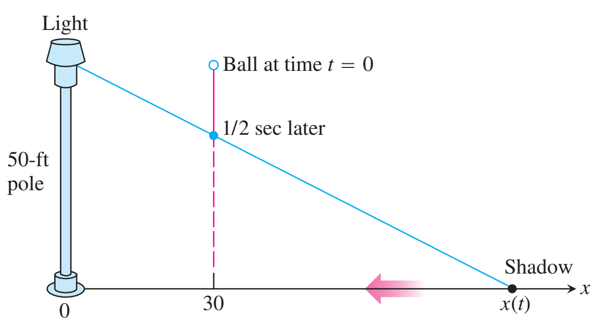
26. ***Particle Motion*** A particle moves from right to left along the parabolic curve $y = \sqrt{-x}$ in such a way that its x-coordinate (in meters) decreases at the rate of 8 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = -4$? $\frac{2}{5}$ radian/sec

27. ***Melting Ice*** A spherical iron ball is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 8 mL/min, how fast is the outer surface area of ice decreasing when the outer diameter (ball plus ice) is 20 cm? 1.6 cm^2/min

28. ***Particle Motion*** A particle $P(x, y)$ is moving in the coordinate plane in such a way that $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)? -5 m/sec

29. ***Moving Shadow*** A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the base of the light? -3 ft/sec

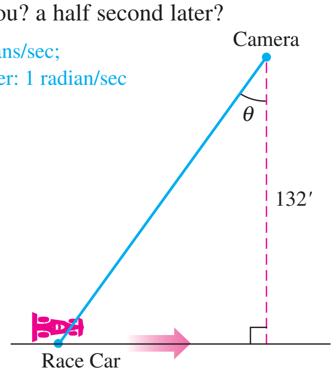
30. ***Moving Shadow*** A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light as shown below. How fast is the ball's shadow moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ in t sec.) -1500 ft/sec



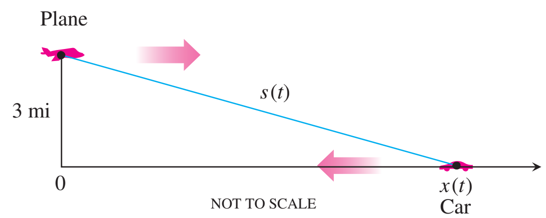
NOT TO SCALE

31. ***Moving Race Car*** You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mph (264 ft/sec) as shown in the figure. About how fast will your camera angle θ be changing when the car is right in front of you? a half second later?

In front: 2 radians/sec;
Half second later: 1 radian/sec

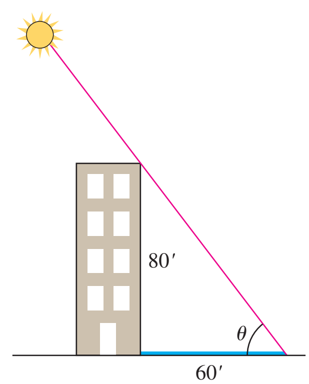


32. ***Speed Trap*** A highway patrol airplane flies 3 mi above a level, straight road at a constant rate of 120 mph. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi the line-of-sight distance is decreasing at the rate of 160 mph. Find the car's speed along the highway. 80 mph

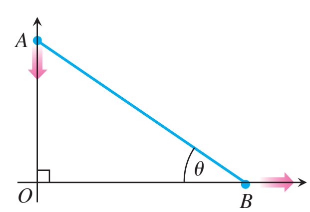


NOT TO SCALE

33. ***Building's Shadow*** On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long as shown in the figure. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow length decreasing? Express your answer in in./min, to the nearest tenth. (Remember to use radians.) 7.1 in./min



34. ***Walkers*** A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2 m/sec and B moves away from the intersection at 1 m/sec as shown in the figure. At what rate is the angle θ changing when A is 10 m from the intersection and B is 20 m from the intersection? Express your answer in degrees per second to the nearest degree. -6 deg/sec



35. ***Moving Ships*** Two ships are steaming away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yards). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles? 29.5 knots

36. True. Since $\frac{dC}{dt} = 2\pi \frac{dr}{dt}$, a constant $\frac{dr}{dt}$ results in a constant $\frac{dC}{dt}$.

37. False. Since $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$, the value of $\frac{dA}{dt}$ depends on r.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

36. True or False If the radius of a circle is expanding at a constant rate, then its circumference is increasing at a constant rate. Justify your answer.

37. True or False If the radius of a circle is expanding at a constant rate, then its area is increasing at a constant rate. Justify your answer.

38. Multiple Choice If the volume of a cube is increasing at 24 in^3/ min and each edge of the cube is increasing at 2 in./min, what is the length of each edge of the cube? A

(A) 2 in. **(B)** $2\sqrt{2}$ in. **(C)** $\sqrt[3]{12}$ in. **(D)** 4 in. **(E)** 8 in.

39. Multiple Choice If the volume of a cube is increasing at 24 in^3/ min and the surface area of the cube is increasing at 12 in^2/ min, what is the length of each edge of the cube? E

(A) 2 in. **(B)** $2\sqrt{2}$ in. **(C)** $\sqrt[3]{12}$ in. **(D)** 4 in. **(E)** 8 in.

40. Multiple Choice A particle is moving around the unit circle (the circle of radius 1 centered at the origin). At the point (0.6, 0.8) the particle has horizontal velocity $dx/dt = 3$. What is its vertical velocity dy/dt at that point? C

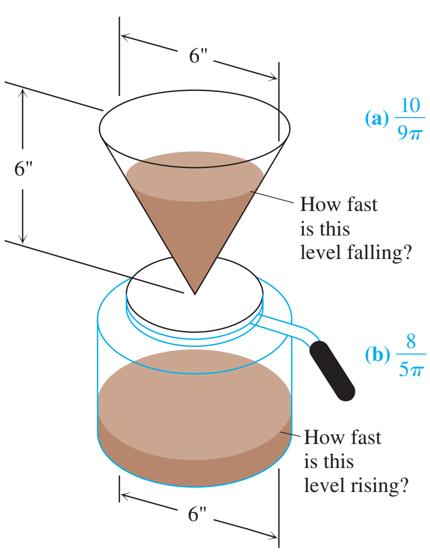
(A) -3.875 **(B)** -3.75 **(C)** -2.25 **(D)** 3.75 **(E)** 3.875

41. Multiple Choice A cylindrical rubber cord is stretched at a constant rate of 2 cm per second. Assuming its volume does not change, how fast is its radius shrinking when its length is 100 cm and its radius is 1 cm? B

(A) 0 cm/sec **(B)** 0.01 cm/sec **(C)** 0.02 cm/sec
(D) 2 cm/sec **(E)** 3.979 cm/sec

Explorations

42. *Making Coffee* Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.



(a) $\frac{10}{9\pi} \approx 0.354$ in./min

(b) $\frac{8}{5\pi} \approx 0.509$ in./min

(a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

(b) How fast is the level in the cone falling at that moment?

43. *Cost, Revenue, and Profit* A company can manufacture x items at a cost of $c(x)$ dollars, a sales revenue of $r(x)$ dollars, and a profit of $p(x) = r(x) - c(x)$ dollars (all amounts in thousands). Find dc/dt, dr/dt, and dp/dt for the following values of x and dx/dt.

(a) $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$. $\frac{dc}{dt} = 0.3$ $\frac{dr}{dt} = 0.9$ $\frac{dp}{dt} = 0.6$

(b) $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$. $\frac{dc}{dt} = -1.5625$ $\frac{dr}{dt} = 3.5$ $\frac{dp}{dt} = 5.0625$

44. Group Activity *Cardiac Output* In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würtzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 liters a minute. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (mL/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ mL/min and $D = 97 - 56 = 41$ mL/L,

$$y = \frac{233 \text{ mL/min}}{41 \text{ mL/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

$\frac{dy}{dt} = \frac{466}{1681} \approx 0.277$ L/min^2

Extending the Ideas

45. *Motion along a Circle* A wheel of radius 2 ft makes 8 revolutions about its center every second.

(a) Explain how the parametric equations

$$x = 2\cos\theta, \quad y = 2\sin\theta$$

can be used to represent the motion of the wheel.

(b) Express θ as a function of time t.

(c) Find the rate of horizontal movement and the rate of vertical movement of a point on the edge of the wheel when it is at the position given by $\theta = \pi/4$, $\pi/2$, and π.

46. *Ferris Wheel* A Ferris wheel with radius 30 ft makes one revolution every 10 sec.

(a) Assume that the center of the Ferris wheel is located at the point (0, 40), and write parametric equations to model its motion. [*Hint:* See Exercise 45.]

(b) At $t = 0$ the point P on the Ferris wheel is located at (30, 40). Find the rate of horizontal movement, and the rate of vertical movement of the point P when $t = 5$ sec and $t = 8$ sec.

47. *Industrial Production* **(a)** Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.) 9% per year

Let $v = g(t)$ be the average production per person in the labor force at time t. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y.

(b) Suppose that the labor force in part (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate? Increasing at 1% per year

9. (a) $\dfrac{dA}{dt} = 14\text{ cm}^2/\text{sec}$ **(b)** $\dfrac{dP}{dt} = 0\text{ cm/sec}$

(c) $\dfrac{dD}{dt} = -\dfrac{14}{13}\text{ cm/sec}$

(d) The area is increasing, because its derivative is positive. The perimeter is not changing, because its derivative is zero. The diagonal length is decreasing, because its derivative is negative.

12. $V = \dfrac{4}{3}\pi r^3$, so $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. But $S = 4\pi r^2$, so we are given that $\dfrac{dV}{dt} = kS = 4k\pi r^2$. Substituting, $4k\pi r^2 = 4\pi r^2\dfrac{dr}{dt}$ which gives $\dfrac{dr}{dt} = k$.

Quick Quiz for AP* Preparation: Sections 4.4–4.6

You may use a graphing calculator to solve the following problems.

1. Multiple Choice If Newton's method is used to approximate the real root of $x^3 + 2x - 1 = 0$, what would the third approximation, x_3, be if the first approximation is $x_1 = 1$? B

(A) 0.453 **(B)** 0.465 **(C)** 0.495 **(D)** 0.600 **(E)** 1.977

2. Multiple Choice The sides of a right triangle with legs x and y and hypotenuse z increase in such a way that $dz/dt = 1$ and $dx/dt = 3\,dy/dt$. At the instant when $x = 4$ and $y = 3$, what is dx/dt? B

(A) $\dfrac{1}{3}$ **(B)** 1 **(C)** 2 **(D)** $\sqrt{5}$ **(E)** 5

3. Multiple Choice An observer 70 meters south of a railroad crossing watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection? A

(A) 57.60 **(B)** 57.88 **(C)** 59.20 **(D)** 60.00 **(E)** 67.40

4. Free Response **(a)** Approximate $\sqrt{26}$ by using the linearization of $y = \sqrt{x}$ at the point (25, 5). Show the computation that leads to your conclusion.

(b) Approximate $\sqrt{26}$ by using a first guess of 5 and one iteration of Newton's method to approximate the zero of $x^2 - 26$. Show the computation that leads to your conclusion.

(c) Approximate $\sqrt[3]{26}$ by using an appropriate linearization. Show the computation that leads to your conclusion.

Chapter 4 Key Terms

absolute change (p. 240)
absolute maximum value (p. 187)
absolute minimum value (p. 187)
antiderivative (p. 200)
antidifferentiation (p. 200)
arithmetic mean (p. 204)
average cost (p. 224)
center of linear approximation (p. 233)
concave down (p. 207)
concave up (p. 207)
concavity test (p. 208)
critical point (p. 190)
decreasing function (p. 198)
differential (p. 237)
differential estimate of change (p. 239)
differential of a function (p. 239)
extrema (p. 187)
Extreme Value Theorem (p. 188)
first derivative test (p. 205)
first derivative test for local extrema (p. 205)
geometric mean (p. 204)
global maximum value (p. 177)
global minimum value (p. 177)
increasing function (p. 198)
linear approximation (p. 233)
linearization (p. 233)
local linearity (p. 233)
local maximum value (p. 189)
local minimum value (p. 189)
logistic curve (p. 210)
logistic regression (p. 211)
marginal analysis (p. 223)
marginal cost and revenue (p. 223)
Mean Value Theorem (p. 196)
monotonic function (p. 198)
Newton's method (p. 235)
optimization (p. 219)
percentage change (p. 240)
point of inflection (p. 208)
profit (p. 223)
quadratic approximation (p. 245)
related rates (p. 246)
relative change (p. 240)
relative extrema (p. 189)
Rolle's Theorem (p. 196)
second derivative test for local extrema (p. 211)
standard linear approximation (p. 233)