NEURČITÝ INTEGRÁL

1) Základní vzorce (platí na definičním oboru integrandu)

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C; \quad x > 0, \quad n \in \mathbf{R}, \quad n \neq -1$$

$$\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + C, \quad x \neq 0$$

$$\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + C,$$

$$\int a^x \, \mathrm{d}x = \frac{a^x}{\ln a} + C, \quad a > 0, \quad a \neq 1$$

$$\int \sin x \, \mathrm{d}x = -\cos x + C,$$

$$\int \cos x \, \mathrm{d}x = \sin x + C,$$

$$\int \frac{1}{\cos^2 x} \, \mathrm{d}x = \operatorname{tg} x + C, \quad x \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbf{Z}$$

$$\int \frac{1}{\sin^2 x} \, \mathrm{d}x = -\operatorname{cotg} x + C, \quad x \neq k\pi, \quad k \in \mathbf{Z}$$

Linearita integrálu

$$\int (\alpha\, f(x) + \beta\, g(x)) \, \mathrm{d}x = \alpha \int f(x) \, \mathrm{d}x + \beta \int g(x) \, \mathrm{d}x$$

Pomocí základních vzorců a linearity spočtěte integrály z následujících funkcí:

a) $x^3 - 3\sqrt{x} + \dfrac{x}{\sqrt[4]{x^5}}$, b) $\dfrac{1+x^2}{\sqrt[3]{x}}$, c) $\operatorname{cotg}^2 x$.

2) Substituce

$$\int f(\varphi(x))\, \varphi'(x) \, \mathrm{d}x = \left[\begin{array}{c} \varphi(x) = t \\ \varphi'(x)\mathrm{d}x = \mathrm{d}t \end{array} \right] = \int f(t) \, \mathrm{d}t = F(t) = F(\varphi(x)).$$

a) $\displaystyle\int x\sqrt{1+2x^2} \, \mathrm{d}x$, b) $\displaystyle\int x^3 \sqrt{x^4+3} \, \mathrm{d}x$, c) $\displaystyle\int \sin^4 x \, \cos x \, \mathrm{d}x$

d) $\displaystyle\int a^{\sin x} \cos x \, \mathrm{d}x$, e) $\displaystyle\int x\, \mathrm{e}^{2x^2} \, \mathrm{d}x$, f) $\displaystyle\int \operatorname{tg} x \, \mathrm{d}x$.

g) $\displaystyle\int \frac{x^4}{\sqrt{3x^5+7}} \, \mathrm{d}x$, h) $\displaystyle\int \frac{2x^3}{\sqrt{(5x^4-1)^3}} \, \mathrm{d}x$, i) $\displaystyle\int \frac{3x^4}{\sqrt[3]{(3x^5-1)}} \, \mathrm{d}x$,

j) $\displaystyle\int 2x^2 \sqrt[4]{(x^3+2)^3} \, \mathrm{d}x$, k) $\displaystyle\int 2x^3 \sqrt[3]{(x^4+1)^2} \, \mathrm{d}x$, l) $\displaystyle\int \frac{2x}{\sqrt[4]{x^2+1}} \, \mathrm{d}x$.

Vzorce

$$\int f(ax+b) \, \mathrm{d}x = \frac{1}{a} F(ax+b) + C, \quad \text{kde} \int f(x) \, \mathrm{d}x = F(x) + C$$

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C$$

3) a) $\displaystyle\int \frac{x^2}{x^3+4}\,\mathrm{d}x$, b) $\displaystyle\int \frac{\sin x}{1+\cos x}\,\mathrm{d}x$, c) $\displaystyle\int (2x+5)^{10}\,\mathrm{d}x$,

d) $\displaystyle\int \cos(4x+2)\,\mathrm{d}x$, e) $\displaystyle\int \frac{1}{x\,\ln x}\,\mathrm{d}x$, f) $\displaystyle\int \frac{1}{\sqrt[3]{4x-3}}\,\mathrm{d}x$.

g) $\displaystyle\int a^{2x-7}\,\mathrm{d}x$, h) $\displaystyle\int \frac{2x+3}{x^2+3x+8}\,\mathrm{d}x$, i) $\displaystyle\int \frac{\mathrm{e}^x}{\mathrm{e}^x+3}\,\mathrm{d}x$

j) $\displaystyle\int \frac{\mathrm{e}^{3x-1}}{\mathrm{e}^{3x-1}+3}\,\mathrm{d}x$, k) $\displaystyle\int \frac{\cos(x+2)}{2-\sin(x+2)}$, l) $\displaystyle\int \frac{x^3+1}{x^4+4x+5}\,\mathrm{d}x$.

Vzorce

$$\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\,\mathrm{arctg}\frac{x}{a} + C,$$

$$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\,\ln\left|\frac{x-a}{x+a}\right| + C,$$

$$\int \frac{1}{\sqrt{x^2+a}}\,\mathrm{d}x = \ln|x+\sqrt{x^2+a}| + C,$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + C.$$

4) a) $\displaystyle\int \frac{1}{x^2+2x+2}\,\mathrm{d}x$, b) $\displaystyle\int \frac{1}{x^2+3x+3}\,\mathrm{d}x$, c) $\displaystyle\int \frac{1}{x^2-3x+2}\,\mathrm{d}x$,

d) $\displaystyle\int \frac{1}{\sqrt{x^2+4x+5}}\,\mathrm{d}x$, e) $\displaystyle\int \frac{1}{\sqrt{6x-x^2}}\,\mathrm{d}x$, f) $\displaystyle\int \frac{1}{\sqrt{9x^2+6x}}\,\mathrm{d}x$.

5) Per partes

$$\int f(x)\,g'(x)\,\mathrm{d}x = f(x)\,g(x) - \int f'(x)\,g(x)\,\mathrm{d}x$$

a) $\displaystyle\int x\ \cos 2x\,\mathrm{d}x$, b) $\displaystyle\int x\ \mathrm{e}^{-2x}\,\mathrm{d}x$, c) $\displaystyle\int \arccos x\,\mathrm{d}x$,

d) $\displaystyle\int \cos^2 2x\,\mathrm{d}x$, e) $\displaystyle\int \mathrm{e}^x\ \cos 2x\,\mathrm{d}x$, f) $\displaystyle\int x^2\ \sin x\,\mathrm{d}x$

6) Parciální zlomky

a) $\displaystyle\int \frac{1}{x-4}\,\mathrm{d}x$, b) $\displaystyle\int \frac{1}{5x+6}\,\mathrm{d}x$, c) $\displaystyle\int \frac{x-1}{x^2+1}\,\mathrm{d}x$,

d) $\displaystyle\int \frac{3x+2}{x^2+4x+5}\,\mathrm{d}x$, e) $\displaystyle\int \frac{x}{(x+1)(x+2)}\,\mathrm{d}x$, f) $\displaystyle\int \frac{x+4}{x\,(x-2)^2}\,\mathrm{d}x$.

g) $\displaystyle\int \frac{x^2+3x}{(x+1)(x+2)(x-1)}\,\mathrm{d}x$, h) $\displaystyle\int \frac{1-2x}{(x+1)(x^2+1)}\,\mathrm{d}x$, i) $\displaystyle\int \frac{x+2}{x\,(x^2+4x+5)}\,\mathrm{d}x$.

Výsledky (integrační konstantu vynecháváme)

1) a) $\dfrac{x^4}{4} - 2\sqrt{x^3} + \dfrac{4}{3}\sqrt[4]{x^3}$, b) $\dfrac{3}{2}\sqrt[3]{x^2} + \dfrac{3}{8}\sqrt[3]{x^8}$, c) $-\mathrm{cotg}\, x - x$.

2) a) $\dfrac{1}{6}\sqrt{(1+2x^2)^3}$, b) $\dfrac{1}{6}\sqrt{(x^4+3)^3}$, c) $\dfrac{\sin^5 x}{5}$,

d) $\dfrac{a^{\sin x}}{\ln a}$, e) $\dfrac{1}{4}\,\mathrm{e}^{2x^2}$, f) $-\ln|\cos x|$.

g) $\dfrac{2}{15}\sqrt{3x^5+7}$, h) $-\dfrac{1}{5}\,\dfrac{1}{\sqrt{(5x^4-1)}}$ i) $\dfrac{3}{10}\sqrt[3]{(3x^5-1)^2}$,

j) $\dfrac{8}{21}\sqrt[4]{(x^3+2)^7}$ k) $\dfrac{3}{10}\sqrt[3]{(x^4+1)^5}$ l) $\dfrac{4}{3}\sqrt[4]{(x^2+1)^3}$.

3) a) $\dfrac{1}{3}\ln|x^3+4|$, b) $-\ln|1+\cos x|$, c) $\dfrac{(2x+5)^{11}}{22}$, d) $\dfrac{1}{4}\sin(4x+2)$,

e) $\ln|\ln x|$, f) $\dfrac{3}{8}\sqrt[3]{(4x-3)^2}$, g) $\dfrac{a^{2x-7}}{2\ln a}$, h) $\ln|x^2+3x+8|$,

i) $\ln|\mathrm{e}^x+3|$, j) $\dfrac{1}{3}\ln|\mathrm{e}^{3x-1}+3|$, k) $-\ln|2-\sin(x+2)|$, l) $\dfrac{1}{4}\ln|x^4+4x+5|$.

4) a) $\mathrm{arctg}(x+1)$, b) $\dfrac{2}{\sqrt{3}}\,\mathrm{arctg}\,\dfrac{2x+3}{\sqrt{3}}$, c) $\ln\left|\dfrac{x-2}{x-1}\right|$

d) $\ln|x+2+\sqrt{(x+2)^2+1}|$, e) $\arcsin\dfrac{x-3}{3}$, f) $\dfrac{1}{3}\ln|3x+1+\sqrt{(3x+1)^2-1}|$.

5) a) $\dfrac{x\,\sin 2x}{2} + \dfrac{\cos 2x}{4}$, b) $-\dfrac{\mathrm{e}^{-2x}}{4}(1+2x)$, c) $x\arccos x - \sqrt{1-x^2}$,

d) $\dfrac{2x+\sin 2x\,\cos 2x}{4}$, e) $\dfrac{\mathrm{e}^x(\cos 2x + 2\sin 2x)}{5}$, f) $(2-x^2)\cos x + 2x\,\sin x$.

6) a) $\ln|x-4|$, b) $\dfrac{1}{5}\ln|5x+6|$, c) $\dfrac{1}{2}\ln|x^2+1| - \mathrm{arctg}\, x$,

d) $\dfrac{3}{2}\ln|x^2+4x+5| - 4\,\mathrm{arctg}(x+2)$, e) $2\ln|x+2| - \ln|x+1|$, f) $\ln\left|\dfrac{x}{x-2}\right| - \dfrac{3}{x-2}$,

g) $\ln|x+1| + \dfrac{2}{3}\ln\left|\dfrac{x-1}{x+2}\right|$, h) $\dfrac{3}{2}\ln|x+1| - \dfrac{3}{4}\ln|x^2+1| - \dfrac{1}{2}\mathrm{arctg}\, x$,

i) $\dfrac{2}{5}\ln|x| - \dfrac{1}{5}\ln|x^2+4x+5| + \dfrac{1}{5}\mathrm{arctg}(x+2)$.

Řešení vybraných příkladů

2) a) $$\int x\sqrt{1+2x^2}\,\mathrm{d}x = \left[\begin{array}{c} t^2 = 1+2x^2 \\ 2\,t\,\mathrm{d}t = 4x\,\mathrm{d}x \end{array}\right] = \frac{1}{2}\int t^2\,\mathrm{d}t = \frac{1}{2}\cdot\frac{t^3}{3} = \frac{\sqrt{(1+2x^2)^3}}{6}.$$

Můžeme rovněž použít následující substituce

$$\int x\sqrt{1+2x^2}\,\mathrm{d}x = \left[\begin{array}{c} t = 1+2x^2 \\ \mathrm{d}t = 4x\,\mathrm{d}x \end{array}\right] = \frac{1}{4}\int \sqrt{t}\,\mathrm{d}t = \frac{1}{4}\cdot\frac{t^{3/2}}{\frac{3}{2}} = \frac{\sqrt{(1+2x^2)^3}}{6},$$

integrování je pro někoho možná maličko složitější.

3) a) $\displaystyle\int \frac{x^2}{x^3+4}\,\mathrm{d}x = \left[\begin{array}{c} f(x)=x^3+4 \\ f'(x)=3x^2 \end{array}\right] = \frac{1}{3}\ln|x^3+4|.$

4) b) $\displaystyle\int \frac{1}{x^2+3x+3}\,\mathrm{d}x = \left[\begin{array}{c} x^2+3x+3 = \left(x+\frac{3}{2}\right)^2 - \frac{9}{4}+3 = \\ = \left(x+\frac{3}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left(\left(\frac{2x+3}{\sqrt{3}}\right)^2+1\right) \end{array}\right] = \frac{4}{3}\int \frac{1}{\left(\frac{2x+3}{\sqrt{3}}\right)^2+1}\,\mathrm{d}x =$

$\displaystyle= \left[\begin{array}{c} t = \frac{2x+3}{\sqrt{3}} \\ \mathrm{d}t = \frac{2}{\sqrt{3}}\mathrm{d}x \end{array}\right] = \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{1}{t^2+1}\,\mathrm{d}t = \frac{2}{\sqrt{3}}\operatorname{arctg} t = \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x+3}{\sqrt{3}}.$

5) c) $\displaystyle\int \arccos x\,\mathrm{d}x = \left[\begin{array}{c} u'=1,\ v=\arccos x \\ u=x,\ v'=-\frac{1}{\sqrt{1-x^2}} \end{array}\right] = \underbrace{x\arccos x}_{A} + \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = \left[\begin{array}{c} t^2 = 1-x^2 \\ 2t\,\mathrm{d}t = -2x\,\mathrm{d}x \end{array}\right]$

$\displaystyle= A - \int \frac{t\,\mathrm{d}t}{t} = A - t = x\arccos x - \sqrt{1-x^2}.$

6) c) $\displaystyle\frac{x-1}{x^2+1} = \frac{1}{2}\cdot\frac{2x}{x^2+1} - \frac{1}{x^2+1} \Rightarrow \int \frac{x-1}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\ln(1+x^2) - \operatorname{arctg} x.$

6) f) Rozložíme integrand na parciální zlomky: $\displaystyle\frac{x+4}{x\,(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2},$

převedeme na společného jmenovatele: $\displaystyle\frac{x+4}{x\,(x-2)^2} = \frac{A\,(x-2)^2 + B\,x(x-2) + C\,x}{x\,(x-2)^2}$

a porovnáme koeficienty u stejných mocnin v čitatelích. Dostaneme soustavu pro neznámé $A,\ B,\ C$:
$x^2:\ A+B=0,$
$x^1:\ -4A-2B+C=1,$
$x^0:\ 4A=4,$ kterou řeší $A=1,\quad B=-1,\quad C=3.$

Nebo do čitatelů dosadíme kořeny jmenovatele.
$x=0:\ 4=4A \ \Rightarrow\ A=1,$
$x=2:\ 6=2C \ \Rightarrow\ C=3$ Konstantu B vypočítáme např. z rovnice, kterou jsme dostali porovnáním koeficientů u $x^2:\ A+B=0.$
Odtud již snadno dospějeme k výsledku:

$$\int \frac{x+4}{x\,(x-2)^2}\,\mathrm{d}x = \int \left(\frac{1}{x} - \frac{1}{x-2} + \frac{3}{(x-2)^2}\right)\mathrm{d}x = \ln\left|\frac{x}{x-2}\right| - \frac{3}{x-2}.$$